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NULL CONTROLLABILITY OF DEGENERATE NONAUTONOMOUS PARABOLIC EQUATIONS

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Abstract. In this paper we are interested in the study of the null controllability for the one dimensional degenerate nonautonomous parabolic equation

$$u_t - M(t)(a(x)u_x)_x = h\chi_\omega, \qquad (x,t) \in Q = (0,1) \times (0,T),$$

where $\omega = (x_1, x_2)$ is a small nonempty open subset in $(0,1)$, $h \in L^2(\omega \times (0,T))$, the diffusion coefficients $a(\cdot)$ is degenerate at $x = 0$ and $M(\cdot)$ is nondegenerate on $[0,T]$. Also, the boundary conditions are considered to be Dirichlet- or Neumann-type related to the degeneracy rate of $a(\cdot)$. Under some conditions on the functions $a(\cdot)$ and $M(\cdot)$, we prove some global Carleman estimates which will yield the observability inequality of the associated adjoint system and, equivalently, the null controllability of our parabolic equation.

Keywords. Null controllability; nonautonomous parabolic equation; Carleman estimates.

1. Introduction

The purpose of this paper is to establish the null controllability for the linear nonautonomous degenerate parabolic equation

$$(1.1) \qquad \begin{cases} u_t - M(t)(a(x)u_x)_x = h\chi_\omega, & (x,t) \in Q \\ u(1,t) = u(0,t) = 0, \quad t \in (0,T) \\ \text{or} \\ u(1,t) = (au_x)(0,t) = 0, \quad t \in (0,T) \\ u(x,0) = u_0(x), \qquad x \in (0,1), \end{cases}$$

where $\omega = (x_1, x_2)$ is a nonempty open subinterval of $(0,1)$, $Q = (0,1) \times (0,T)$, $a(\cdot)$ and $M(\cdot)$ are time and space diffusion coefficients, the initial condition u_0 is given

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in $L^2(0,1)$, and $h \in L^2(\omega \times (0,T))$ is the control function acting on ω.
The null controllability of nondegenerate parabolic equations have been widely studied in the last years (see in particular [6], [13], [14], [18], [20]). On the other hand, very few results are known in the case of autonomous ($M(t) = 1$) degenerate equations; see [3], [4], [5], [8], [19]. The main tool to study the null controllability of the above parabolic equations is the Carleman estimates. These last estimates are used to show the observability inequality of the adjoint parabolic equations, which is equivalent to the null controllability of the above parabolic equations. The Carleman estimates are the main results of the above references. Recently in [21], the authors established a new Carleman estimate for the autonomous degenerate equations under some general conditions on the degenerate diffusion coefficient a.

The main objective of this paper is the null controllability of a one-dimensional parabolic equation when the diffusion coefficient is allowed to be degenerate at the boundary point $x = 0$ of the interval $I = (0,1)$, and it might be non-autonomous. This can help to study a local null controllability result for a nonlinear degenerate parabolic PDE with nonlocal nonlinearities which has important physical motivations. In particular there exists several examples of real world physical models where nonlocal terms appear naturally:

• In the case of migration of populations, for instance bacteria in a container, we may have instead of M:

$$M(t) = \tilde{M}\left(\int_0^1 u(x,t)\,dx\right)$$

Other more general M can also be found in practice, for instance

$$M(t) = \tilde{M}\left(\int_0^1 u(x,t)\,dx, \int_0^1 u_x(x,t)\,dx\right)$$

• In the context of reaction-diffusion systems, terms of this kind

$$M(t) = \tilde{M}\left(\int_0^1 |u_x(x,t)|^2\,dx\right)$$

appear in the parabolic Kirchhoff equation (see **[10]**).

2. Assumptions and Preliminary Results

In order to study the null controllability of equations 1.1, we make the following assumptions on the coefficients $M(\cdot)$ and $a(\cdot)$.

Hypothesis 1.

1. *M is continuous on $(0,T)$ and there exist two positive constants α_0, β_0 independent of T such that*

$$0 < \alpha_0 \le M(t) \le \beta_0, \quad t \in (0,T),$$

2. *M is derivable on $(0,T)$ and there exists a positive constant γ_0 independent of T such that*
$$|M'(t)| \leq \gamma_0, \quad t \in (0,T).$$

Hypothesis 2.

1. *$a \in C([0,1]) \cap C^1((0,1])$, $a(x) > 0$ in $(0,1]$ and $a(0) = 0$,*
2. *there exists $\alpha \in (0,2)$ such that $xa'(x) \leq \alpha a(x)$ for every $x \in [0,1]$,*
3. *if $\alpha \in [1,2)$, there exist $m > 0$ and $\delta_0 > 0$ such that for every $x \in [0,\delta_0]$, we have*
$$a(x) \geq m \sup_{0\leq y\leq x} a(y).$$

Remark 2.1. It should be noted that Hypothesis 2 appeared for the first time in [21]. It is weaker than the condition given in [5]. In [21] the author also proved that under Hypothesis 2 the classical Hardy-inequality does not hold in general, (see [21, Example 3]) and they proposed an improved Hardy inequality (see Proposition 2.2).

