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# COMMON FIXED POINTS OF A PAIR OF SELFMAPS SATISFYING CERTAIN WEAKLY CONTRACTIVE INEQUALITY INVOLVING RATIONAL TYPE EXPRESSIONS VIA TWO AUXILIARY FUNCTIONS IN PARTIALLY ORDERED METRIC SPACES

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**Abstract.** In this paper, we prove the existence of coincidence and common fixed points of a pair of selfmaps satisfying a certain weakly contractive inequality with two auxiliary functions involving rational type expressions in partially ordered metric spaces. These results extend some of the known existing results in the literature from a single selfmap to a pair of selfmaps. Examples are provided in support of our results.
**Keywords**: common fixed points, partially ordered metric spaces, rational type contraction mappings, auxiliary functions

## 1. Introduction

The Banach contraction principle is one of the pivotal results in fixed point theory. It is a very popular tool for solving existence problems in many different fields of mathematics. Ran and Reurings [15] extended the Banach contraction principle in partially ordered sets. For more work on the existence of fixed points in partially ordered metric spaces, we refer the reader to [1, 3, 7, 8, 9, 13, 16].

In 1975, Dass and Gupta [6] extended the Banach contraction principle through rational expression as follows.

**Theorem 1.1.** (Dass and Gupta [6]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping such that there exist $\alpha,\ \beta \geq 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) \tag{1.1}$$

for all $x,\ y\ \in X$.
Then $T$ has a unique fixed point.

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**Definition 1.1.** Let $(X, \preceq)$ be a partially ordered set. A mapping $T : X \to X$ is said to be non-decreasing if for any $x, y \in X$, $x \preceq y$ implies that $Tx \preceq Ty$.

In 2013, Cabrera, Harjani and Sadarangani [4] proved the above theorem in the context of partially ordered metric spaces as follows.

**Theorem 1.2.** (Cabrera, Harjani and Sadarangani [4]) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that (1.1) is satisfied for all $x,\ y\ \in X$ with $x \preceq y$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point.

**Theorem 1.3.** (Cabrera, Harjani and Sadarangani [4]) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in $X$ such that $x_n \to x$ then $x_n \preceq x$ for all $n \in N$. Let $T : X \to X$ be a non-decreasing mapping such that (1.1) is satisfied for all $x,\ y\ \in X$ with $x \preceq y$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ then $T$ has a fixed point.

**Theorem 1.4.** (Cabrera, Harjani and Sadarangani [4]) In addition to the hypotheses of Theorem 1.2 (Theorem 1.3), suppose that for every $x,\ y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then $T$ has a unique fixed point.

We write

$\Phi = \{\varphi : [0,\ \infty) \to [0,\ \infty) : \varphi$ is monotonic non-decreasing, continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$.

$\Psi = \{\psi : [0,\ \infty) \to [0,\ \infty) :$ for any sequence $\{t_n\}$ in $[0, \infty)$ with $t_n \to t > 0$ implies that $\underline{\lim}\psi(t_n) > 0\}$.

**Remark 1.1.** If $\psi \in \Psi$ then $\psi(t) > 0$ for $t > 0$.

**Remark 1.2.** If $t_n \to t$ and $\psi(t_n) \to 0$ implies that $t = 0$.

In 2014, Chandok, Choudhury and Metiya [5] improved Theorem 1.2 and Theorem 1.3 by using the functions of $\Phi$ and $\Psi$.

**Theorem 1.5.** (Chandok, Choudhury and Metiya [5]) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that for all $x,\ y\ \in X$ with $x \preceq y$,

$$\varphi(d(Tx, Ty)) \leq \varphi(M(x, y)) - \psi(N(x, y)) \tag{1.2}$$

for some $\varphi \in \Phi$ and $\psi \in \Psi$, where

$M(x, y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)},\ \frac{d(y,Tx)[1+d(x,Ty\})]}{1+d(x,y)}\ ,\ d(x, y)\}$ and

$N(x, y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)},\ d(x, y)\}$.

If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point.

**Theorem 1.6.** (Chandok, Choudhury and Metiya [5]) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in $X$ such that $x_n \to x$ then $x_n \preceq x$ for all $n \in N$. Let $T : X \to X$ be a non-decreasing mapping. Suppose that (1.2) holds, where $M(x, y)$, $N(x, y)$ and the conditions upon $\varphi$ and $\psi$ are the same as in Theorem 1.5. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ then $T$ has a fixed point.

**Theorem 1.7.** (Chandok, Choudhury and Metiya [5]) In addition to the hypotheses of Theorem 1.5 (Theorem 1.6), suppose that for every $x,\ y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then $T$ has a unique fixed point.

Recently, Sastry, Babu, Sarma and Krishna [17] improved Theorem 1.5, Theorem 1.6 and Theorem 1.7 by relaxing the continuity of $\varphi$ and replacing $M(x, y)$ by $M_1(x, y)$ and $N(x, y)$ by $N_1(x, y)$.

**Theorem 1.8.** (Sastry, Babu, Sarma and Krishna [17]) Let $(X, \preceq)$ be a partially ordered set and $(X, d)$ be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping. Suppose there exists $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi$ is non-decreasing and $\varphi(t) = 0 \iff t = 0$, and $\psi \in \Psi$ such that
$\varphi(d(Tx, Ty)) \leq \varphi(M_1(x, y)) - \ \psi(N_1(x, y))$, where
$M_1(x, y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty\})]}{1+d(x,y)}, \ \frac{d(y,Tx)[1+d(x,Ty\})]}{1+d(x,y)}\ , d(x, y)\}$
and
$N_1(x, y)= \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty\})]}{1+d(x,y)}\ , d(x, y)\}$, for all $x,\, y \in X$ with $x\ \preceq y$.
i.e. $M_1(x, y)= \max\{N_1(x, y),\ \frac{d(y,Tx)[1+d(x,Ty\})]}{1+d(x,y)}\ \}$.
If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$ is a Cauchy sequence.

**Theorem 1.9.** (Sastry, Babu, Sarma and Krishna [17]) In addition to the hypotheses of Theorem 1.8, suppose that $T$ is continuous. Then $T$ has a fixed point.

**Theorem 1.10.** (Sastry, Babu, Sarma and Krishna [17]) In addition to the hypotheses of Theorem 1.8, assume the following:

$(i)$ $x, y, z \in X$, such that $x < y < z \Rightarrow d(x, y) < d(x, z)$, and $d(y, z) < d(x, z)$

$(ii)$ if $\{x_n\}$ is an increasing sequence in $X$ such that $x_n \to z$, then $x_n \preceq z$ for all $n \in \mathbb{N}$.

Further " for every $u, v \in X$, there exists $z \in X$ which is comparable to both $u$ and $v$".
Then $T$ has a unique fixed point in $X$.

In 1986, Jungck [11] defined the concept of compatible mappings.

**Definition 1.2.** [11] A pair $(S, T)$ of self-mappings of a metric space $(X, d)$ is said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

In 1998, Pant introduced a new continuity condition, known as reciprocal continuity and obtained a common fixed point theorem by using the compatibility in a metric space. The notion of reciprocal continuity is weaker than the continuity of one of the mappings.

**Definition 1.3.** [14] Two self-mappings $S$ and $T$ of a metric space $(X, d)$ are called reciprocally continuous if $\lim_{n\to\infty} STx_n = Sz$ and $\lim_{n\to\infty} TSx_n = Tz$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z$ in $X$.

**Definition 1.4.** [12] Two self-maps $S$ and $T$ of a metric space $(X, d)$ are said to be weakly compatible if they commute at their coincidence points. i.e. if for any $x$ in $X$ with $Sx = Tx$ then $STx = TSx$.

**Definition 1.5.** [10] Let $(X, \preceq)$ be a partially ordered set and $T$ and $S : X \to X$ be two selfmaps. $T$ is said to be $S$-non-decreasing if for all $x, y \in X$, $Sx \preceq Sy$ implies $Tx \preceq Ty$.

In this paper, $(X, \preceq, d)$ denotes a partially ordered metric space, where $(X, \preceq)$ is a partially ordered set, and $d$ is a metric on $X$. If $X$ is complete with respect to the metric $d$ then we call $(X, \preceq, d)$ a partially ordered complete metric space.

The following lemma is useful in our subsequent discussion.

**Lemma 1.1.** [2]. Let $(X,\ d)$ be a metric space. Let $\{x_n\}$ be a sequence in $X$ such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. For each $k > 0$, corresponding to $n(k)$, we can choose $m(k)$ to be the smallest integer with $m(k) > n(k) > k$ satisfying $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. Hence for such $m(k)$ and $n(k)$, we have $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$.

It can be shown that the following identities are satisfied.

$$(i)\ \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon, \qquad (ii)\ \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon,$$

$$(iii)\ \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon, \text{ and } (iv)\ \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon.$$

In Section 2, we prove the existence of coincidence and common fixed points of a pair of maps satisfying certain generalized contractive mappings with auxiliary functions $\varphi \in \Phi$ and $\psi \in \Psi$ involving rational type expressions in partially ordered metric spaces. In Section 3, we draw some corollaries from our main results and give examples in support of our results.

