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SOME CURVES ASSOCIATED WITH α-SASAKIAN MANIFOLDS

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Abstract. The aim of the present paper is to study biharmonic magnetic curves on three-dimensional α-Sasakian manifolds. We also characterize biminimal curves on a hypersurface of a three-dimensional α-Sasakian manifold with constant curvature.

Keywords: Magnetic curves, Biminimal curves, ξ-vertical hypersurfaces, α-Sasakian manifolds, signed curvature

1. Introduction

In [3], Cabrerizo, Fernandez and Gomez introduced a geometric approach to the study of magnetic fields on three-dimensional Sasakian manifolds. A magnetic curve is the trajectory of magnetic fields. Geodesics on a manifold are curves which do not experience any kind of forces where the magnetic curves experience force due to magnetic fields. If the magnetic field disappears, the magnetic curves become geodesics. In this way a magnetic curve is a generalization of a geodesic.

Nowadays, the study of geometric variational problem of curves has become a subject of growing interest. Elastic curves are concerned with the classically known geometrical variational problems. A plane curve is called an elastic curve if it is a critical point of elastic energy [10]. In [9], Inoguchi and Lee studied another geometric variational problem in Riemannian 2-manifolds of constant curvature. They have investigated biminimal curves in 2-dimensional space forms. A smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is said to be biharmonic if it is a critical point of the bienergy functional $E_2(\phi) = \int_M |\tau(\phi)^2| dv_g$, where $\tau(\phi) = tr\nabla d\phi$ is the tension field of ϕ. For further details we refer [4], [9], [12]. Loubeau and Montaldo introduced the notion of biminimal immersions [11]. An isometric immersion $\phi : (M, g) \to (N, h)$ is said to be biminimal if it is a critical point of the bienergy functional under all normal variations. Thus, the biminimality is weaker than biharmonicity for isometric immersions, in general. For a unit speed curve $\gamma(s)$ in a Riemannian 2-manifold M, its tension field is given by $\mathcal{T}(\gamma) = \nabla_{\gamma'}\gamma'$.

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Thus the bienergy of γ is the elastic energy $E_2(\gamma) = \frac{1}{2}\int k(s)^2 ds$, where $k(s)$ is signed curvature of γ. Loubeau and Montaldo [11] proved that *a unit speed curve $\gamma(s)$ in a Riemannian 2-manifold of Gaussian curvature K is biminimal if and only if its signed curvature $k(s)$ satisfies*

$$k'' - k^3 + kK = 0. \tag{1.1}$$

γ is biharmonic if and only if γ is biminimal and additionally satisfies $kk' = 0$. The concept of ξ-vertical and ξ-horizontal can be found in the paper [8]. Following it we shall call the hypersurface of a three-dimensional α-Sasakian manifold as ξ-vertical if the tangent vector fields of the hypersurface is orthogonal to ξ. Here, we investigate the biharmonic magnetic curves on three-dimensional α-Sasakian manifolds. We are also interested in studying biminimal curves in ξ-vertical hypersurfaces of three-dimensional α-Sasakian manifolds with constant curvature. We mainly calculate the signed curvature of the biminimal curves because it is well known that a plane curve is completely determined by its signed curvature. The notion of trans-Sasakian manifolds was introduced by J. A. Oubina [13] in 1985. A trans-Sasakian manifold with α as a constant and $\beta = 0$ is known as α-Sasakian manifold. The present paper is organized as follows:
After the introduction, we recollect some preliminaries in Section 2 for the subsequent use. Section 3 is devoted to the study of biharmonic magnetic curves on three-dimensional α-Sasakian manifolds. In Section 4, as in the paper [9], we calculate the signed curvature of biminimal curves on a ξ-vertical hypersurface of a three-dimensional α-Sasakian manifold.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M)$ [2]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if

$$[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0.$$

For an α-Sasakian manifold we know

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X), \tag{2.5}$$

for a constant α. Here ∇ is the Levi-Civita connection on M. From (2.5) it follows that

$$\nabla_X \xi = -\alpha\phi X. \tag{2.6}$$

An α-Sasakian manifold of dimension three is a trans-Sasakian manifold of dimension three with α as a constant and $\beta = 0$. The Riemannian curvature tensor R with respect to the Levi-Civita connection of a three-dimensional α-Sasakian manifold is given by [7]