As in [5, 21, 24], for the well-posedness of the problem, the natural setting involves the space

$$H^1_a(0,1) := \{u \in L^2(0,1) \cap H^1_{loc}(0,1) : \int_0^1 a(x)u_x^2 dx < \infty\},$$

which is a Hilbert space for the scalar product

$$\langle u,v\rangle := \int_0^1 uv + a(x)u_x v_x dx, \quad u,v \in H^1_a(0,1). \tag{2.1}$$

For any $u \in H^1_a(0,1)$, the trace of u at $x = 1$ obviously makes sense, which allows us to consider the homogeneous Dirichlet condition at $x = 1$. On the other hand, the trace of u at $x = 0$ only makes sense when $0 \leq \alpha < 1$. However, for $\alpha \geq 1$, the trace at $x = 0$ does not make sense anymore, so one chooses a suitable Neumann boundary condition in this case (see, for example, Lemma 10 of [21]). This leads to the introduction of the following space $H^1_{a,0}(0,1)$ depending on the value of α:

1. For $0 \leq \alpha < 1$,
$$H^1_{a,0}(0,1) := \{u \in H^1_a(0,1) : u(1) = u(0) = 0\}.$$
2. For $1 \leq \alpha < 2$,
$$H^1_{a,0}(0,1) := \{u \in H^1_a(0,1) : u(1) = 0\}.$$

In order to study the well-posedeness of 1.1, we define the operator $(A(t), D(A(t)))$ by

$$A(t)u := M(t)Au := M(t)(a(x)u_x)_x, \tag{2.2}$$

endowed with the domain

$$D(A(t)) = D(A) = \{u \in H^1_{a,0}(0,1) \cap H^2_{\text{loc}}((0,1]) : (a(x)u_x)_x \in L^2(0,1)\}, t \in [0,T].$$

Remark 2.2. The domain $D(A)$ may also be characterized in the case of $\alpha \in [0,1)$ by

$$D(A) := \{u \in L^2(0,1) \cap H^2_{\text{loc}}((0,1]) : a(x)u_x \in H^1(0,1) \text{ and } \quad u(0) = u(1) = 0\},$$

and in the case of $\alpha \in [1,2)$ by

$$D(A) := \{u \in L^2(0,1) \cap H^2_{\text{loc}}((0,1]) : a(x)u_x \in H^1(0,1) \text{ and } \quad (a(x)u_x)(0) = 0 = u(1)\}.$$

Some properties of the operator A are given in the following proposition, see [7].

Proposition 2.1. *The operator $(A, D(A))$ is closed, self-adjoint and negative with the dense domain in $L^2(0,1)$. Hence A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on $L^2(0,1)$.*

From the assumptions on $M(\cdot)$, we can check that the family of operators $(A(t), D(A(t))), 0 \le t \le T$, satisfies the Acquistapace-Terreni conditions (see [1, 2]), thereby generating an evolution family $U(t,s), t \ge s \ge 0$. More precisely, for $t \ge s$ the map $(t,s) \mapsto U(t,s) \in \mathcal{L}(L^2(0,1))$ is continuous and continuously differentiable in t, $U(t,s)L^2(0,1) \subset D(A(t))$, and $\partial U(t,s) = A(t)U(t,s)$. We further have $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = I$ for $t \ge s \ge r \ge 0$. Moreover, for $s \in \mathbb{R}$ and $x \in D(A(s))$, the function $t \mapsto u(t) = U(t,s)x$ is continuous at $t = s$ and u is the unique solution in $C([s,\infty), L^2(0,1)) \cap C^1((s,\infty), L^2(0,1))$ of the Cauchy problem $u'(t) = A(t)u(t), t > s, u(s) = x$. These facts have been established in [1, 2].

The problem 1.1 is well-posed in the sense of the following theorem.

Theorem 2.1. *For all $h \in L^2(\omega \times (0,T))$ and $u_0 \in L^2(0,1)$, the problem 1.1 has a unique weak solution*

$$u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_a(0,1)).$$

Moreover, if $u_0 \in D(A)$, then

$$u \in H^1(0,T; L^2(0,1)) \cap L^2(0,T; D(A)) \cap C([0,T]; H^1_a(0,1)).$$

Throughout this paper we use the following improved Hardy inequality taken from [21, Theorem 2.1], which will be the key ingredient in the proof of our Carleman estimate.

Proposition 2.2. *For all $\eta > 0$ and $0 < \gamma < 2 - \alpha$, there exists some positive constant $C_0(a, \alpha, \gamma, \eta) > 0$ such that for all $u \in H^1_{a,0}(0,1)$, the following inequality holds*

$$\int_0^1 a(x)u_x^2 dx + C_0 \int_0^1 u^2 dx \ge \frac{a(1)(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha}} dx + \eta \int_0^1 \frac{u^2}{x^\gamma} dx. \tag{2.3}$$

3. Carleman Estimates

In this section, we prove a crucial Carleman estimate, which will be useful for proving the observability inequality for the adjoint problem of 1.1. For this purpose, let us consider the parabolic problem

$$(3.1)\qquad \begin{cases} v_t + A(t)v = f, & (x,t) \in Q \\ v(1,t) = v(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in (0,1) \\ v(1,t) = (av_x)(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in [1,2), \\ v(x,T) = v_T(x), \quad x \in (0,1). \end{cases}$$

Now, we consider $0 < \gamma < 2 - \alpha$ and $\varphi(x,t) = \theta(t)p(x)$. Here

$$(3.2)\qquad \theta(t) = [t(T-t)]^{-k}, k = 1 + 2/\gamma, \quad p(x) = \frac{c_1}{2-\alpha}\Big(\int_0^x \frac{y}{a(y)}dy - c_2\Big)$$

where $c_1 > 0$ and $c_2 > \frac{1}{a(1)(2-\alpha)}$ such that $p(x) < 0$ for all $x \in [0,1]$. Observe that there exists some constant $c = c(T) > 0$ such that

$$(3.3)\qquad |\theta_t| \le c\theta^{1+1/k}, \quad |\theta_{tt}| \le c\theta^{1+2/k} \quad \text{in} \quad (0,T).$$

We have the following main result.