## 2. Main Results

**Theorem 2.1.** Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $S,\ T : X \to X$ be self maps of $X$, and $T$ is $S$ non-decreasing. Suppose that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(d(Tx, Ty)) \leq \varphi(M(x, y)) - \psi(N(x, y)) \tag{2.1}$$

where

$$M(x, y) = \max\{\frac{d(Sy, Ty)[1 + d(Sx, Tx)]}{1 + d(Sx, Sy)},\ \frac{d(Sx, Tx)[1 + d(Sy, Ty\})]}{1 + d(Sx, Sy)}, \frac{d(Sy, Tx)[1 + d(Sx, Ty\})]}{1 + d(Sx, Sy)},\ d(Sx, Sy)\}$$

and
$N(x, y)=\max\{\frac{d(Sy,Ty)[1+d(Sx,Tx)]}{1+d(Sx,Sy)}, \frac{d(Sx,Tx)[1+d(Sy,Ty\})]}{1+d(Sx,Sy)}, d(Sx, Sy)\}$
for all $x,\ y \in X$ with $Sx \preceq Sy$.
Furthermore, assume that

(i) $T(X) \subseteq S(X)$;

(ii) there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$;

(iii) $S(X)$ is a closed subset of $X$; and

(iv) if any non-decreasing sequence $\{x_n\}$ in $X$ converges to $x$ then $x_n \preceq x$ for all $n = 0, 1, 2, ....$

Then $S$ and $T$ have a coincident point in $X$.

*Proof.* By (ii), let $x_0 \in X$ be such that $Sx_0 \preceq Tx_0$. Since $T(X) \subseteq S(X)$, we choose $x_1 \in X$ such that $Tx_0 = Sx_1$. Since $Sx_0 \preceq Tx_0 = Sx_1$, and $T$ is $S$ non-decreasing, we have $Tx_0 \preceq Tx_1$. Again, using $T(X) \subseteq S(X)$, we have $Tx_1 = Sx_2$ for some $x_2 \in X$ so that $Tx_0 \preceq Sx_2$ i.e.$Sx_1 \preceq Sx_2$. By using a similar argument we choose a sequence $\{x_n\}$ in $X$ with $Tx_n = Sx_{n+1}$ and $Sx_n \preceq Sx_{n+1}$ for each $n = 0, 1, 2, ...$ .
If $Sx_n = Sx_{n+1}$ for some $n \geq 0$ then $Sx_n = Tx_n$ so that $x_n$ is a coincidence point of $S$ and $T$.
Hence, with out loss of generality, we assume that $Sx_n \neq Sx_{n+1}$ for each $n \geq 0$.
Since $Sx_{n-1} \preceq Sx_n$, by (2.1) we have,

$$\begin{aligned}\varphi(d(Sx_n, Sx_{n+1})) &= \varphi(d(Tx_{n-1}, Tx_n)) \\ &\leq \varphi(M(x_{n-1}, x_n)) - \psi(N(x_{n-1}, x_n)),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
&M(x_{n-1},x_n)\\
&=\max\{\frac{d(Sx_n,Tx_n)[1+d(Sx_{n-1},Tx_{n-1})]}{1+d(Sx_{n-1},Sx_n)},\frac{d(Sx_{n-1},Tx_{n-1})[1+d(Sx_n,Tx_n)]}{1+d(Sx_{n-1},Sx_n)},\\
&\quad\frac{d(Sx_n,Tx_{n-1})[1+d(Sx_{n-1},Tx_n)]}{1+d(Sx_{n-1},Sx_n)},d(Sx_{n-1},Sx_n)\}\\
&=\max\{\frac{d(Sx_n,Sx_{n+1})[1+d(Sx_{n-1},Sx_n)]}{1+d(Sx_{n-1},Sx_n)},\frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},\\
&\quad\frac{d(Sx_n,Sx_n)[1+d(Sx_{n-1},Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},d(Sx_{n-1},Sx_n)\}\\
&=\max\{d(Sx_n,Sx_{n+1}),\frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},d(Sx_{n-1},Sx_n)\}
\end{aligned}$$

and

$$\begin{aligned}
&N(x_{n-1},x_n)\\
&=\max\{\frac{d(Sx_n,Tx_n)[1+d(Sx_{n-1},Tx_{n-1})]}{1+d(Sx_{n-1},Sx_n)},\\
&\quad\frac{d(Sx_{n-1},Tx_{n-1})[1+d(Sx_n,Tx_n)]}{1+d(Sx_{n-1},Sx_n)},d(Sx_{n-1},Sx_n)\}\\
&=\max\{\frac{d(Sx_n,Sx_{n+1})[1+d(Sx_{n-1},Sx_n)]}{1+d(Sx_{n-1},Sx_n)},\\
&\quad\frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},d(Sx_{n-1},Sx_n)\}\\
&=\max\{d(Sx_n,Sx_{n+1}),\frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},d(Sx_{n-1},Sx_n)\}.
\end{aligned}$$

If $\max\{d(Sx_n,Sx_{n+1}),d(Sx_{n-1},Sx_n)\}=d(Sx_n,Sx_{n+1})$, then

$$\begin{aligned}
M(x_{n-1},x_n)&=\max\{d(Sx_n,Sx_{n+1}),\frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)}\}\\
&=d(Sx_n,Sx_{n+1})
\end{aligned}$$

and $N(x_{n-1},x_n)=d(Sx_n,Sx_{n+1})$.
Now from (2.1), we have

$$\begin{aligned}
\varphi(d(Sx_n,Sx_{n+1}))&\leq\varphi(d(Sx_n,Sx_{n+1}))-\psi(d(Sx_n,Sx_{n+1}))\\
&<\varphi(d(Sx_n,Sx_{n+1})),
\end{aligned}$$

a contradiction.
Hence $\max\{d(Sx_n,Sx_{n+1}),d(Sx_{n-1},Sx_n)\}=d(Sx_{n-1},Sx_n)$. In this case

$$\begin{aligned}
M(x_{n-1},x_n)&=\max\{\frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},d(Sx_{n-1},Sx_n)\}\\
&=d(Sx_{n-1},Sx_n)
\end{aligned}$$

and $N(x_{n-1}, x_n) = d(Sx_{n-1}, Sx_n)$.
Therefore from (2.2), we have

$$\varphi(d(Sx_n, Sx_{n+1})) \leq \varphi(d(Sx_{n-1}, Sx_n)) - \psi(d(Sx_{n-1}, Sx_n)) \tag{2.3}$$

$$< \varphi(d(Sx_{n-1}, Sx_n)). \tag{2.4}$$

Thus it follows that $\{\varphi(d(Sx_n, Sx_{n+1}))\}$ is a strictly decreasing sequence of positive real numbers and so $\lim_{n\to\infty} \varphi(d(Sx_n, Sx_{n+1}))$ exists and it is $r$ (say).
i.e. $\lim_{n\to\infty} \varphi(d(Sx_n, Sx_{n+1})) = r \geq 0$.
From (2.4), since $\varphi$ is non-decreasing, it follows that $\{d(Sx_n, Sx_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and so $\lim_{n\to\infty} d(Sx_n, Sx_{n+1})$ exists and it is $r^{'}$ (say). i.e. $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = r^{'} \geq 0$.
Suppose that $r^{'} > 0$.
From (2.3), we have
$0 \leq \psi(d(Sx_{n-1}, Sx_n)) \leq \varphi(d(Sx_{n-1}, Sx_n)) - \varphi(d(Sx_n, Sx_{n+1}))$.
On taking limit supremum as $n \to \infty$ on both sides, we have

$$\begin{aligned} 0 \leq \overline{\lim}\psi(d(Sx_{n-1}, Sx_n)) &\leq \overline{\lim}\varphi(d(Sx_{n-1}, Sx_n)) - \underline{\lim}\varphi(d(Sx_n, Sx_{n+1})) \\ &= r - r = 0 \text{ as } n \to \infty \end{aligned}$$

so that $\overline{\lim}\psi(d(Sx_{n-1}, Sx_n)) = 0$. Hence $\underline{\lim}\psi(d(Sx_{n-1}, Sx_n)) = 0$.
Therefore $\lim_{n\to\infty} \psi(d(Sx_{n-1}, Sx_n)) = 0$, which is a contradiction.
Therefore, $r^{'} = 0$. i.e. $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0$.
We now show that $\{Sx_n\}$ is Cauchy.
Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then by Lemma 1.1 there exists an $\epsilon > 0$ for which we can find sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(Sx_{m(k)}, Sx_{n(k)}) \geq \epsilon$ and $d(Sx_{m(k)-1}, Sx_{n(k)}) < \epsilon$ and the following identities satisfied.

