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THE GENERALIZED NON-ABSOLUTE TYPE OF TRIPLE Γ^3 SEQUENCE SPACES DEFINED MUSIELAK-ORLICZ FUNCTION *

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Abstract. In this paper we introduce the notion of $\lambda_{mnk} - \Gamma^3$ and Λ^3 sequences. Further, we introduce the spaces
$\left[\Gamma^{3\lambda}_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ and
$\left[\Lambda^{3\lambda}_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$, which are of non-absolute type and we prove that these spaces are linearly isomorphic to the spaces Γ^3 and Λ^3, respectively. Moreover, we establish some inclusion relations between these spaces.

Keywords: Analytic sequence, Γ^3 space, difference sequence space, Musielak-Orlicz function, p-metric space.

1. Introduction

Let (x_{mnk}) be a triple sequence of real or complex numbers. A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [10,11], Esi et al. [3-5], Datta et al. [1],Subramanian et al. [12], Debnath et al. [2] and many others.
A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

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The space of all triple entire sequences are usually denoted by Γ^3. The space Λ^3 and Γ^3 is a metric space with the metric

$$d(x,y) = sup_{m,n,k}\left\{|x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,...\right\}, \tag{1.1}$$

for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ $in\,\Lambda^3(\Gamma^3)$.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [7] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Let $w^3, \chi^3(\Delta), \Lambda^3(\Delta)$ be denote the spaces of all, triple gai difference sequence space and triple analytic difference sequence space respectively. The difference triple sequence space was introduced by Debnath et al. (see [2]) and is defined as $\Delta x_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$ and $\Delta^0 x_{mnk} = \langle x_{mnk}\rangle$.

2. Definitions and Preliminaries

Throughout the article $w^3, \chi^3(\Delta), \Lambda^3(\Delta)$ denote the spaces of all, triple gai difference sequence spaces and triple analytic difference sequence spaces respectively.

For a triple sequence $x \in w^3$, Subramanian et al. introduced by ([12]), the spaces $\Gamma^3(\Delta), \Lambda^3(\Delta)$ as follows:

$$\Gamma^3(\Delta) = \left\{x \in w^3 : |\Delta x_{mnk}|^{1/m+n+k} \to 0\, as\, m,n,k \to \infty\right\}$$

$$\Lambda^3(\Delta) = \left\{x \in w^3 : sup_{m,n,k}|\Delta x_{mnk}|^{1/m+n+k} < \infty\right\}.$$

The spaces $\Gamma^3(\Delta), \Lambda^3(\Delta)$ are metric spaces with the metric

$$d(x,y) = sup_{m,n,k}\left\{|\Delta x_{mnk} - \Delta y_{mnk}|^{1/m+n+k} : m,n,k = 1,2,\cdots\right\}$$

for all $x = (x_{mnk})$ and $y = (y_{mnk})$ in $\Gamma^3(\Delta), \Lambda^3(\Delta)$. □

Definition 2.1. An Orlicz function ([see [6]) is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([8]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - (f_{mnk})(u) : u \geq 0\}, m, n, k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, [see [9]] the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0\, as\, m, n, k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an exteneded real number.

Definition 2.2. Let X, Y be a real vector space of dimension w, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,
(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ for $1 \leq p < \infty$; (or)
(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces (see [13]) .

3. Main Results

Let $\eta = (\lambda_{mnk})$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_{111} = 1$ and $\lambda_{m+n+k+3} \leq \lambda_{m+n+k+3} + 1$, for all $m, n, k \in \mathbb{N}$.

The generalized de la Vallée-Poussin means is defined by
$t_{mnk}(x) = \lambda_{mnk}^{-1} \sum_{m,n,k \in I_{mnk}} x_{mnk}$, where $I_{mnk} = [mnk - \lambda_{mnk} + 1, mnk]$. A sequence $x = (x_{mnk})$ is said to $(V, \lambda)-$ summable to a number L if $t_{mnk}(x) \to L$, as $mnk \to \infty$.

The notion of $\lambda-$ triple entire and triple analytic sequences as follows: Let $\lambda = (\lambda_{mnk})_{m,n,k=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity.

Consider $B_\eta^\mu(x) = \frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}} \lambda_{mnk} x_{mnk} - \lambda_{m,n+1,k} x_{m,n+1,k} - \lambda_{m,n,k+1} x_{m,n,k+1} + \lambda_{m,n+1,k+1} x_{m,n+1,k+1} - \lambda_{m+1,n,k} x_{m+1,n,k} + \lambda_{m+1,n+1,k} x_{m+1,n+1,k} + \lambda_{m+1,n,k+1} x_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1} x_{m+1,n+1,k+1}$.