Theorem 3.1. *Assume that the functions $a(\cdot)$ and $M(\cdot)$ satisfy Hypotheses 1 and 2 and let $T > 0$. For every $0 < \gamma < 2-\alpha$ there exists $s_0 = s_0(T, a, \alpha, \gamma, \beta_0, \alpha_0, \gamma_0) > 0$ such that for all $s \ge s_0$ and all solutions v of (3.1), we have*

$$\frac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3 \frac{x^2}{a(x)} v^2 e^{2s\varphi} dxdt + s\int\int_Q \theta a(x) v_x^2 e^{2s\varphi} dxdt + sa(1)(1-\alpha)^2 \int\int_Q \theta \frac{v^2}{x^{2-\alpha}} e^{2s\varphi} dxdt$$
$$+ s\int\int_Q \theta \frac{v^2}{x^\gamma} e^{2s\varphi} dxdt \le \frac{18}{\alpha_0^2}\Big(\int\int_Q f^2 e^{2s\varphi} dxdt + \frac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta v_x^2(1,t) e^{2s\varphi(1,t)} dt\Big).$$

Proof For the proof, let us define the function $w = e^{s\varphi}v$, where $s > 0$ and v is the solution to (3.1). Then w satisfies

$$\begin{cases} (e^{-s\varphi}w)_t + M(t)\Big(a(x)(e^{-s\varphi}w)_x\Big)_x = f, & (x,t) \in Q, \\ w(1,t) = w(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in (0,1), \\ w(1,t) = (aw_x)(0,t) = s(\varphi_x a w)(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in [1,2), \\ w(x,T) = w(x,0) = 0, \quad x \in (0,1). \end{cases}$$
(3.4)

Set

$$Lv := v_t + M(t)(a(x)v_x)_x, \quad L_s w := e^{s\varphi} L(e^{-s\varphi}w).$$

$$L_s w := L_1 w + L_2 w$$

where

$$\begin{aligned} L_1 w &:= M(t)(a(x)w_x)_x - s\varphi_t w + s^2 M(t)a(x)\varphi_x^2 w, \\ L_2 w &:= w_t - 2sM(t)a(x)\varphi_x w_x - sM(t)(a(x)\varphi_x)_x w. \end{aligned} \tag{3.5}$$

Therefore, we have

$$2\langle L_1 w, L_2 w\rangle \le \|L_1 w + L_2 w\|^2 = \|f e^{s\varphi}\|^2, \tag{3.6}$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the usual norm and scalar product in $L^2(Q)$, respectively. The proof of Theorem 3.1 is based on the computation of the scalar product $(L_1 w, L_2 w)$ which comes in the following lemma.

Lemma 3.1. *The scalar product $\langle L_1 w, L_2 w\rangle$ may be written as a sum of the distributed term $(d.t)$ and boundary term $(b.t)$, where the distributed term $(d.t)$ is given by*

$$\begin{aligned} (d.t) \;=\; & -2s^2 \int\int_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt \\ & + s\int\int_Q \theta(2ap_{xx} + a'p_x)a(x)M^2(t)w_x^2 dxdt \\ & + s^3 \int\int_Q \theta^3(2ap_{xx} + a'p_x)a(x)p_x^2 M^2(t)w^2 dxdt \\ & + \frac{1}{2}\int\int_Q M'(t)a(x)w_x^2 dxdt - \frac{s^2}{2}\int\int_Q M'(t)\theta^2 a(x)p_x^2 w^2 dxdt \end{aligned} \tag{3.7}$$

whereas the boundary term $(b.t)$ is given by

$$(b.t) = -s\int_0^T \Big[M^2(t)\theta p_x (a(x)w_x)^2\Big]_0^1 dt. \tag{3.8}$$

Proof To simplify the notation, we will denote by $(L_i w)_j, (1 \le i \le 2, 1 \le j \le 3)$ the j^{th} term in the expression of $L_i w$ given in (3.5). We will develop nine terms appearing in the product scalar $\langle L_1 w, L_2 w\rangle$. For this, we will integrate by parts several times respect to the space and time variables. First we have

$$\begin{aligned} \langle (L_1 w)_1, (L_2 w)_1\rangle &= \int\int_Q M(t)(a(x)w_x)_x w_t dxdt \\ &= \int_0^T \Big[M(t)a(x)w_x w_t\Big]_0^1 dt - \int\int_Q M(t)a(x)w_x w_{tx} dxdt \qquad (3.9) \\ &= \int_0^T \Big[M(t)a(x)w_x w_t\Big]_0^1 dt - \frac{1}{2}\int_0^1 \Big[M(t)a(x)w_x^2\Big]_0^T dx + \frac{1}{2}\int\int_Q M'(t)a(x)w_x^2 dxdt. \end{aligned}$$

Then

$$\begin{aligned}\langle (L_1w)_2,(L_2w)_1\rangle &= -s\int\int_Q \varphi_t w w_t dxdt\\ &= -\frac{s}{2}\int_0^1 \Big[\varphi_t w^2\Big]_0^T dx + \frac{s}{2}\int\int_Q \varphi_{tt} w^2 dxdt \qquad (3.10)\\ &= -\frac{s}{2}\int_0^1 \Big[\varphi_t w^2\Big]_0^T dx + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt.\end{aligned}$$