$$(i)\ \lim_{k\to\infty} d(Sx_{m(k)}, Sx_{n(k)}) = \epsilon \qquad (ii)\ \lim_{k\to\infty} d(Sx_{m(k)-1}, Sx_{n(k)-1}) = \epsilon$$

$$(iii)\ \lim_{k\to\infty} d(Sx_{m(k)-1}, Sx_{n(k)}) = \epsilon \text{ and } (iv)\ \lim_{k\to\infty} d(Sx_{n(k)-1}, Sx_{m(k)}) = \epsilon.$$

By (2.1), we have

$$\begin{aligned} \varphi(d(Sx_{n(k)}, Sx_{m(k)})) &= \varphi(d(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\leq \varphi(M(x_{n(k)-1}, x_{m(k)-1})) - \psi(N(x_{n(k)-1}, x_{m(k)-1})), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
&M(x_{n(k)-1}, x_{m(k)-1})\\
&= \max\{\frac{d(Sx_{m(k)-1}, Tx_{m(k)-1})[1 + d(Sx_{n(k)-1}, Tx_{n(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{n(k)-1}, Tx_{n(k)-1})[1 + d(Sx_{m(k)-1}, Tx_{m(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{m(k)-1}, Tx_{n(k)-1})[1 + d(Sx_{n(k)-1}, Tx_{m(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})}, d(Sx_{n(k)-1}, Sx_{m(k)-1})\}\\
&= \max\{\frac{d(Sx_{m(k)-1}, Sx_{m(k)})[1 + d(Sx_{n(k)-1}, Sx_{n(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{n(k)-1}, Sx_{n(k)})[1 + d(Sx_{m(k)-1}, Sx_{m(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{m(k)-1}, Sx_{n(k)})[1 + d(Sx_{n(k)-1}, Sx_{m(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})}, d(Sx_{n(k)-1}, Sx_{m(k)-1})\},
\end{aligned}$$

and

$$\begin{aligned}
&N(x_{n(k)-1}, x_{m(k)-1})\\
&= \max\{\frac{d(Sx_{m(k)-1}, Tx_{m(k)-1})[1 + d(Sx_{n(k)-1}, Tx_{n(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{n(k)-1}, Tx_{n(k)-1})[1 + d(Sx_{m(k)-1}, Tx_{m(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})}, d(Sx_{n(k)-1}, Sx_{m(k)-1})\}\\
&= \max\{\frac{d(Sx_{m(k)-1}, Sx_{m(k)})[1 + d(Sx_{n(k)-1}, Sx_{n(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{n(k)-1}, Sx_{n(k)})[1 + d(Sx_{m(k)-1}, Sx_{m(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})}, d(Sx_{n(k)-1}, Sx_{m(k)-1})\}.
\end{aligned}$$

Hence $\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \max\{0,\ 0,\ \frac{\epsilon(1+\epsilon)}{1+\epsilon)},\ \epsilon\} = \epsilon$,
$\lim_{k\to\infty} N(x_{n(k)-1}, x_{m(k)-1}) = \max\{0,\ 0, \epsilon\} = \epsilon$.
Since $\varphi$ is continuous, we have $\overline{\lim}\ \varphi(d(Sx_{n(k)}, Sx_{m(k)})) = \varphi(\epsilon)$.
From (2.5) and taking limit supremum as $n \to \infty$, we have
$\varphi(\epsilon) \leq \varphi(\epsilon) - \underline{\lim}\psi(N(x_{n(k)-1}, x_{m(k)-1}))$, and it implies that
$\underline{\lim}\psi(N(x_{n(k)-1}, x_{m(k)-1})) \leq 0$,
a contradiction.
Therefore $\{Sx_n\}$ is a Cauchy sequence in $X$.
Since $S(X)$ is complete, there exists $y \in X$ such that $\lim_{n\to\infty} Sx_n = Sy$.

(2.6) Hence $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = Sy$ for some $y \in X$.

Now we show that $Sy = Ty$.
Suppose that $Sy \neq Ty$. i.e. $d(Sy, Ty) > 0$.

Since $\{Sx_n\}$ is a non-decreasing sequence, $Sx_n \to Sy$ for some $y \in X$ and by condition $(iv)$, we have $Sx_n \preceq Sy$ for all $n \geq 0$.
Now, from (2.1), we have

$$\varphi(d(Tx_n, Ty)) \leq \varphi(M(x_n, y)) - \psi(N(x_n, y)), \tag{2.7}$$

where

$$\begin{aligned} &M(x_n, y) \\ &= \max\{\frac{d(Sy, Ty)[1 + d(Sx_n, Tx_n)]}{1 + d(Sx_n, Sy)}, \frac{d(Sx_n, Tx_n)[1 + d(Sy, Ty)]}{1 + d(Sx_n, Sy)}, \\ &\qquad \frac{d(Sy, Tx_n)[1 + d(Sx_n, Ty)]}{1 + d(Sx_n, Sy)}, d(Sx_n, Sy)\} \\ &= \max\{\frac{d(Sy, Ty)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_n, Sy)}, \frac{d(Sx_n, Sx_{n+1})[1 + d(Sy, Ty)]}{1 + d(Sx_n, Sy)}, \\ &\qquad \frac{d(Sy, Sx_{n+1})[1 + d(Sx_n, Ty)]}{1 + d(Sx_n, Sy)}, d(Sx_n, Sy)\} \end{aligned}$$

and

$$\begin{aligned} &N(x_n, y) \\ &= \max\{\frac{d(Sy, Ty)[1 + d(Sx_n, Tx_n)]}{1 + d(Sx_n, Sy)}, \frac{d(Sx_n, Tx_n)[1 + d(Sy, Ty)]}{1 + d(Sx_n, Sy)}, d(Sx_n, Sy)\} \\ &= \max\{\frac{d(Sy, Ty)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_n, Sy)}, \frac{d(Sx_n, Sx_{n+1})[1 + d(Sy, Sy)]}{1 + d(Sx_n, Sy)}, \\ &\qquad d(Sx_n, Sy)\}. \end{aligned}$$

Also, $\lim_{n\to\infty} M(x_n, y) = d(Sy, Ty)$ and $\lim_{n\to\infty} N(x_n, y) = d(Sy, Ty)$.
Now on taking limit supremum as $n \to \infty$ on both sides of (2.7) we have
$\overline{\lim}\varphi(d(Tx_n, Ty)) \leq \overline{\lim}\varphi(M(x_n, y)) - \underline{\lim}\psi(N(x_n, y))$,
which implies that $\varphi(d(Sy, Ty)) \leq \varphi(d(Sy, Ty)) - \underline{\lim}\psi(N(x_n, y))$
so that $\underline{\lim}\psi(N(x_n, y)) \leq 0$,
a contradiction.
Hence $Ty = Sy$ so that $S$ and $T$ have a coincidence point $y$. □.

**Theorem 2.2.** In addition to the hypotheses of Theorem 2.1, assume that

$(i)$ $S$ and $T$ are weakly compatible,

$(ii)$ $Sx = Tx$ implies $Sx \preceq SSx$ for any $x \in X$.

Then $T$ and $S$ have common fixed point in $X$.

Furthermore, assume the following: Condition(H): there exists $u \in X$ such that $Su \preceq Tu$ and $Tu$ is comparable to $Tx$ and $Ty$, for all $x, y \in X$.
Then $S$ and $T$ have a unique common fixed point in $X$.

*Proof.* From the proof of Theorem 2.1, we have $\{Sx_n\}$ is a non-decreasing sequence that converges to $Sy$ for some $y \in X$ with $Sy = Ty$.
Let $w = Ty = Sy$.
Since $S$ and $T$ are weakly compatible, $Tw = TSy = STy = Sw$.
Suppose that $w \neq Tw$.
By hypothesis $(ii)$ we have $Sy \preceq SSy = STy$.
Therefore, from (2.1), we have

$$\begin{aligned} \varphi(d(w,Tw)) &= \varphi(d(Ty,TTy)) \\ &\leq \varphi(M(y,Ty)) - \psi(N(y,Ty)) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &M(y,Ty) \\ &= \max\{\frac{d(STy,TTy)[1+d(Sy,Ty)]}{1+d(Sy,STy)}, \ \frac{d(Sy,Ty)[1+d(STy,TTy)]}{1+d(Sy,STy)}, \\ &\qquad \frac{d(STy,Ty)[1+d(Sy,TTy)]}{1+d(Sy,STy)} \ , \ d(Sy,STy)\} \\ &= \max\{\frac{d(Sw,TTy)}{1+d(Sy,Sw)}, \ 0, \frac{d(Sw,Ty)[1+d(Sy,TTy)]}{1+d(Sy,Sw)} \ , \ d(Sy,Sw)\} \\ &= \max\{\frac{d(Tw,TTy)}{1+d(w,Tw)}, \ 0, \frac{d(Tw,w)[1+d(w,TTy)]}{1+d(w,Tw)} \ , \ d(w,Tw)\} \\ &= \max\{\frac{d(Tw,Tw)}{1+d(w,Tw)}, \ 0, \frac{d(Tw,w)[1+d(w,Tw)]}{1+d(w,Tw)} \ , \ d(w,Tw)\} \\ &= d(w,Tw), \end{aligned}$$

and

$$\begin{aligned} &N(y,Ty) \\ &= \max\{\frac{d(STy,TTy)[1+d(Sy,Ty)]}{1+d(Sy,STy)}, \ \frac{d(Sy,Ty)[1+d(STy,TTy)]}{1+d(Sy,STy)} \ , \ d(Sy,STy)\} \\ &= \max\{\frac{d(Sw,TTy)}{1+d(Sy,Sw)}, \ 0, \ d(Sy,Sw)\} \\ &= \max\{\frac{d(Tw,TTy)}{1+d(w,Tw)}, \ 0, \ d(w,Tw)\} \\ &= \max\{\frac{d(Tw,Tw)}{1+d(w,Tw)}, \ 0, \ d(w,Tw)\} \\ &= d(w,Tw). \end{aligned}$$