Definition 3.1. A sequence $x = (x_{mnk}) \in w^3$ is said to be $\lambda-$ convergent to a, if $B^{\mu}_{\eta}(x) \to a \; as \; m,n,k \to \infty$ and write $\lambda - lim(x) = a$.
Definition 3.2. A sequence $x = (x_{mnk}) \in w^3$ is said to be $\lambda-$ triple entire sequence if $B^{\mu}_{\eta}(x) \to 0 \; as \; m,n,k \to \infty$.

Definition 3.3. A sequence $x = (x_{mnk}) \in w^3$ is said to be $\lambda-$ triple analytic sequence if $sup_{mnk} B^{\mu}_{\eta}(x) < \infty$. We have
$lim_{m,n,k\to\infty} \left|B^{\mu}_{\eta}(x) - a\right| = lim_{m,n,k\to\infty} \frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}}$
$\lambda_{mnk}x_{mnk} - \lambda_{m,n+1,k}x_{m,n+1,k} - \lambda_{m,n,k+1}x_{m,n,k+1} + \lambda_{m,n+1,k+1}x_{m,n+1,k+1} - \lambda_{m+1,n,k}x_{m+1,n,k} + \lambda_{m+1,n+1,k}x_{m+1,n+1,k} + \lambda_{m+1,n,k+1}x_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1}x_{m+1,n+1,k+1} = a$. So we can say that $lim_{m,n,k\to\infty} \left|B^{\mu}_{\eta}(x)\right| = a$. Hence x is $\lambda_{mnk}x_{mnk}-$ convergent to a.
Lemma 3.4. Every convergent sequence is $\lambda_{mnk}-$ convergent to the same ordinary limit.

Proof. Omitted. □

Lemma 3.5. If a $\lambda_{mnk}-$ Musielak-convergent sequence converges in the ordinary sense, then it must Musielak-converge to the same $\lambda_{mnk}-$ limit.

Proof. Let $x = (x_{mnk}) \in w^3$ and $m,n,k \geq 1$. We have
$|\Delta^m x_{mnk}|^{1/m+n+k} - B^{\mu}_{\eta}(x) = |\Delta^m x_{mnk}|^{1/m+n+k} -$
$\frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} \lambda_{mnk}x_{mnk} - \lambda_{m,n+1,k}x_{m,n+1,k} - \lambda_{m,n,k+1}x_{m,n,k+1} + \lambda_{m,n+1,k+1}x_{m,n+1,k+1} - \lambda_{m+1,n,k}x_{m+1,n,k} + \lambda_{m+1,n+1,k}x_{m+1,n+1,k} + \lambda_{m+1,n,k+1}x_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1}x_{m+1,n+1,k+1}$.
Therefore we have for every $x = (x_{mnk}) \in w^3$ that $|\Delta^m x_{mnk}|^{1/m+n+k} - B^{\mu}_{\eta}(x) = S_{mnk}(x)\,(m,n,k \in \mathbb{N})$. where the sequence $S(x) = (S_{mnk}(x))^{\infty}_{m,n,k=0}$ is defined by $S_{000}(x) = 0$ and $S_{mnk}(x) = \frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}}$
$\lambda_{mnk} - \lambda_{m,n+1,k} - \lambda_{m,n,k+1} + \lambda_{m,n+1,k+1} - \lambda_{m+1,n,k} + \lambda_{m+1,n+1,k} + \lambda_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1}, (m,n,k \geq 1)$. □

Lemma 3.6. A $\lambda_{mnk}-$ Musielak-convergent sequence $x = (x_{mnk})$ converges if and only if $S(x) \in \left[\Gamma^3_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$

Proof. Let $x = (x_{mnk})$ be $\lambda_{mnk}-$ Musielak-convergent sequence. Then from Lemma 3.2, we have $x = (x_{mnk})$ converges to the same $\lambda_{mnk}-$ limit. We obtain $S(x) \in \left[\Gamma^3_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. Conversely, let $S(x) \in \left[\Gamma^3_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. We have

$$lim_{m,n,k\to\infty} |\Delta^m \lambda_{mnk}x_{mnk}|^{1/m+n+k} = lim_{m,n,k\to\infty} B^{\mu}_{\eta}(x).$$

From the above equation, we deduce that $\lambda_{mnk}-$ convergent sequence $x = (x_{mnk})$ converges. □

Lemma 3.7. Every triple analytic sequence is $\lambda_{mnk}-$ triple analytic.