We also have

$$\begin{aligned}\langle (L_1w)_3,(L_2w)_1\rangle &= s^2\int\int_Q a(x)M(t)\varphi_x^2 w w_t dxdt = \tfrac{s^2}{2}\int_0^1 \Big[a(x)M(t)\varphi_x^2 w^2\Big]_0^T dx\\ &-s^2\int\int_Q a(x)M(t)\varphi_x\varphi_{xt} w^2 dxdt - \tfrac{s^2}{2}\int\int_Q a(x)M'(t)\varphi_x^2 w^2 dxdt\\ &= \tfrac{s^2}{2}\int_0^1 \Big[a(x)M(t)\varphi_x^2 w^2\Big]_0^T dx - s^2\int\int_Q a(x)M(t)p_x^2\theta\theta_t w^2 dxdt \qquad (3.11)\\ &\quad -\tfrac{s^2}{2}\int\int_Q a(x)M'(t)\theta^2 p_x^2 w^2 dxdt.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle (L_1w)_1,(L_2w)_2\rangle &= -2s\int\int_Q M^2(t)\varphi_x(a(x)w_x)(a(x)w_x)_x dxdt\\ &= -s\int_0^T \Big[M^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1 dt + s\int\int_Q M^2(t)\varphi_{xx}a^2(x)w_x^2 dxdt \qquad (3.12)\\ &= -s\int_0^T \Big[M^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1 dt + s\int\int_Q M^2(t)\theta p_{xx}a^2(x)w_x^2 dxdt.\end{aligned}$$

We also have

$$\begin{aligned}\langle (L_1w)_2,(L_2w)_2\rangle &= 2s^2\int\int_Q M(t)a(x)\varphi_x\varphi_t w w_x dxdt\\ &= s^2\int_0^T \Big[M(t)a(x)\varphi_t\varphi_x w^2\Big]_0^1 dt - s^2\int\int_Q M(t)a(x)\varphi_{tx}\varphi_x w^2 dxdt\\ &\quad -s^2\int\int_Q M(t)\varphi_t(a(x)\varphi_x)_x w^2 dxdt\end{aligned}$$

$$= s^2 \int_0^T \Big[M(t)a(x)\varphi_t\varphi_x w^2\Big]_0^1 dt - s^2 \int\int_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt \tag{3.13}$$
$$-s^2 \int\int_Q M(t)\theta_t p(a(x)\varphi_x)_x w^2 dxdt.$$

Additionally, we find that

$$\langle (L_1 w)_3, (L_2 w)_2\rangle = -2s^3 \int\int_Q M^2(t)a^2(x)\varphi_x^3\varphi_t w w_x dxdt \tag{3.14}$$

$$= -s^3 \int_0^T \Big[M^2(t)a^2(x)\varphi_x^3 w^2\Big]_0^1 dt + s^3 \int\int_Q M^2(t)\Big[2aa'\varphi_x + 3a^2\varphi_{xx}\Big]\varphi_x^2 w^2 dxdt.$$

Let us now consider the scalar product

$$\langle (L_1 w)_1, (L_2 w)_3\rangle = -s \int\int_Q M^2(t)(a(x)w_x)_x(a(x)\varphi_x)_x w dxdt \tag{3.15}$$

$$= -s \int_0^T \Big[M^2(t)(a(x)\varphi_x)_x a(x)w_x w\Big]_0^1 dt + s \int\int_Q M^2(t)(a(x)\varphi_x)_{xx} a(x) w w_x dxdt$$
$$+s \int\int_Q M^2(t)(a(x)\varphi_x)_x a(x) w_x^2 dxdt$$

$$= -s \int_0^T \Big[M^2(t)(a(x)\varphi_x)_x a(x) w w_x\Big]_0^1 dt + s \int\int_Q M^2(t)(a(x)\varphi_x)_x a(x) w_x^2 dxdt,$$

since $(a(x)\varphi_x)_{xx} = 0$.
Furthemore

$$\langle (L_1 w)_2, (L_2 w)_3\rangle = s^2 \int\int_Q M(t)\varphi_t(a(x)\varphi_x)_x w^2 dxdt. \tag{3.16}$$

Finally, we have

$$\langle (L_1 w)_3, (L_2 w)_3\rangle = -s^3 \int\int_Q M^2(t)a(x)\varphi_x^2(a(x)\varphi_x)_x w^2 dxdt. \tag{3.17}$$

Additionally (3.9)-(3.17), we find that

$$(d.t) \quad = \quad -2s^2 \int\int_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt$$

$$+s\int\int_Q \theta(2ap_{xx}+a'p_x)a(x)M^2(t)w_x^2dxdt$$

$$+s^3\int\int_Q \theta^3(2ap_{xx}+a'p_x)a(x)p_x^2M^2(t)w^2dxdt \tag{3.18}$$

$$+\frac{1}{2}\int\int_Q M'(t)a(x)w_x^2dxdt-\frac{s^2}{2}\int\int_Q M'(t)\theta^2a(x)p_x^2w^2dxdt,$$

and

$$(b.t)\quad =\int_0^T\Big[M(t)a(x)w_xw_t-sM^2(t)\varphi_x(a(x)w_x)^2+s^2M(t)a(x)\varphi_t\varphi_xw^2$$

$$-s^3M^2(t)a^2(x)\varphi_x^3w^2-sM^2(t)(a(x)\varphi_x)_xa(x)ww_x\Big]_0^1dt \tag{3.19}$$

$$+\int_0^1\Big[-\tfrac{1}{2}M(t)a(x)w_x^2-\tfrac{s}{2}\varphi_tw^2+\tfrac{s^2}{2}a(x)M(t)\varphi_x^2w^2\Big]_0^Tdx$$

$$=-\int_0^T\Big[sM^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1dt.$$

The proof of (3.19) is similar to that in [5] and the fact was used that $M(\cdot)$ is a bounded function. Now we put $(d.t)=A+B$, where

$$A=\quad -2s^2\int\int_Q M(t)a(x)\theta\theta_tp_x^2w^2dxdt+\tfrac{s}{2}\int\int_Q \theta_{tt}pw^2dxdt$$

$$+s\int\int_Q \theta(2ap_{xx}+a'p_x)a(x)M^2(t)w_x^2dxdt$$

$$+s^3\int\int_Q \theta^3(2ap_{xx}+a'p_x)a(x)p_x^2M^2(t)w^2dxdt, \tag{3.20}$$

and

$$B=\frac{1}{2}\int\int_Q M'(t)a(x)w_x^2dxdt-\frac{s^2}{2}\int\int_Q M'(t)\theta^2a(x)p_x^2w^2dxdt. \tag{3.21}$$

Observe that

$$A+B\le\frac{1}{2}\|fe^{s\varphi}\|^2-(b.t). \tag{3.22}$$

The crucial step is to prove the following estimate.