Hence, from (2.8),

$$\begin{aligned} \varphi(d(w,Tw)) &\leq \varphi(d(w,Tw)) - \psi(d(w,Tw)) \\ &< \varphi(d(w,Tw)) \end{aligned}$$

is a contradiction.
Therefore $w = Tw$. Hence $w = Tw = Sw$.
Therefore $w$ is a common fixed point of $S$ and $T$.
We now prove the uniqueness of common fixed point of $S$ and $T$.
Let $z$ and $w$ be two common fixed points of $S$ and $T$. i.e. $Sz = Tz = z$ and $Sw = Tw = w$ with $z \neq w$.
Case $(I)$: $z$ and $w$ are comparable. With out loss of generality we assume that $z \preceq w$. i.e. $Sz \preceq Sw$
From (2.1), we have

$$\begin{aligned} \varphi(d(z,w)) &= \varphi(d(Tz,Tw)) \\ &\leq \varphi(M(z,w)) - \psi(N(z,w)) \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &M(z,w) \\ &= \max\{\frac{d(Sw,Tw)[1+d(Sz,Tz)]}{1+d(Sz,Sw)}, \ \frac{d(Sz,Tz)[1+d(Sw,Tw)]}{1+d(Sz,Sw)}, \\ &\qquad \frac{d(Sw,Tz)[1+d(Sz,Tw)]}{1+d(Sz,Sw)}, \ d(Sz,Sw)\} \\ &= \max\{\frac{d(w,w)}{1+d(z,w)}, \ 0, \frac{d(w,z)[1+d(z,w)]}{1+d(z,w)}, \ d(z,w)\} \\ &= \max\{0, \ 0, d(z,w), \ d(z,w)\} \\ &= d(z,w). \end{aligned}$$

$$\begin{aligned} N(z,w) &= \max\{\frac{d(Sw,Tw)[1+d(Sz,Tz)]}{1+d(Sz,Sw)}, \ \frac{d(Sz,Tz)[1+d(Sw,Tw)]}{1+d(Sz,Sw)}, \ d(Sz,Sw)\} \\ &= \max\{\frac{d(w,w)}{1+d(z,w)}, \ 0, \ d(z,w)\} \\ &= \max\{0, \ 0,, \ d(z,w)\} \\ &= d(z,w). \end{aligned}$$

Hence, from (2.9), we have

$$\begin{aligned} \varphi(d(z,w)) &\leq \varphi(d(z,w)) - \psi(d(z,w)) \\ &< \varphi(d(z,w)), \end{aligned}$$

a contradiction.
Therefore $z = w$. This shows that $S$ and $T$ have a unique common fixed point in $X$.

Case $(II)$ : $z$ and $w$ are not comparable.

In this case, by assumption, there exists $u \in X$ such that $Su \preceq Tu$ and $Tu$ is comparable to $Tz$ and $Tw$.

Subcase (i) : We assume that $Tz \preceq Tu, Tw \preceq Tu$ and $Su \preceq Tu$. Now we set $u = u_0$. Since $T(X) \subseteq S(X)$, there exists $u_1 \in X$ such that

$$Tu_0 = Su_1. \tag{2.10}$$

Since $Tz \preceq Tu$, $Tz = Sz$ and $Tu = Tu_0 = Su_1$, we have

$$Sz \preceq Su_1. \tag{2.11}$$

Since $Su_0 \preceq Tu_0 = Su_1$, we have

$$Su_0 \preceq Su_1. \tag{2.12}$$

Since $T$ is $S$ non-decreasing, from (2.11) and (2.12) we get

$$Tz \preceq Tu_1 \quad \text{and} \tag{2.13}$$

$$Tu_0 \preceq Tu_1. \tag{2.14}$$

Since $T(X) \subseteq S(X)$, there exists $u_2 \in X$ such that

$$Tu_1 = Su_2. \tag{2.15}$$

From (2.10), (2.14) and (2.15) we have

$$Su_1 \preceq Su_2. \tag{2.16}$$

From (2.13)and (2.15), it follows that

$$Sz \preceq Su_2, \qquad \text{since } \ Tz = Sz. \tag{2.17}$$

Since $T$ is $S$ non-decreasing, from (2.16) and (2.17) we get

$$Tu_1 \preceq Tu_2 \quad \text{and} \tag{2.18}$$

$$Tz \preceq Tu_2. \tag{2.19}$$

On continuing this process, we can construct a sequence$\{u_n\}$ in $X$ such that

$$Su_{n+1} = Tu_n, \ Sz \preceq Su_{n+1} \text{ and } Su_n \preceq Su_{n+1} \text{ for } n = 0, 1, 2.... \tag{2.20}$$

(2.21) Also, we can easily see that $Sw \preceq Su_{n+1}$ for $n = 0, 1, 2....$

Since $Su_n \preceq Su_{n+1}$, by using the inequality (2.1), it is easy to see that $\{Su_n\}$ is Cauchy as in the proof of Theorem 2.1. Since $S(X)$ is complete, there exists $v \in X$ such that $Su_n \to Sv$ as $n \to \infty$.

We now show that $Sz = Sv$. Suppose that $Sz \neq Sv$.
Since $Sz \preceq Su_n$, from (2.1) we have

$$\varphi(d(Sz, Su_{n+1})) = \varphi(d(Tz, Tu_n)) \leq \varphi(M(z, u_n)) - \psi(N(z, u_n)) \tag{2.22}$$

where

$$M(z, u_n) = \max\{\frac{d(Su_n, Tu_n)[1 + d(Sz, Tz)]}{1 + d(Sz, Su_n)}, \frac{d(Sz, Tz)[1 + d(Su_n, Tu_n)]}{1 + d(Sz, Su_n)}, \frac{d(Su_n, Tz)[1 + d(Sz, Tu_n)]}{1 + d(Sz, Su_n)}, d(Sz, Su_n)\}$$

and

$$N(z, u_n) = \max\{\frac{d(Su_n, Tu_n)[1 + d(Sz, Tz)]}{1 + d(Sz, Su_n)}, \frac{d(Sz, Tz)[1 + d(Su_n, Tu_n)]}{1 + d(Sz, Su_n)}, d(Sz, Su_n)\}.$$

Hence $\lim\limits_{n\to\infty} M(z, u_n) = \max\{0, 0, d(Sv, Tz), d(Sz, Sv)\} = d(Sv, Sz)$ and
$\lim\limits_{n\to\infty} N(z, u_n) = \max\{0, 0, d(Sz, Sv)\} = d(Sv, Sz)$.
Taking limit supremum on (2.22), we have

$$\varphi(d(Sz, Sv)) \leq \varphi(d(Sz, Sv)) - \underline{\lim}\psi(N(z, u_n)) \tag{2.23}$$

so that $\underline{\lim}\psi(N(z, u_n)) \leq 0$,
a contradiction.
Therefore, $Sz = Sv$.
Similarly we can prove that $Sw = Sv$. Hence $Sz = Sw$, which implies that $z = w$.