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\xi\beta - 2\alpha^2)(g(Y,Z)X - g(X,Z)Y) \\ &\quad -g(Y,Z)(\frac{r}{2} - 3\alpha^2)\eta(X)\xi \\ &\quad +g(X,Z)(\frac{r}{2} - 3\alpha^2)\eta(Y)\xi \\ &\quad +(\frac{r}{2} - 3\alpha^2)\eta(Y)\eta(Z)X \\ &\quad +(\frac{r}{2} - 3\alpha^2)\eta(X)\eta(Z))Y, \end{aligned} \tag{2.7}$$

where S is the Ricci tensor of type $(0,2)$, and r is the scalar curvature of the manifold M with respect to the Levi-Civita connection. Again it is well known that the Gaussian curvature of a 2-dimensional space is the half of the scalar curvature of the space.

Let M be a $3-$dimensional Riemannian manifold. Let $\gamma : I \to M$, I being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot\gamma}$ denote the covariant differentiation along γ with respect to the Levi-Civita connection on M. It is said that γ is a Frenet curve if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\nabla_t t = 0$ (geodesic), $t = \dot\gamma$. Here, . denotes differentiation with respect to the arc parameter.

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(= \dot\gamma)$, n and a non-negative function k (curvature) along γ such that $\nabla_t t = kn$, $\nabla_t n = -kt$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(= \dot\gamma)$, n, b and two non-negative functions k(curvature) and τ(torsion) along γ such that

$$\nabla_t t = kn, \tag{2.8}$$

$$\nabla_t n = -kt + \tau b, \tag{2.9}$$

$$\nabla_t b = -\tau n. \tag{2.10}$$

With respect to the Levi-Civita connection, a Frenet curve of osculating order 3 for which k is a positive constant and $\tau = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if k and τ both are positive constants and the curve is called a generalized helix if $\frac{k}{\tau}$ is a constant.

3. Biharmonic magnetic curves in three-dimensional α-Sasakian manifolds

In [6] Cho and Lee studied biharmonic Legendre curves. Biharmonic curves have also been studied by the first author of the present paper [15]. Following [6], we give the following definition.

Definition 3.1. *A unit speed curve γ on a smooth manifold is called biharmonic with respect to Levi-Civita connection if it satisfies*

$$\nabla_t^3 t + R(\nabla_t t, t)t = 0, \tag{3.1}$$

where $t = \dot{\gamma}$.

Following [3], we give the following definition.

Definition 3.2. *A smooth curve γ on a three-dimensional α-sasakian manifold is called magnetic curve if it satisfies*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \phi\dot{\gamma}. \tag{3.2}$$

Let us consider a magnetic curve γ on a three-dimensional α-Sasakian manifold. So, we have $\nabla_{\dot{\gamma}}\dot{\gamma} = \phi\dot{\gamma}$. Differentiating it along $\dot{\gamma}$, we obtain

$$\nabla_{\dot{\gamma}}^2\dot{\gamma} = (\nabla_{\dot{\gamma}}\phi)\dot{\gamma} - \dot{\gamma} + \eta(\dot{\gamma})\xi.$$

By virtue of (2.5), the above equation yields

$$\nabla_{\dot{\gamma}}^2\dot{\gamma} = \alpha g(\dot{\gamma}, \dot{\gamma})\xi + \eta(\dot{\gamma})(\xi - \alpha\dot{\alpha}) - \dot{\gamma}. \tag{3.3}$$

Differentiating the above equation along γ and using (2.6) we have

$$\nabla_{\dot{\gamma}}^3\dot{\gamma} = -(\alpha^2 + 2\alpha\eta(\dot{\gamma}) + 1)\phi\dot{\gamma} + \nabla_{\dot{\gamma}}\eta(\dot{\gamma})\xi - \alpha\nabla_{\dot{\gamma}}\eta(\dot{\gamma})\dot{\gamma}. \tag{3.4}$$