Lemma 3.2. *There exists a positive constant $s_1=s_1(T,a,\alpha,\alpha_0,\beta_0,\gamma,\gamma_0)>0$ such that for all $s\ge s_1$ we have,*

$$A+B\ge\quad \tfrac{s^3\alpha_0^2}{4(2-\alpha)^2}\int\int_Q \theta^3\tfrac{x^2}{a(x)}w^2dxdt+s\tfrac{\alpha_0^2}{4}\int\int_Q \theta a(x)w_x^2dxdt$$

$$+\tfrac{sa(1)(1-\alpha)^2\alpha_0^2}{4}\int\int_Q \theta\tfrac{w^2}{x^{2-\alpha}}dxdt+\tfrac{s}{4}\alpha_0^2\int\int_Q \theta\tfrac{w^2}{x^{\gamma}}dxdt. \tag{3.23}$$

Proof By the assumption $xa'(x) \le \alpha a(x)$ and the fact that $p_x = \frac{c_1 x}{(2-\alpha)a(x)}$, and the observation that

$$\begin{aligned} 2ap_{xx} + a'p_x &= \tfrac{c_1}{2-\alpha}\Big(\tfrac{2a(x)-xa'(x)}{a(x)}\Big) \\ &\ge \tfrac{c_1}{2-\alpha}\Big(\tfrac{2a(x)-\alpha a(x)}{a(x)}\Big) = c_1 \end{aligned} \tag{3.24}$$

one can estimate A in the following way

$$\begin{aligned} A \ge & -\frac{2s^2c_1^2}{(2-\alpha)^2}\beta_0 \int\int_Q \theta\theta_t \frac{x^2}{a(x)} w^2 dxdt + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt \\ & + sc_1\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt + \frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2}\int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt. \end{aligned} \tag{3.25}$$

According to the relation (3.3), we know that $|\theta\theta_t| \le c\theta^{2+1/k} \le c'\theta^3$ and we obtain

$$\begin{aligned} A \ge & \Big(\frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2} - \frac{2s^2c_1^2c'}{(2-\alpha)^2}\beta_0\Big)\int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt \\ & + sc_1\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt. \end{aligned} \tag{3.26}$$

Let

$$A_1 = c_1\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt + \int\int_Q \theta_{tt} p w^2 dxdt. \tag{3.27}$$

Therefore

$$\begin{aligned} A \ge & \Big(\frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2} - \frac{2s^2c_1^2c'}{(2-\alpha)^2}\beta_0\Big)\int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt \\ & + \frac{s}{2}c_1\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt + \frac{s}{2}A_1. \end{aligned} \tag{3.28}$$

We apply the improved Hardy inequality (2.3), with $\eta = 1$, which gives

$$\int_0^1 a(x) w_x^2 dx + c_0 \int_0^1 w^2 dx \ge \frac{a(1)(1-\alpha)^2}{4}\int_0^1 \frac{w^2}{x^{2-\alpha}} dx + \int_0^1 \frac{w^2}{x^\gamma} dx, \tag{3.29}$$

for suitable $c_0 = c_0(a, \alpha, \gamma)$. Therefore, we can write

$$\begin{aligned} A_1 \ge & \frac{a(1)(1-\alpha)^2 c_1\alpha_0^2}{4}\int\int_Q \theta \frac{w^2}{x^{2-\alpha}} dxdt + c_1\alpha_0^2 \int\int_Q \theta \frac{w^2}{x^\gamma} dxdt \\ & - c_0c_1\alpha_0^2 \int\int_Q \theta w^2 dxdt + \int\int_Q \theta_{tt} p w^2 dxdt. \end{aligned} \tag{3.30}$$

Finally, we need to estimate the term

$$A_2 = \int\int_Q \theta_{tt} p w^2 dxdt - c_0 c_1 \alpha_0^2 \int\int_Q \theta w^2 dxdt. \tag{3.31}$$

By (3.3), there exists a positive constant c_3 such that

$$|A_2| \le c_3 \int\int_Q \theta^{1+2/k} w^2 dxdt. \tag{3.32}$$

Now, we consider $q = \frac{k}{k-1}$ and $q' = k$, so that $\frac{1}{q} + \frac{1}{q'} = 1$. Using the Young inequality, we have for all $\varepsilon > 0$

$$\begin{aligned} |A_2| &\le c_3 \int\int_Q \left(\theta^{1+2/k-\frac{3}{q'}} a^{\frac{1}{q'}} x^{\frac{-2}{q'}} w^{\frac{2}{q}}\right)\left(\theta^{\frac{3}{q'}} a^{\frac{-1}{q'}} x^{\frac{2}{q'}} w^{\frac{2}{q'}}\right) dxdt \\ &\le c_3\varepsilon \int\int_Q \theta^{(1+2/k-\frac{3}{q'})q} a^{\frac{q}{q'}} x^{\frac{-2q}{q'}} w^2 dxdt + c_3 c(\varepsilon) \int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt, \end{aligned} \tag{3.33}$$

where $c(\varepsilon) = \frac{1}{q'}(\varepsilon q)^{\frac{-q'}{q}}$. Observe that

$$(1 + 2/k - \frac{3}{q'})q = 1, \quad \frac{2q}{q'} = \gamma. \tag{3.34}$$

Using the fact that $a(\cdot)$ is continuous on $[0, 1]$, there exists a positive constant c_4 such that $(a(x))^{\frac{q}{q'}} \le c_4$ for every $x \in [0, 1]$, and then

$$A_2 \ge -c_3 c_4 \varepsilon \int\int_Q \theta \frac{w^2}{x^\gamma} dxdt - c_3 c(\varepsilon) \int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt. \tag{3.35}$$