Subcase (ii) : We assume that $Tu \preceq Tz, Tu \preceq Tw$ and $Su \preceq Tu$. Now we set $u = u_0$. Since $T(X) \subseteq S(X)$, there exists $u_1 \in X$ such that

$$Tu_0 = Su_1. \tag{2.24}$$

Since $Tu \preceq Tz$, $Tz = Sz$ and $Tu = Tu_0 = Su_1$, we have

$$Su_1 \preceq Sz. \tag{2.25}$$

Since $Su_0 \preceq Tu_0 = Su_1$, we have

$$Su_0 \preceq Su_1. \tag{2.26}$$

Since $T$ is $S$ non-decreasing, from (2.25) and (2.26) we get

$$Tu_1 \preceq Tz \quad \text{and} \tag{2.27}$$

$$Tu_0 \preceq Tu_1. \tag{2.28}$$

Since $T(X) \subseteq S(X)$, there exists $u_2 \in X$ such that

$$Tu_1 = Su_2. \tag{2.29}$$

From (2.24), (2.28) and (2.29) we have

$$Su_1 \preceq Su_2. \tag{2.30}$$

From (2.27)and (2.29), it follows that

$$Su_2 \preceq Sz, \quad \text{since } Tz = Sz. \tag{2.31}$$

Since $T$ is $S$ non-decreasing, from (2.30) and (2.31) we get

$$Tu_1 \preceq Tu_2 \quad \text{and} \tag{2.32}$$

$$Tu_2 \preceq Tz. \tag{2.33}$$

On continuing this process, we can construct a sequence$\{u_n\}$ in $X$ such that

$$Su_{n+1} = Tu_n,\ Su_n \preceq Sz \text{ and } Su_n \preceq Su_{n+1} \text{ for } n = 0, 1, 2.... \tag{2.34}$$

(2.35) Also we can easily see that $Su_n \preceq Sw$ for $n = 0, 1, 2....$

Since $Su_n \preceq Su_{n+1}$, by using the inequality (2.1), it is easy to see that $\{Su_n\}$ is Cauchy as in the proof of Theorem 2.1. Since $S(X)$ is complete, there exists $v \in X$ such that $Su_n \to Sv$ as $n \to \infty$.
We now show that $Sz = Sv$. Suppose that $Sz \neq Sv$.
Since $Su_n \preceq Sz$, from (2.1) we have

$$\varphi(d(Su_{n+1}, Sz)) = \varphi(d(Tu_n, Tz)) \leq \varphi(M(u_n, z)) - \psi(N(u_n, z)) \tag{2.36}$$

where

$$M(u_n, z) = \max\{\frac{d(Sz, Tz)[1 + d(Su_n, Tu_n)]}{1 + d(Su_n, Sz)}, \frac{d(Su_n, Tu_n)[1 + d(Sz, Tz)]}{1 + d(Su_n, Sz)}, \frac{d(Sz, Tu_n)[1 + d(Su_n, Tz)]}{1 + d(Su_n, Sz)}, d(Su_n, Sz)\}$$

and

$$N(u_n, z) = \max\{\frac{d(Sz, Tz)[1+d(Su_n, Tu_n)]}{1 + d(Su_n, Sz)}, \frac{d(Su_n, Tu_n)[1+d(Sz, Tz)]}{1 + d(Su_n, Sz)}, d(Su_n, Sz)\}.$$

Hence $\lim_{n\to\infty} M(u_n, z) = \max\{0, 0, d(Sz, Sv), d(Sv, Sz)\} = d(Sv, Sz)$ and
$\lim_{n\to\infty} N(u_n, z) = \max\{0, 0, d(Sv, Sz)\} = d(Sv, Sz)$.
On taking limit supremum as $n \to \infty$ on (2.36), we have

$$\varphi(d(Sv, Sz)) \leq \varphi(d(Sv, Sz)) - \underline{\lim}\psi(N(u_n, z)) \tag{2.37}$$

so that $\underline{\lim}\psi(N(u_n, z)) \leq 0$,
a contradiction.
Therefore, $Sz = Sv$.
Similarly we can prove that $Sw = Sv$. Hence $Sz = Sw$, which implies that $z = w$.
Subcase $(iii)$ : We assume that $Tu \preceq Tz, Tw \preceq Tu$ and $Su \preceq Tu$.

In this case, $Tw \preceq Tz$ i.e., $w \preceq z$. By case(i)the uniqueness follows.

Subcase $(iv)$ : We assume that $Tz \preceq Tu, Tu \preceq Tw$ and $Su \preceq Tu$.

In this case, $Tz \preceq Tw$ i.e. $z \preceq w$. By case(i) the uniqueness follows.
Hence in either of the two cases $S$ and $T$ have a unique common fixed point. □

Now we relax the closedness of $S(X)$ and condition $(iv)$ of Theorem 2.1, but by imposing the compatible property and reciprocal continuity of a pair of maps and prove the following.

**Theorem 2.3.** Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $S,\ T : X \to X$ be self maps of $X$ and $T$ is $S$ non-decreasing. Suppose that there exist $\varphi \in \Phi,\ \psi \in \Psi$ and satisfying the inequality (2.1). Assume that

$(i)$ $T(X) \subseteq S(X)$;

$(ii)$ there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$;

$(iii)$ $S$ and $T$ are reciprocally continuous;

$(iv)$ the pair $(S, T)$ is compatible;

$(v)$ $Sz = Tz$ implies $Sz \preceq SSz$ for any $z \in X$.

Then $S$ and $T$ have a common fixed point.

Furthermore, assume that Condition(H) of Theorem 2.2, then $S$ and $T$ have a unique common fixed point in $X$.

*Proof.* The sequence $\{x_n\}$ is constructed such that $Sx_{n+1} = Tx_n$ for all $n \geq 0$ and the proof of the Cauchy part of the sequence $\{Sx_n\}$ is the same as that one mentioned in the proof of Theorem 2.1.
Since $(X, d)$ is complete, there exists $z \in X$ such that $\lim\limits_{n\to\infty} Sx_n = z$ and consequently we have $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} Sx_{n+1} = z$.
Since $S$ and $T$ are reciprocally continuous, we have
$\lim\limits_{n\to\infty} STx_n = Sz$ and $\lim\limits_{n\to\infty} TSx_n = Tz$.
Again, since $S$ and $T$ are compatible, it follows that
$\lim\limits_{n\to\infty} d(STx_n, TSx_n) = 0$, *i.e.*, $d(Sz,\ Tz) = 0$ so that $Sz = Tz$.
Now, since every compatible pair is weakly compatible, by using the compatibility of $S$ and $T$ we have $STz = TSz = TTz$.
Suppose that $Tz \neq TTz$. Now

$\varphi(d(Tz,TTz)) \leq \varphi(M(z,Tz)) - \psi(N(z,Tz))$
where

$$\begin{aligned} M(z,Tz) &= \max\{\frac{d(STz,TTz)[1+d(Sz,Tz)]}{1+d(Sz,STz)},\ \frac{d(Sz,Tz)[1+d(STz,TTz)]}{1+d(Sz,STz)}, \\ &\qquad \frac{d(STz,Tz)[1+d(Sz,TTz)]}{1+d(Sz,STz)},\ d(Sz,STz)\} \\ &= \max\{0,0,d(Tz,TTz),d(Tz,TTz)\} \\ &= d(Tz,TTz), \text{ and in a similar way it is easy to see that } N(z,Tz) = d(Tz,TTz). \end{aligned}$$

Therefore

$$\begin{aligned} \varphi(d(Tz,TTz)) &\leq \varphi(d(Tz,TTz)) - \psi(d(Tz,TTz)) \\ &< \varphi(d(Tz,TTz)), \end{aligned}$$

a contradiction.
Hence $Tz = TTz$ so that $Tz$ is a fixed point of $T$.
Therefore, $Tz$ is a common fixed point of $S$ and $T$.
We now prove the uniqueness of the common fixed point of $S$ and $T$.
Let $z$ and $w$ be two common fixed points of $S$ and $T$. i.e. $Sz = Tz = z$ and $Sw = Tw = w$, with $z \neq w$.
If $z$ and $w$ are comparable then by Case $(I)$ of the proof of Theorem 2.2, the conclusion follows.

We now suppose $z$ and $w$ are not comparable. In this case, by following the line of the Subcase $(i)$ of Case $(II)$ of Theorem 2.2, we reach at (2.20) and (2.21). i.e., there exists a sequence $\{u_n\}$ in $X$ such that
$Su_{n+1} = Tu_n,\ Sz \preceq Su_{n+1}, Sw \preceq Su_{n+1}$ and $Su_n \preceq Su_{n+1}$,
for all $n = 0, 1, 2, ....$
Since $Su_n \preceq Su_{n+1}$, by using the inequality (2.1), it is easy to see that $\{Su_n\}$ is Cauchy as in the proof of Theorem 2.1.
Since $X$ is complete, there exists $v \in X$ such that $Su_n \to v$ as $n \to \infty$.
We now show that $Sz = v$. Suppose that $Sz \neq v$.
Since $Sz \preceq Su_n$, from (2.1) we have

$$\varphi(d(Sz,Su_{n+1})) = \varphi(d(Tz,Tu_n)) \leq \varphi(M(z,u_n)) - \psi(N(z,u_n)) \tag{2.38}$$

where

$$\begin{aligned} M(z,u_n) = \max\{&\frac{d(Su_n,Tu_n)[1+d(Sz,Tz)]}{1+d(Sz,Su_n)}, \frac{d(Sz,Tz)[1+d(Su_n,Tu_n)]}{1+d(Sz,Su_n)}, \\ &\frac{d(Su_n,Tz)[1+d(Sz,Tu_n)]}{1+d(Sz,Su_n)}, d(Sz,Su_n)\} \end{aligned}$$

and

$$N(z,u_n) = \max\{\frac{d(Su_n,Tu_n)[1+d(Sz,Tz)]}{1+d(Sz,Su_n)}, \frac{d(Sz,Tz)[1+d(Su_n,Tu_n)]}{1+d(Sz,Su_n)}, d(Sz,Su_n)\}.$$