Proof. let

$$S(x) \in \left[\Lambda^3_{fB^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right].$$

Then there exists $M > 0$, We have

$$sup_{mnk} \left|\Delta^m \lambda_{mnk} x_{mnk}\right|^{1/m+n+k} = sup_{mnk} B^{\mu}_{\eta}(x) < M.$$

From the above equation, we deduce that $\lambda_{mnk}-$ analytic sequence $x = (x_{mnk})$ analytic. □

Lemma 3.8. A $\lambda_{mnk}-$ Musielak-analytic sequence $x = (x_{mnk})$ is analytic if and only if $S(x) \in \left[\Lambda^3_{f B^{\mu}_{\eta}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$

Proof. From Lemma 3.4 and $S_{000}(x) = 0$ and

$$S_{mnk}(x) = \frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}} \lambda_{mnk} x_{mnk} - \lambda_{m,n+1,k} x_{m,n+1,k} - \lambda_{m,n,k+1} x_{m,n,k+1}$$
$$+ \lambda_{m,n+1,k+1} x_{m,n+1,k+1} - \lambda_{m+1,n,k} x_{m+1,n,k} + \lambda_{m+1,n+1,k} x_{m+1,n+1,k}$$
$$+ \lambda_{m+1,n,k+1} x_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1} x_{m+1,n+1,k+1}, \quad (m,n,k \geq 1).$$

4. The spaces of $\lambda_{mnk}-$ triple entire and triple analytic sequences

In this section we introduce the sequence space: If

$$\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right],$$
$$\left[\Lambda^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$$

then the sets of λ_{mnk} triple entire, triple analytic sequences respectively.

$$\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$$
$$= lim_{m,n,k\to\infty} \left[B^{\mu}_{\eta}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = 0$$

$$\left[\Lambda^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$$
$$= sup_{mnk} \left[B^{\mu}_{\eta}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] < \infty.$$

Theorem 4.1. *The sequence spaces* $\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *and*
$\left[\Lambda^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *are isomorphic to the spaces*
$\left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *and*
$\left[\Lambda^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$

Proof. *We only consider the case* $\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \cong$

$\left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *and*

$\left[\Lambda^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \cong$
$\left[\Lambda^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *can be shown similarly.*
Consider the transformation T *defined,*
$Tx = B^{\mu}_{\eta} \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *for every*
$x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. *The linearity of* T *is obvious. It is trivial that* $x = 0$ *whenever* $Tx = 0$ *and hence* T *is injective.*
To show surjective we define the sequence $x = \{x_{mnk}(\lambda)\}$ *by*

$$B^{\mu}_{\eta}(x) = \frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} (\Delta\lambda_{mnk}x_{mnk}) = y_{mnk} \tag{4.1}$$

We can say that $B^{\mu}_{\eta}(x) = y_{mnk}$ *from (2) and*
$x \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$, *hence*
$B^{\mu}_{\eta}(x) \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. *We deduce from that* $x \in$
$\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *and* $Tx = y$. *Hence* T *is surjective. We have for every* $x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *that*
$d(Tx,0)_{\chi^3} = d(Tx,0)_{\Lambda^3} = d(x,0)_{\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]}$.
Hence $\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ *and*
$\left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *are ismorphic. Similarly obtain other sequence spaces.* The proof is completed. □

Theorem 4.2. *The inclusion* $\left[\Gamma^3_{f\Delta_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \subset$
$\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *holds*

Proof.Let $\left[\Gamma^3_{f\Delta_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. *Then we deduce that*
$\frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} (\Delta\lambda_{mnk}x_{mnk}) \leq$
$\frac{1}{\varphi_{rst}} lim_{m,n,k\to\infty} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} (\Delta\lambda_{mnk}x_{mnk}) =$
$lim_{m,n,k\to\infty} |\Delta^m\lambda_{mnk}x_{mnk}|^{1/m+n+k} = 0$. *Hence*
$x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. The proof is completed.
□

Theorem 4.3. *The inclusion* $\left[\Lambda^3_{f\Delta_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \subset$
$\left[\Lambda^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *holds.*

Proof. It is obvious. Therefore omit the proof. The proof is completed. □

Theorem 4.4. *The inclusion* $\left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \subset$
$\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *hold. Furthermore, the equalities*

hold if and only if $S(x) \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *for every sequence x in the space* $\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$

Proof.Consider

$$\left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \subset \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \quad (7)$$

is obvious from Lemma 3.4. Then, we have for every $x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *that* $x \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *and hence* $S(x) \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *by Lemma 3.6. Conversely, let* $x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. *Then, we have that* $S(x) \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. *Thus, it follows by Lemma 3.5 and then Lemma 3.6, that* $x \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. *We get*

$$\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \subset \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \quad (8)$$

From the equation (7) and (8) we get $\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. The proof is completed.

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