Putting the estimate (3.35) in (3.30) and using (3.28), we obtain

$$\begin{aligned} A \ &\ge \left(\tfrac{s^3 c_1^3 \alpha_0^2}{(2-\alpha)^2} - \tfrac{2s^2 c_1^2 c'}{(2-\alpha)^2}\beta_0 - \tfrac{s c_3 c(\varepsilon)}{2}\right) \int\int_Q \theta^3 \tfrac{x^2}{a(x)} w^2 dxdt + \tfrac{s}{2} c_1 \alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt \\ &\quad + \tfrac{s a(1)(1-\alpha)^2 c_1 \alpha_0^2}{8} \int\int_Q \theta \tfrac{w^2}{x^{2-\alpha}} dxdt + \tfrac{s}{2}\left(c_1\alpha_0^2 - c_3 c_4 \varepsilon\right) \int\int_Q \theta \tfrac{w^2}{x^\gamma} dxdt. \end{aligned} \tag{3.36}$$

Now, take $c_1 = 2$ and $\varepsilon = \varepsilon(a, \alpha, \alpha_0, \gamma) = \frac{3\alpha_0^2}{2c_3c_4}$. Thus there exists $s_2 = s_2(T, a, \alpha, \alpha_0, \beta_0, \gamma) > 0$ such that for all $s \ge s_2$

$$\begin{aligned} A \ &\ge \tfrac{s^3 \alpha_0^2}{(2-\alpha)^2} \int\int_Q \theta^3 \tfrac{x^2}{a(x)} w^2 dxdt + s\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt \\ &\quad + \tfrac{s a(1)(1-\alpha)^2 \alpha_0^2}{4} \int\int_Q \theta \tfrac{w^2}{x^{2-\alpha}} dxdt + \tfrac{s}{4}\alpha_0^2 \int\int_Q \theta \tfrac{w^2}{x^\gamma} dxdt. \end{aligned} \tag{3.37}$$

On the other hand, we have

$$\begin{aligned}|B| &\leq \tfrac{1}{2}\int\int_Q |M'(t)|a(x)w_x^2dxdt + \tfrac{s^2}{2}\int\int_Q |M'(t)|\theta^2 a(x)p_x^2 w^2dxdt\\ &\leq \tfrac{\gamma_0}{2}\int\int_Q a(x)w_x^2dxdt + \tfrac{2s^2\gamma_0}{(2-\alpha)^2}\int\int_Q \theta^2\tfrac{x^2}{a(x)}w^2dxdt\\ &\leq 2\gamma_0\Big(\int\int_Q a(x)w_x^2dxdt + \tfrac{s^2}{(2-\alpha)^2}\int\int_Q \theta^2\tfrac{x^2}{a(x)}w^2dxdt\Big)\\ &\leq 2c_5\gamma_0\Big(\int\int_Q \theta a(x)w_x^2dxdt + \tfrac{s^2}{(2-\alpha)^2}\int\int_Q \theta^3\tfrac{x^2}{a(x)}w^2dxdt\Big)\\ &\leq \tfrac{3\alpha_0^2}{4}\Big(s\int\int_Q \theta a(x)w_x^2dxdt + \tfrac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3\tfrac{x^2}{a(x)}w^2dxdt\Big) \qquad (3.38)\end{aligned}$$

for all $s \geq \frac{8c_5\gamma_0}{3\alpha_0^2}$. Therefore,

$$B \geq -s\frac{3\alpha_0^2}{4}\int\int_Q \theta a(x)w_x^2dxdt - \frac{3s^3\alpha_0^2}{4(2-\alpha)^2}\int\int_Q \theta^3\frac{x^2}{a(x)}w^2dxdt. \qquad (3.39)$$

By adding (3.37) and (3.39), for $s \geq s_1(a,\alpha,\gamma,\beta_0,\alpha_0,\gamma_0) > 0$, with $s_1 = \max\{s_2, \frac{8c_5\gamma_0}{3\alpha_0^2}\}$, we obtain the complet proof of Lemma 3.2.

Now, using the fact that $\int_0^T \Big[sM^2(t)\varphi_x(a(x)w_x)^2\Big]_0 dt$ is non-negative, the right hand of (3.22) becomes

$$\frac{1}{2}\|fe^{s\varphi}\|^2 - (b.t) \leq \frac{1}{2}\int\int_Q f^2e^{2s\varphi}dxdt + \frac{2sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta w_x^2(1,t)dt. \qquad (3.40)$$

From (3.22), (3.40) and Lemma 3.2, we obtain

$$\begin{aligned}&\tfrac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3\tfrac{x^2}{a(x)}w^2dxdt + s\int\int_Q \theta a(x)w_x^2dxdt + sa(1)(1-\alpha)^2\int\int_Q \theta\tfrac{w^2}{x^{2-\alpha}}dxdt\\ &+s\int\int_Q \theta\tfrac{w^2}{x^\gamma}dx \leq \tfrac{2}{\alpha_0^2}\Big(\int\int_Q f^2e^{2s\varphi}dxdt + \tfrac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta w_x^2(1,t)dt\Big)\end{aligned} \qquad (3.41)$$

for all $s \geq s_1$. Finally, we turn back to our original function $v = e^{-s\varphi}w$. Using that

$$v_x = \Big(-s\theta\frac{2}{2-\alpha}\frac{x}{a(x)}w + w_x\Big)e^{-s\varphi},$$

by the Young inequality, we find

$$\begin{aligned}s\int\int_Q \theta a(x)v_x^2e^{2s\varphi}dxdt &\leq 8\frac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3\frac{x^2}{a(x)}w^2dxdt\\ &+2s\int\int_Q \theta a(x)w_x^2dxdt. \qquad (3.42)\end{aligned}$$