Hence $\lim_{n\to\infty} M(z,u_n) = \max\{0, 0, d(v, Sz), d(Sz, v)\} = d(v, Sz)$ and
$\lim_{n\to\infty} N(z,u_n) = \max\{0, 0, d(Sz, v)\} = d(v, Sz)$.
On taking limit supremum as $n \to \infty$ on (2.38), we have

$$\varphi(d(Sz,v)) \le \varphi(d(Sz,v)) - \underline{\lim}\psi(N(z,u_n)) \tag{2.39}$$

so that $\underline{\lim}\psi(N(z,u_n)) \le 0$,
a contradiction.
Therefore, $Sz = v$.
Similarly, we can prove that $Sw = v$. Hence $Sz = Sw$, which implies that $z = w$.
In all other cases we prove the uniqueness of the theorem as in the proof of Theorem 2.2. □

## 3. Corollaries and Examples

By choosing $S = I_X$ in Theorem 2.1, we have the following corollary.

**Corollary 3.1.** Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a self map of $X$ and $T$ is non-decreasing. Suppose that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(d(Tx,Ty)) \le \varphi(M(x,y)) - \psi(N(x,y)), \tag{3.1}$$

where
$M(x,y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty\})]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty\})]}{1+d(x,y)}, d(x,y)\}$
and
$N(x,y)= \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty\})]}{1+d(x,y)}, d(x,y)\}$
for all $x, y \in X$ with $x \preceq y$.
Furthermore, assume that

- $(i)$ there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- $(ii)$ if any non-decreasing sequence $\{x_n\}$ in $X$ converges to $x$ then $x_n \preceq x$ for all $n = 0, 1, 2, ....$

Then $T$ has a fixed point.

We now consider the following examples in support of our main results.

**Example 3.1.** Let $X = [0, 3]$ with the usual metric. We define partial order $\preceq$ on $X$ as follows:
$\preceq := \{(x,y) \in X \times X : x = y\} \cup \{(0, \frac{1}{2}), (0, \frac{3}{4}), (\frac{1}{2}, \frac{3}{4})\}$, where $x \preceq y$ means $x \le y$ in the usual sense.
Then $(X, \preceq, d)$ is a partially ordered complete metric space. We define
$T : X \to X$ by $T(x) = \begin{cases} x + \frac{1}{2} & if\ x \in [0,1) - \{\frac{1}{2}, \frac{3}{4}\} \\ \frac{3}{4} & if\ x = \frac{3}{4} \\ 2 & otherwise, \end{cases}$ and

$S : X \to X$ by $S(x) = \begin{cases} 2x & if\ x \in [0,1] - \{\frac{3}{4}, \frac{3}{8}\} \\ \frac{3}{2} & if\ x = \frac{3}{8} \\ \frac{3}{4} & if\ x = \frac{3}{4} \\ 2 & otherwise. \end{cases}$

Clearly $T(X) \subseteq S(X)$, $SX$ is closed and $T$ is $S$ non-decreasing.
We choose $x_0 = 0 \in X$ then $Sx_0 \preceq Tx_0$. We define
$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = 2t^2, t \geq 0$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \frac{t}{4}$ , $t \geq 0$.
We now verify the inequality (2.1).

Case (i) : Let $(x,\ y) = (0, \frac{1}{4})$ such that $S(0) \preceq S(\frac{1}{4})$.

In this case, $\varphi(d(T(0), T(\frac{1}{4}))) = \varphi(d(\frac{1}{2}, \frac{3}{4})) = \varphi(\frac{1}{4}) = \frac{1}{8}$,
$M(0, \frac{1}{4}) = \frac{1}{2}$ and $N(0, \frac{1}{4}) = \frac{1}{2}$;
Now $\varphi(M(0, \frac{1}{4})) = \varphi(\frac{1}{2}) = \frac{1}{2}, \psi(N(0, \frac{1}{4})) = \psi(\frac{1}{2}) = \frac{1}{8}$.
Therefore
$\varphi(d(T(0), T(\frac{1}{4}))) = \frac{1}{8} \leq \frac{1}{2} - \frac{1}{8} = \varphi(M(0, \frac{1}{4})) - \psi(N(0, \frac{1}{4}))$.

Case (ii) : Let $(x,\ y) = (0, \frac{3}{4})$ such that $S(0) \preceq S(\frac{3}{4})$.

In this case, $\varphi(d(T(0), T(\frac{3}{4}))) = \varphi(d(\frac{1}{2}, \frac{3}{4})) = \varphi(\frac{1}{4}) = \frac{1}{8}$,
$M(0, \frac{3}{4}) = \frac{3}{4}$ and $N(0, \frac{3}{4}) = \frac{3}{4}$;
Now $\varphi(M(0, \frac{3}{4})) = \varphi(\frac{3}{4}) = \frac{9}{8}, \psi(N(0, \frac{3}{4})) = \psi(\frac{3}{4}) = \frac{3}{16}$.
Therefore
$\varphi(d(T(0), T(\frac{3}{4}))) = \frac{1}{8} \leq \frac{9}{8} - \frac{3}{16} = \varphi(M(0, \frac{3}{4})) - \psi(N(0, \frac{3}{4}))$.

Case (iii) : Let $(x,\ y) = (\frac{1}{4}, \frac{3}{4})$ such that $S(\frac{1}{4}) \preceq S(\frac{3}{4})$.

In this case, $\varphi(d(T(\frac{1}{4}), T(\frac{3}{4}))) = \varphi(d(\frac{3}{4}, \frac{3}{4})) = \varphi(0) = 0$,
$M(\frac{1}{4}, \frac{3}{4}) = \frac{1}{4}$ and $N(\frac{1}{4}, \frac{3}{4}) = \frac{1}{4}$;
Now $\varphi(M(\frac{1}{4}, \frac{3}{4})) = \varphi(\frac{1}{4}) = \frac{1}{8}, \psi(N(\frac{1}{4}, \frac{3}{4})) = \psi(\frac{1}{4}) = \frac{1}{16}$.
Therefore
$\varphi(d(T(\frac{1}{4}), T(\frac{3}{4}))) = 0 \leq \frac{1}{8} - \frac{1}{16} = \varphi(M(\frac{1}{4}, \frac{3}{4})) - \psi(N(\frac{1}{4}, \frac{3}{4}))$.
In the remaining cases, the inequality (2.1) holds trivially.
Therefore $S$ and $T$ satisfy all the hypotheses of Theorem 2.1 and
$S$ and $T$ have infinitely many coincident points.
Furthermore, we note that clearly $S$ and $T$ are weakly compatible, and
$Sx = Tx \Rightarrow Sx \preceq SSx\ \forall x \in X$, so that $(i)$and $(ii)$ of Theorem 2.2 hold and $\frac{3}{4}$ and 2 are common fixed points of $S$ and $T$.
Further, we observe that $S$ and $T$ do not satisfy 'Condition H'.
For

Case (i) : If $u = 0$ then $Su = 0, Tu = \frac{1}{2}$, clearly $Su \preceq Tu$.

In this case, for any $x, y \in [0,3) - \{0, \frac{1}{4}, \frac{3}{4}\}$, neither $Tx$ nor $Ty$ is comparable to $\frac{1}{2} = Tu$.

Case (ii) : If $u = \frac{1}{4}$ then $Su = \frac{1}{2}, Tu = \frac{3}{4}$, clearly $Su \preceq Tu$.

In this case, for any $x, y \in [0,3) - \{0, \frac{1}{4}, \frac{3}{4}\}$, neither $Tx$ nor $Ty$ is comparable to $\frac{3}{4} = Tu$.

Case (iii) : If $u = \frac{3}{4}$ then $Su = \frac{3}{4}, Tu = \frac{3}{4}$, clearly $Su \preceq Tu$.

In this case, for any $x, y \in [0,3) - \{0, \frac{1}{4}, \frac{3}{4}\}$, neither $Tx$ nor $Ty$ is comparable to $\frac{3}{4} = Tu$.

Case (iv) : If $u = [1,3)$ then $Su = 2 = Tu$, clearly $Su \preceq Tu$.

In this case, for any $x, y \in [0,1) - \{\frac{1}{2}\}$, neither $Tx$ nor $Ty$ is comparable to $2 = Tu$.

Case (v) : If $u \in [0,3) - \{0, \frac{1}{4}, \frac{3}{4}\}$ then clearly $Su \npreceq Tu$.
Hence 'Condition(H)' fails to hold.