In view of (2.7)

$$R(\phi\dot{\gamma}, \dot{\gamma}) = (\frac{r}{2} - 2\alpha^2 + \eta(\dot{\gamma})^2(3\alpha^2 - \frac{r}{2}))\phi\dot{\gamma}. \tag{3.5}$$

From (3.4) and (3.5) and considering the curve biharmonic, we obtain

$$\begin{aligned} \nabla_{\dot{\gamma}}^3\dot{\gamma} + R(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma})\dot{\gamma} &= (\frac{r}{2} - 2\alpha^2 + \eta(\dot{\gamma})^2(3\alpha^2 - \frac{r}{2}) \\ &- \alpha^2 - 2\alpha\eta(\dot{\gamma}) - 1)\phi\dot{\gamma} + \nabla_{\dot{\gamma}}\eta(\dot{\gamma})\xi \\ &- \alpha\nabla_{\dot{\gamma}}\eta(\dot{\gamma})\dot{\gamma}. \end{aligned} \tag{3.6}$$

If the curve is biharmonic, we have

$$\begin{aligned} &(\frac{r}{2} - 2\alpha^2 + \eta(\dot{\gamma})^2(3\alpha^2 - \frac{r}{2}) - \alpha^2 - 2\alpha\eta(\dot{\gamma}) - 1)\phi\dot{\gamma} \\ + \; &\nabla_{\dot{\gamma}}\eta(\dot{\gamma})\xi - \alpha\nabla_{\dot{\gamma}}\eta(\dot{\gamma})\dot{\gamma} \\ = \; &0. \end{aligned} \tag{3.7}$$

Taking inner product with ξ in both sides of the equation (3.7) we obtain $\nabla_{\dot\gamma}\eta(\dot\gamma) = 0$. Consequently, $\eta(\dot\gamma)$ = a constant. Taking inner product in both sides of (3.7) with respect to $\phi\dot\gamma$, it follows that

$$\frac{r}{2} - 2\alpha^2 + \eta(\dot\gamma)^2(3\alpha^2 - \frac{r}{2}) - \alpha^2 - 2\alpha\eta(\dot\gamma) - 1 = 0.$$

The above equation yields r = a constant. A three-dimensional trans-Sasakian Sasakian manifold with constant structure function is locally ϕ-symmetric [7] if and only if the scalar curvature of the manifold is constant. It is so for three-dimensional α-Sasakian manifolds. Thus we are in a position to state the following:

Theorem 3.1. *If a three-dimensional α-Sasakian manifold admits a biharmonic magnetic curve, then the manifold is locally ϕ-symmetric.*

4. Biminimal curves on a hypersurface of a three-dimensional α-Sasakian manifold.

Consider an α-Sasakian manifold of dimension three. If the manifold is of constant curvature then, the Riemannian curvature of the manifold is given by

$$R(X,Y)Z = \lambda(g(Y,Z)X - g(X,Z)Y), \tag{4.1}$$

for the tangent vector fields X, Y, Z of the manifold and a constant λ. Comparing (4.1) with (2.7) it follows that $\lambda = \frac{r}{2} - 2\alpha^2$. Hence (4.1) takes the form

$$R(X,Y)Z = (\frac{r}{2} - 2\alpha^2)(g(Y,Z)X - g(X,Z)Y), \tag{4.2}$$

for the tangent vector fields X, Y, Z of the manifold. The Ricci tensor of the manifold is given by

$$S(X,W) = 2(\frac{r}{2} - 2\alpha^2)g(X,W). \tag{4.3}$$

And the scalar curvature of the manifold is given by

$$r = -6\alpha^2. \tag{4.4}$$

Let S_1 be the Ricci tensor and r_1 be the scalar curvature of the ξ vertical hypersurface of the manifold. Then we have

$$S_1(X,W) = (\frac{r}{2} - 2\alpha^2)g(X,W), \tag{4.5}$$

where X, W are orthogonal to ξ. $r_1 = r - 4\alpha^2$. Using (4.4) in (4.3) we obtain

$$r_1 = 2\alpha^2. \tag{4.6}$$

Let G_1 be the Gaussian curvature of the ξ-vertical hypersurface of the manifold. Then $G_1 = \alpha^2$. Thus we obtain the following:

Proposition 4.1. *The Gaussian curvature of the ξ-vertical hypersurface of a three-dimensional α-Sasakian manifold with constant curvature is α^2.*

Let us consider an α-Sasakian manifold of dimension three, where α is a constant. Let M be a ξ vertical hypersurface of a three-dimensional α-Sasakian manifold. Now the Gaussian curvature of M is α^2. Consider a biminimal curve γ on M. Let $k(s)$ be the signed curvature of γ. Hence we have

$$k'' - k^3 + k\alpha^2 = 0. \tag{4.7}$$

Multiplying both sides of the above equation by $2\frac{dk}{ds}$ and integrating we get

$$(k')^2 - \frac{1}{2}k^4 + \alpha^2 k^2 = d, \tag{4.8}$$

where d is constant of integration. The above equation yields

$$\int \frac{dk}{\sqrt{k^4 - 2\alpha^2 k^2 + 2d}} = \int \frac{ds}{\sqrt{2}},$$

or,

$$\int \frac{dk}{\sqrt{k^4 - 2\alpha^2 k^2 + 2d}} = \frac{1}{\sqrt{2}}(s - s_0),$$

where s_0 is a constant. The left-hand side of the above equation is an elliptic integral. The integration process is given in the paper [9]. Following the process we can get the values of k for different values of the integration constants as the following.

(i) when $\alpha^4 - 2d < 0$,

$$k = \alpha\left(\frac{1 - \mathrm{cn}(\sqrt{2}\alpha(s - s_0)); \sin\theta}{1 + \mathrm{cn}(\sqrt{2}\alpha(s - s_0)); \sin\theta}\right)^{\frac{1}{2}}, \tag{4.9}$$

where θ is given by $\cos 2\theta = \frac{\alpha^4}{\sqrt{2d}}$.

(ii) when $\alpha^4 - 2d = 0$,

$$k = -\alpha \tanh\left(\frac{\alpha(s - s_0)}{\sqrt{2}}\right). \tag{4.10}$$

(iii) when $\alpha^4 - 2d > 0$ and $d > 0$

$$k = b\,\mathrm{sn}\left(\frac{a(s - s_0)}{\sqrt{2}}; \frac{b}{a}\right), \text{or}, k = \frac{a}{\mathrm{sn}\left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{b}{a}\right)}, \tag{4.11}$$

where a and b are given by $a^2 = \alpha^2 + \sqrt{\alpha^4 - 2d}$ and $b^2 = \alpha^2 - \sqrt{\alpha^4 - 2d}$.

(iv) when $\alpha^4 - 2d > 0$, $d < 0$

$$k = \frac{b}{\mathrm{cn}\left(\frac{\sqrt{a^2+b^2}}{\sqrt{2}}; \frac{a}{\sqrt{a^2+b^2}}\right)}. \tag{4.12}$$

In this case a and b are given by $a^2 = \sqrt{\alpha^4 - 2d} - \alpha^2$ and $b^2 = \sqrt{\alpha^4 - 2d} + \alpha^2$.

From the above values of k, we see that the signed curvature of biminimal curves on the ξ-vertical hypersurface of a three-dimensional α-Sasakian manifold depends on the choice of d and the value of α. In the above k are periodic but not of a constant period. Now we know that [1] if a curve in a 2-dimensional space is a periodic curve, then its signed curvature is so. Thus we are in a position to state the following:

Theorem 4.1. *No biminimal curve on a ξ-vertical hypersurface of a three-dimensional α-Sasakian manifold with constant curvature is a curve of constant period.*

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