Also, we have

$$w_x(1,t) = \Big(s\varphi_x v(1,t) + v_x(1,t)\Big)e^{s\varphi(1,t)}$$

$$= v_x(1,t)e^{s\varphi(1,t)}. \tag{3.43}$$

Consequently, from 3.41-3.43, we have

$$\frac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3 \frac{x^2}{a(x)} v^2 e^{2s\varphi}dxdt + s\int\int_Q \theta a(x) v_x^2 e^{2s\varphi}dxdt + sa(1)(1-\alpha)^2\int\int_Q \theta \frac{v^2}{x^{2-\alpha}} e^{2s\varphi}dxdt$$

$$+s\int\int_Q \theta \frac{v^2}{x^\gamma} e^{2s\varphi}dxdt \leq \frac{18}{\alpha_0^2}\Big(\int\int_Q f^2 e^{2s\varphi}dxdt + \frac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta v_x^2(1,t)e^{2s\varphi(1,t)}dt\Big)$$

for all $s \geq s_0$, with $s_0 = s_1$

4. Observability Inequality and null controllability

In order to prove the controllability of (1.1), we first need to derive the observability inequality for the following adjoint problem

$$\begin{cases} v_t + A(t)v = 0, & (x,t) \in Q \\ v(1,t) = v(0,t) = 0, & \text{in the case} \quad \alpha \in (0,1) \quad t \in (0,T) \\ v(1,t) = (av_x)(0,t) = 0, & \text{in the case} \quad \alpha \in [1,2) \quad t \in (0,T) \\ v(x,T) = v_T(x), & x \in (0,1). \end{cases} \tag{4.1}$$

More precisely, we need to prove the following inequality

Proposition 4.1. *Assume that the coefficients $a(\cdot)$ and $M(\cdot)$ satisfiy the hypothesis (2) and (1), respectivly, and let $T > 0$ be given and ω be a nonempty subinterval of $(0,1)$. Then there existe a positive constant $C = C(T,a,\alpha,M)$ such that the following observability inequality is valid for every solution v of (4.1)*

$$\int_0^1 v^2(x,0)dx \leq C\int_0^T\int_\omega v^2(x,t)dxdt. \tag{4.2}$$

Now, by standard arguments, a null controllability result follows.

Theorem 4.1. *Let $T > 0$ be given, and ω be a nonempty subinterval of $(0,1)$. Then for all $u_0 \in L^2(0,1)$, there exists $h \in L^2(\omega \times (0,T))$ such that the solution u of (1.1) satisfies $u(x,T) = 0$, for every $x \in (0,1)$. Furthermore, we have the estimate*

$$\|h\|_{L^2(\omega\times(0,T))} \leq C\|u_0\|_{L^2(0,1)} \tag{4.3}$$

for some constant C.

To prove the observability inequality, we need the following lemma.

Lemma 4.1. *(Caccioppoli's inequality) Let $\omega_0 \Subset \omega$ be a nonempty open set. Then, there exists a positive constant $\tilde{c}$ such that for every solution of (4.1)*

$$\int_0^T \int_{\omega_0} v_x^2 e^{2s\varphi} dxdt \leq \tilde{c} \int_0^T \int_\omega v^2 dxdt.$$

Proof Let us consider a smooth function $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & \forall x \in \mathbb{R}, \\ \xi(x) = 1, & x \in \omega_0 \\ \xi(x) = 0, & x \notin \bar{\omega} \end{cases} \tag{4.4}$$

and $\xi > 0$ for $x \in \omega$. Then

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \int_0^1 \xi^2 e^{2s\varphi} v^2 dxdt \\ &= 2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + 2 \int\int_Q \xi^2 e^{2s\varphi} v v_t dxdt \\ &= 2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt - 2 \int\int_Q \xi^2 M(t) e^{2s\varphi} v (a(x) v_x)_x dxdt \\ &= 2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + 2 \int\int_Q M(t) (\xi^2 e^{2s\varphi})_x a(x) v v_x dxdt + 2 \int\int_Q M(t) \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt. \end{aligned}$$

Hence,

$$\begin{aligned} 2 \int\int_Q M(t) \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt &= -2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt - 2 \int\int_Q M(t) (\xi^2 e^{2s\varphi})_x a(x) v v_x dxdt \\ &\leq -2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + \frac{\beta_0^2}{\alpha_0} \int\int_Q \left(\sqrt{a} \frac{(\xi^2 e^{2s\varphi})_x}{\xi e^{s\varphi}} v \right)^2 dxdt \\ &\quad + \alpha_0 \int\int_Q \left(\sqrt{a} \xi e^{s\varphi} v_x \right)^2 dxdt. \end{aligned} \tag{4.5}$$

In other hand we have

$$2\alpha_0 \int\int_Q \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt \leq 2 \int\int_Q M(t) \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt. \tag{4.6}$$

Using (4.5) and (4.6), we obtain

$$\begin{aligned} &\alpha_0 \int\int_Q \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt \\ &\leq -2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + \frac{\beta_0^2}{\alpha_0} \int\int_Q \left(\sqrt{a} \frac{(\xi^2 e^{2s\varphi})_x}{\xi e^{s\varphi}} \right) v^2 dxdt. \end{aligned} \tag{4.7}$$