The following is an example in support of Theorem 2.2.

**Example 3.2.** Let $X = \{0, 1, 2, 5\}$ with the usual metric. We define partial order $\preceq$ on $X$ as follows:
$\preceq := \{(0,0), (1,1), (2,2), (5,5), (0,1), (0,2), (0,5), (1,2), (1,5), (2,5)\}$, where $x \preceq y$ means $x \leq y$ in the usual sense.
Then $(X, \preceq, d)$ is a partially ordered metric space. We define $S, T : X \to X$ by $S0 = 0, S1 = 1, S2 = 5, S5 = 2$ and

$$T0 = T1 = T5 = 1, T2 = 2.$$

Clearly, $T(X) \subseteq S(X)$, and $T$ is $S$ non-decreasing.
We choose $x_0 = 0 \in X$. Then $Sx_0 \preceq Tx_0$. We define

$$\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ by } \varphi(t) = t^3,\ t \geq 0, \quad \text{and}$$

$$\psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ by } \psi(t) = \begin{cases} \frac{4}{5}t & if\ t \in \mathbb{Q}^+ \\ 1 & otherwise. \end{cases}$$

We now verify the inequality (2.1).
*Case (i)*: Let $(x,\ y) = (1, 2)$ such that $S1 \preceq S2$.

In this case, $\varphi(d(T1, T2)) = \varphi(d(1, 2)) = \varphi(1) = 1$, $M(1, 2) = 4$ and $N(1, 2) = 4$. Now $\varphi(M(1, 2)) = \varphi(4) = 64$, $\psi(N(1, 2)) = \psi(4) = \frac{16}{5}$.
Therefore
$\varphi(d(T1, T2)) = 1 \leq 64 - \frac{16}{5} = \varphi(M(1, 2)) - \psi(N(1, 2))$.
*Case (ii)* : Let $(x,\ y) = (0, 2)$ such that $S0 \preceq S2$.

In this case, $\varphi(d(T0, T2)) = \varphi(d(1, 2)) = \varphi(1) = 1$, $M(0, 2) = 5$ and $N(0, 2) = 5$. Now $\varphi(M(0, 2)) = \varphi(5) = 125$, $\psi(N(0, 2)) = \psi(5) = 4$.
Therefore
$\varphi(d(T0, T2)) = 1 \leq 125 - 4 = \varphi(M(0, 2)) - \psi(N(0, 2))$.
*Case (iii)* : Let $(x,\ y) = (5, 2)$ such that $S5 \preceq S2$.

In this case, $\varphi(d(T5, T2)) = \varphi(d(1, 2)) = \varphi(1) = 1$, $M(5, 2) = 3$ and $N(5, 2) = 3$. Now $\varphi(M(5, 2)) = \varphi(3) = 27$, $\psi(N(5, 2)) = \psi(3) = \frac{12}{5}$.
Therefore
$\varphi(d(T5, T2)) = 1 \leq 27 - \frac{12}{5} = \varphi(M(5, 2)) - \psi(N(5, 2))$.
In the remaining cases the inequality (2.1) holds trivially.
Also, $S$ and $T$ are weakly compatible, and $(ii)$ of Theorem 2.2 hold. Further, by choosing $u = 0$ with $S0 \preceq T0$ and $T0$ is comparable with $Tx$ and $Ty$ for all $x, y \in X$ so that 'Condition (H)' holds.
Therefore, $S$ and $T$ satisfy all the hypotheses of Theorem 2.2 and $S$ and $T$ have a unique common fixed point 1.

The following is an example in support of Theorem 2.3.

**Example 3.3.** Let $X = [0, 2]$ with the usual metric. We define partial order $\preceq$ on $X$ as follows:
$\preceq := \{(x, y) \in X \times X : x = y\} \cup \{(\frac{1}{2^{2n}}, 0) : n \geq 1\}$, where $x \preceq y$ means $x \geq y$ in the usual sense.
Then $(X, \preceq, d)$ is a partially ordered complete metric space. We define

$$T : X \to X \text{ by } T(x) = \begin{cases} \frac{x^2}{4} & if\ x \in [0, 1) \\ 2 & if\ x \in [1, 2], \end{cases} \quad \text{and}$$

$$S : X \to X \text{ by } S(x) = \begin{cases} x^2 & if\ x \in [0, 1) \\ 2 & if\ x \in [1, 2]. \end{cases}$$

Clearly $T(X) \subseteq S(X)$, and $T$ is $S$ non-decreasing.
We choose $x_0 = 0 \in X$. Then $Sx_0 \preceq Tx_0$ and clearly $S$ and $T$ are reciprocally continuous

and the pair $(S,T)$ is compatible.
We define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t^2,\ t \geq 0,$ and
$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \frac{3}{4}t^2\ if\ t \geq 0.$
We now verify the inequality (2.1).
Case (i) : Let $(x,\ y) = (\frac{1}{2^n}, 0)$ such that $S(\frac{1}{2^n}) \preceq S(0)$, for n= 1,2,3, . . . .

In this case, $\varphi(d(T(\frac{1}{2^n})), T(0)) = \varphi(d(\frac{1}{2^{2n+2}}), 0) = \varphi(\frac{1}{2^{2n+2}}) = (\frac{1}{2^{2n+2}})^2$,
$M(\frac{1}{2^n}, 0) = \frac{1}{2^{2n}}$ and $N(\frac{1}{2^n}, 0) = \frac{1}{2^{2n}}$.
Now $\varphi(M(\frac{1}{2^n}, 0)) = \varphi(\frac{1}{2^{2n}}) = (\frac{1}{2^{2n}})^2, \psi(N(\frac{1}{2^n}, 0)) = \psi(\frac{1}{2^{2n}}) = \frac{3}{4}\frac{1}{(2^{2n})^2}$.
Therefore
$\varphi(d(T(\frac{1}{2^n}), T(0))) = (\frac{1}{2^{2n+2}})^2 \leq (\frac{1}{2^{2n}})^2 - \frac{3}{4}\frac{1}{(2^{2n})^2} = \varphi(M(\frac{1}{2^n}, 0)) - \psi(N(\frac{1}{2^n}, 0))$, for $n = 1, 2, 3, ...$ .
In the remaining cases, the inequality (2.1) holds trivially.
Therefore, $S$ and $T$ satisfy all the hypotheses of Theorem 2.3, and $S$ and $T$ have two common fixed points 0 and 2.
Further, we observe that $S$ and $T$ do not satisfy 'Condition H'.
For,
Case (i) : If $u = 0$ then $Su = 0 = Tu$ so that $Su \preceq Tu$.

In this case for any $x, y \in (0, 2]$, neither $Tx$ nor $Ty$ is comparable to $0 = Tu$.
Case (ii) : If $u \in [1, 2]$ then $Su = 2 = Tu$ so that $Su \preceq Tu$.

In this case for any $x, y \in [0, 2)$, neither $Tx$ nor $Ty$ is comparable to $2 = Tu$.
Case (iii) : If $u \in (0, 1)$ then $Su \npreceq Tu$.
Hence 'Condition(H)' fails to hold.

**Example 3.4.** Let $X = \{1, 2, 4, 5\}$ with the usual metric. We define partial order $\preceq$ on $X$ as follows:
$\preceq := \{(1, 1), (2, 2), (4, 4)(5, 5), (1, 2), (1, 4), (1, 5), (2, 4), (2, 5)\}$, where
$x \preceq y$ means $x \leq y$ in the usual sense.
Then $(X, \preceq, d)$ is a partially ordered metric space. We define
$S, T : X \to X$ by $S1 = 1, S2 = 2, S4 = 5, S5 = 4$ and
$T1 = T2 = 1, T4 = T5 = 2.$
Clearly $T(X) \subseteq S(X)$, and $T$ is $S$ non-decreasing.
We choose $x_0 = 1 \in X$. Then $Sx_0 \preceq Tx_0$ and clearly $S$ and $T$ are compatible and reciprocally continuous.
We define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t^2,\ t \geq 0,$ and
$$\psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ by } \psi(t) = \begin{cases} t & if\ t \in [0, 1] \\ 2 & otherwise. \end{cases}$$
We now verify the inequality (2.1).
Case (i) :Let $(x,\ y) = (1, 5)$ such that $S1 \preceq S5$.

In this case, $\varphi(d(T1, T5)) = \varphi(d(1, 2)) = \varphi(1) = 1$, $M(1, 5) = 3$ and $N(1, 5) = 3$.
Now $\varphi(M(1, 5)) = \varphi(3) = 9$, $\psi(N(1, 5)) = \psi(3) = 2$.
Therefore
$\varphi(d(T1, T2)) = 1 \leq 9 - 2 = \varphi(M(1, 5)) - \psi(N(1, 5))$.
Case (ii) : Let $(x,\ y) = (1, 4)$ such that $S1 \preceq S4$.

In this case, $\varphi(d(T1, T4)) = \varphi(d(1, 2)) = \varphi(1) = 1$, $M(1, 4) = 4$ and $N(1, 4) = 4$.
Now $\varphi(M(1, 4)) = \varphi(4) = 16$, $\psi(N(1, 4)) = \psi(4) = 2$.
Therefore
$\varphi(d(T1, T4)) = 1 \leq 16 - 2 = \varphi(M(1, 4)) - \psi(N(1, 4))$.
Case (iii) : Let $(x,\ y) = (2, 5)$ such that $S2 \preceq S5$.

In this case, $\varphi(d(T2,T5)) = \varphi(d(1,2)) = \varphi(1) = 1$, $M(2,5) = 2$ and $N(2,5) = 2$. Now $\varphi(M(2,5)) = \varphi(2) = 4$, $\psi(N(2,5)) = \psi(2) = 2$.
Therefore
$\varphi(d(T2,T5)) = 1 \leq 4 - 2 = \varphi(M(2,5)) - \psi(N(2,5))$.
*Case (iv)* : Let $(x,\ y) = (2,4)$ such that $S2 \preceq S4$.

In this case, $\varphi(d(T2,T4)) = \varphi(d(1,2)) = \varphi(1) = 1$, $M(2,4) = 3$ and $N(2,4) = 3$. Now $\varphi(M(2,4)) = \varphi(3) = 9$, $\psi(N(2,4)) = \psi(3) = 2$.
Therefore
$\varphi(d(T2,T4)) = 1 \leq 9 - 2 = \varphi(M(2,4)) - \psi(N(2,4))$.
In the remaining cases the inequality (2.1) holds trivially.

Further, by choosing $u = 1$ with $S1 \preceq T1$ and $T1$ is comparable with $Tx$ and $Ty$ for all $x, y \in X$ so that 'Condition (H)' holds.
Therefore, $S$ and $T$ satisfy all the hypotheses of Theorem 2.3 and $S$ and $T$ have a unique common fixed point 1.

**Example 3.5.** Let $X = [0,1]$ with usual metric. We define partial order $\preceq$ on $X$ as follows:
$\preceq := \{(\frac{1}{2^n}, \frac{1}{2^{n+k}})/n = 0,1,2,...,k = 1,2,3,....\} \cup \{(0,x)/x \in X\} \cup \Delta$, where $x \preceq y$ means $x \geq y$ in the usual sense.
Then $(X, \preceq, d)$ is a partially ordered complete metric space. We define

$$S : X \to X \text{ by } Sx = \begin{cases} x & if\ x \in [0,\frac{1}{4}] \cup \{\frac{1}{2},1\} \\ 2x & if\ x \in (\frac{1}{4},\frac{1}{2}) \\ \frac{x}{2} & if\ (\frac{1}{2},1) \text{ and} \end{cases}$$

$T : X \to X$ by $Tx = \frac{x^2}{4}$ for all $x \in [0,1]$.
Clearly $T(X) \subseteq S(X)$, and $T$ is $S$ non-decreasing.
We choose $x_0 = \frac{1}{2} \in X$. Then $Sx_0 \preceq Tx_0$
We define $\varphi,\ \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t,\ t \geq 0$, and $\psi(t) = \frac{t}{4},\ t \geq 0$.
We now verify the inequality (2.1).
*Case (I)* : Let $(x,y) = (\frac{1}{2^n}, \frac{1}{2^{n+k}})$ such that $Sx \preceq Sy$ for $n \geq 0$ and $k \geq 1$.
In this case, we have
$M(\frac{1}{2^n}, \frac{1}{2^{n+k}}) = \max\{a,b,c,d\}$ and $N(\frac{1}{2^n}, \frac{1}{2^{n+k}}) = \max\{a,b,d\}$,
where

$$a = (\frac{d(S(\frac{1}{2^{n+k}}),T(\frac{1}{n+k}))[1+d(S(\frac{1}{2^n}),T(\frac{1}{2^n})]}{1+d(S(\frac{1}{2^n}),S(\frac{1}{2^{n+k}})}),\ b = (\frac{d(S(\frac{1}{2^n}),T(\frac{1}{2^n}))[1+d(S(\frac{1}{2^{n+k}}),T(\frac{1}{n+k}))]}{1+d(S(\frac{1}{2^n}),S(\frac{1}{2^{n+k}})}),$$

$$c = (\frac{d(S(\frac{1}{2^{n+k}}),T(\frac{1}{2^n})[1+d(S(\frac{1}{2^n}),T(\frac{1}{2^{n+k}})]}{1+d(S(\frac{1}{2^n}),S(\frac{1}{2^{n+k}})}),\ d = d(S(\frac{1}{2^n}), S(\frac{1}{2^{n+k}})).$$

We observe the following:

1. $a \leq b$ for all $k \geq 1$ and for all $n \geq 0$,
2. $c \leq b$ for all $k \leq n+2$,
3. $c \leq d$ for all $k \geq n+2$.

Hence $M(x,y) = N(x,y) = b$ or $d$.
*Subcase (i)* : $M(x,y) = N(x,y) = b$.

In this case, we have $(\frac{1}{2^{2n+2}} - \frac{1}{2^{2n+2k+2}}) \leq \frac{\frac{3}{4}(\frac{1}{2^n} - \frac{1}{2^{2n+2}})(1+\frac{1}{2^{n+k}} - \frac{1}{2^{2n+2k+2}})}{(1+\frac{1}{2^n} - \frac{1}{2^{n+k}})}$ for all $n \geq 0$ and $k \geq 1$, which implies that
$\varphi(d(Tx,Ty)) \leq b - \frac{b}{4} = \varphi(b) - \psi(b) = \varphi(M(x,y)) - \psi(N(x,y))$.
*Subcase (ii)* : $M(x,y) = N(x,y) = d$.

In this case, we have $(\frac{1}{2^{2n+2}} - \frac{1}{2^{2n+2k+2}}) \leq \frac{3}{4}(\frac{1}{2^n} - \frac{1}{2^{n+k}})$ which implies that
$\varphi(d(Tx,Ty)) \leq b - \frac{d}{4} = \varphi(d) - \psi(d) = \varphi(M(x,y)) - \psi(N(x,y))$.
In either case, the inequality (2.1) holds.
*Case (II)* : Let $(x,y) = (0,x)$ such that $S0 \preceq Sx$.

In this case, $M(0,x) = N(0,x) = \begin{cases} x & if\ x \in [0,\frac{1}{4}] \cup \{\frac{1}{2},1\} \\ 2x & if\ x \in (\frac{1}{4},\frac{1}{2}) \\ \frac{x}{2} & if\ x \in (\frac{1}{2},1). \end{cases}$

If $x \in [0,\frac{1}{4}] \cup \{\frac{1}{2},1\}$ then
$\varphi(d(T0,Tx)) = \frac{x^2}{4} \leq \frac{3x}{4} = x - \frac{x}{4} = \varphi(M(0,x)) - \psi(N(0,x))$.
Similarly, it is easy to see that the inequality (2.1) holds in all other cases.
*Case (III)* : Let $(x,y) \in \Delta$ such that $x = y$.
In this case, we note that
$M(x,x) = N(x,x) = d(Sx,Tx)(1 + d(Sx,Tx))$ for all $x \in X$.

Now $\varphi(d(Tx,Tx)) = \varphi(0) \leq \frac{3}{4}M(x,x) = M(x,x) - \frac{N(x,x)}{4}$
$= \varphi(M(x,y)) - \psi(N(x,x))$ for all $x \in X$.

Hence $S$ and $T$ satisfy the inequality (2.1).
Also, $S,T$ are reciprocally continuous and compatible.
So let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Therefore, $x_n \to 0$ and $z = 0$. There exists $N \in \mathbb{Z}^+$ such that $n \geq N$ implies $x_n \leqslant \frac{1}{4}$. Therefore $Sx_n = x_n$ and $Tx_n = \frac{x_n^2}{4}$ for all $n \geq N$. Now $TSx_n = Tx_n = \frac{x_n^2}{4}$ and $STx_n = S(\frac{x_n^2}{4}) = \frac{x_n^2}{4}$ for all $n \geq N$.
Therefore $TSx_n = STx_n$ for all $n \geq N$ . There is $d(TSx_n, STx_n) = 0$ for all $n \geq N$. Hence $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$. Therefore, the pair $(S,T)$ is compatible.

Also, $\lim_{n\to\infty} STx_n = \lim_{n\to\infty} \dfrac{x_n^2}{4} = 0 = S0$ and $\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} \dfrac{x_n^2}{4} = 0 = T0$. Therefore, $S,T$ are reciprocally continuous. We observe that 'condition (H)' holds, because by choosing $0 \in X$ we have $S0 \preceq T0$ and $T0 = 0$ is comparable with $Tx$ and $Ty$ for all $x,y \in X$. Hence all the hypotheses of Theorem 2.3 hold and $S$ and $T$ have a unique common fixed point 0.

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