Due to the definition of ξ and the fact that $\varphi_t e^{s\varphi}$ and $\varphi_t e^{s\varphi}$ are bounded functions on $\omega \times (0, T)$, the inequality (4.7) implies that there exists a positive constant $\tilde{c_1}$

such that

$$\min_{x\in\omega_0}(a(x))\int_0^T\int_{\omega_0} v_x^2 e^{2s\varphi}dxdt \le \int_0^T\int_{\omega_0} a(x)v_x^2 e^{2s\varphi}dxdt \le \int\int_Q \xi^2 a(x)v_x^2 e^{2s\varphi}dxdt$$
$$\le \tilde{c_1}\int_0^T\int_\omega v^2 dxdt.$$

We deduce that

$$\int_0^T\int_{\omega_0} v_x^2 e^{2s\varphi}dxdt \le \tilde{c}\int_0^T\int_\omega v^2 dxdt, \tag{4.8}$$

with

$$\tilde{c} = \frac{\tilde{c_1}}{\min_{x\in\omega_0}(a(x))}.$$

The proof of the observability inequality (4.2). The proof can be derived in three steps.
Step 1: We consider $\omega_0 = (x_1', x_2') \Subset \omega = (x_1, x_2)$ and a smooth cut-off function $0 \le \xi \le 1$ such that

$$\begin{cases} \xi(x) = 1, & x \in (0, x_1') \\ \xi(x) = 0, & x \in (x_2', 1)). \end{cases} \tag{4.9}$$

The function $w := \xi v$, where v is the solution to (4.1), satisfies the following problem

$$\begin{cases} w_t + M(t)(a(x)w_x)_x = M(t)(2a(x)\xi' v_x + (a(x)\xi')'v) := f, & (x,t) \in Q \\ w(1,t) = w(0,t) = 0, \quad t \in (0,T), \quad \text{in the case } \alpha \in (0,1), \\ w(1,t) = (aw_x)(0,t) = 0, \quad t \in (0,T), \quad \text{in the case } \alpha \in [1,2), \\ w(x,T) = w_T(x), \quad x \in (0,1). \end{cases} \tag{4.10}$$

Applying Theorem 4.1 with $\gamma = \frac{2-\alpha}{2}$ and observe that $w_x(1,t) = 0$, we get

$$\begin{aligned} s_0\int\int_Q \theta w^2 e^{2s_0\varphi}dxdt &\le s_0\int\int_Q \theta\frac{w^2}{x^\gamma}e^{2s_0\varphi}dxdt \\ &\le \frac{18}{\alpha_0^2}\int\int_Q M^2(t)(2a(x)\xi' v_x + (a(x)\xi')'v)^2 e^{2s_0\varphi}dxdt \\ &\le c\int_0^T\int_{\omega_0}(v_x^2 + v^2)e^{2s_0\varphi}dxdt. \end{aligned}$$

According to Lemma 4.1, we obtain

$$s_0\int\int_Q \theta w^2 e^{2s_0\varphi}dxdt \le \check{c}\int_0^T\int_\omega v^2 dxdt.$$

Next, using the definition of ξ, we obtain

$$\int_0^T \int_0^{x_1} \theta v^2 e^{2s_0\varphi} dxdt \leq \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt.$$

Using the fact that $p(x)$ and θ satisfies the following inequality

$$\theta(t) \leq \Big(\frac{3T^2}{16}\Big)^{-k}, t \in [T/4, 3T/4],$$

and

$$|p(x)| \leq \frac{2c_2}{2-\alpha}, \quad \text{for all} \quad x \in [0,1].$$

Then there exists a positive constant $c = c(T, a, \alpha)$ such that

$$e^{-cs_0} \int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt \leq \Big(\frac{T^2}{4}\Big)^k \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt,$$

which implies

$$\int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt \leq e^{cs_0} \Big(\frac{T^2}{4}\Big)^k \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt.$$

Step 2: We define $z = (1-\xi)v$. Then, z satisfies the folowing problem

$$\begin{cases} z_t + M(t)(a(x)z_x)_x = M(t)(2a(x)(1-\xi)'v_x + (a(x)(1-\xi)')'v) := f, & (x,t) \in (x_1', 1) \times (0,T) \\ z(1,t) = z(x_1', t) = 0, \quad t \in (0,T), \\ z(x,T) = z_T(x), \qquad x \in (x_1', 1). \end{cases} \tag{4.11}$$

In this case, we use classical Carleman estimates, since the operator $(a(x)z_x)_x$ is nondegenerate on $(x_1', 1)$. Then v can be estimated on $(x_2, 1) \subset (x_1', 1)$ in the same way, see [14]. Therefore

$$\begin{aligned} \int_{T/4}^{3T/4} \int_0^1 v^2 dxdt &= \int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt + \int_{T/4}^{3T/4} \int_\omega v^2 dxdt + \int_{T/4}^{3T/4} \int_{x_2}^1 v^2 dxdt \\ &\leq C \int_0^T \int_\omega v^2 dxdt. \end{aligned} \tag{4.12}$$

Step 3: Multiplying both sides of (4.1) by v and integrate on $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt} \int_0^1 v^2 dx = M(t) \int_0^1 a(x) v_x^2 dx \geq 0, \quad t \in (0,T).$$

Hence, we deduce that

$$\|v(\cdot,0)\|^2_{L^2(0,1)} \leq \|v(\cdot,t)\|^2_{L^2(0,1)} \quad \text{for all} \quad t \in (0,T). \tag{4.13}$$

Then integrate (4.13) on $(T/4, 3T/4)$ and use (4.13) to obtain

$$\int_0^1 v^2(x,0) dx \leq \frac{2}{T} \int_{T/4}^{3T/4} \int_0^1 v^2 dxdt \leq \tilde{C} \int_0^T \int_\omega v^2 dxdt. \tag{4.14}$$

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