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α_β-BOUNDED SETS AND α_β-TOPOLOGICALLY NILPOTENT ELEMENTS

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Abstract. The aim of this paper is to define and discuss the properties of α_β-boundedness, α_β-topological divisor of zero and α_β-topologically nilpotent elements.

Keywords: operations, α_β-open set, rings, α_β-boundedness, α_β-topologically nilpotent

1. Introduction

Given a topological space (G, τ), Njastad [16] was the first one to talk about α-open sets and proved that they form a topology finer than τ. Ibrahim [6] introduced a strong form of α-open sets called α_β-open sets, where β is an operation on the family of all α-open sets of G. Later Khalaf and Ibrahim [10, 11, 12, 13, 14] continued studying the properties of such open sets and also introduced $\alpha_{(\beta,\beta)}$-topological abelian groups, $\alpha_{(\beta,\beta)}$-topological rings and $\alpha_{(\beta,\gamma)}$-topological modules. In recent years, topological algebra was applied in both harmonic analysis and complex fractional calculus [7, 8, 4]. One of the central definitions of the present paper is that one of α_β-bounded set for an $\alpha_{(\beta,\beta)}$-topological ring and for an $\alpha_{(\beta,\gamma)}$-topological module. Several properties are proven, like hereditariness, stability by topological closure, stability by taking unions and sums. Hereditariness and stability by unions lead me to think to the definition of boundedness proposed by S. T. Hu, [5]. We recall some of the well known definitions and results which can be found in most of text books of abstract algebra we refer to [1], [2], [3], [9], [15] and [17].

2. Preliminaries

Let A be a subset of a topological space (G, τ). We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$, respectively. A subset A of a topological space (G, τ) is called α-open [16] if $A \subseteq Int(Cl(Int(A)))$. By $\alpha O(G, \tau)$, we denote

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the family of all α-open sets of G. An operation $\beta : \alpha O(G,\tau) \to P(G)$ [6] is a mapping from $\alpha O(G,\tau)$ in to power set $P(G)$ of G satisfying the condition, $V \subseteq V^\beta$ for each $V \in \alpha O(G,\tau)$, where V^β denotes the value of β at V. We call the mapping β an operation on $\alpha O(G,\tau)$. A subset A of G is called an α_β-open set [6] if for each point $x \in A$, there exists an α-open set U of G containing x such that $U^\beta \subseteq A$. The complement of an α_β-open set is said to be α_β-closed. We denote the set of all α_β-open sets of (G,τ) by $\alpha O(G,\tau)_\beta$. The α_β-closure [6] of a subset A of G with an operation β on $\alpha O(G)$ is denoted by $\alpha_\beta Cl(A)$ and is defined to be the intersection of all α_β-closed sets containing A. An operation β on $\alpha O(G,\tau)$ is said to be α-regular if for every α-open sets U and V of each $x \in G$, there exists an α-open set W of x such that $W^\beta \subseteq U^\beta \cap V^\beta$.

Definition 2.1. [10] Let (G,τ) be a topological space and $x \in G$, then a subset N of G is said to be α_β-neighbourhood of x, if there exists an α_β-open set U in G such that $x \in U \subseteq N$.

Definition 2.2. [6] A topological space (G,τ) with an operation β on $\alpha O(G)$ is said to be $\alpha_\beta T_2$ if for any two distinct points $x, y \in G$, there exist two α_β-open sets U and V containing x and y, respectively, such that $U \cap V = \phi$.

Definition 2.3. [11] A function $f : (G,\tau) \to (G^{'},\tau^{'})$ is said to be $\alpha_{(\beta,\beta')}$-open if for any α_β-open set A of (G,τ), $f(A)$ is $\alpha_{\beta'}$-open in $(G^{'},\tau^{'})$.

Definition 2.4. [6] A mapping $f : (G,\tau) \to (G^{'},\tau^{'})$ is said to be $\alpha_{(\beta,\beta')}$-continuous if for each x of G and each $\alpha_{\beta'}$-open set V containing $f(x)$, there exists an α_β-open set U such that $x \in U$ and $f(U) \subseteq V$.

Corollary 2.1. [12] *A function $f : G \to G^{'}$ is $\alpha_{(\beta,\beta')}$-continuous if and only if $f^{-1}(V)$ is α_β-open in G, for every $\alpha_{\beta'}$-open set V in $G^{'}$.*

Definition 2.5. [13] Let $(G,+)$ be abelian group and τ be a topology on G. A triple $(G,+,\tau)$ is said to be an $\alpha_{(\beta,\beta)}$-topological group if the following conditions are satisfied:

1. For any two elements $a, b \in G$ and $U \in \alpha O(G,\tau)_\beta$ such that $a + b \in U$, there exist $V, W \in \alpha O(G,\tau)_\beta$ with $a \in V$, $b \in W$ and $V + W \subseteq U$.
2. For any element $a \in G$ and $U \in \alpha O(G,\tau)_\beta$ such that $-a \in U$, there exists $V \in \alpha O(G,\tau)_\beta$ with $a \in V$ and $-V \subseteq U$.

Definition 2.6. [14] Let $(R,+,\cdot)$ be a ring and (R,τ) be a topological space. Then, $(R,+,\cdot,\tau)$ is called an $\alpha_{(\beta,\beta)}$-topological ring if the following conditions are satisfied:

1. $(R,+,\tau)$ is $\alpha_{(\beta,\beta)}$-topological group.

2. For each elements $a, b \in R$ and $U \in \alpha O(R, \tau)_\beta$ such that $a \cdot b \in U$, there exist $V, W \in \alpha O(R, \tau)_\beta$ with $a \in V$, $b \in W$ and $V \cdot W \subseteq U$.

Definition 2.7. [14] Let $(R, +, \cdot, \tau)$ be an $\alpha_{(\beta,\beta)}$-topological ring. A left R-module M is called an $\alpha_{(\beta,\gamma)}$-topological left R-module if on M is specified a topology such that M is an $\alpha_{(\gamma,\gamma)}$-topological abelian group and the following condition is satisfied:

For any $r \in R$ and $m \in M$ and arbitrary α_γ-open set U containing the element $r \cdot m$ in M, there exist an α_β-open set V containing the element r in R and an α_γ-open set W the element m in M such that $V \cdot W \subseteq U$.

Proposition 2.1. *[13] Let a family B_0 of subsets of an $\alpha_{(\beta,\beta)}$-topological abelian group G be a basis of α_β-neighborhoods of zero in G and β be an α-regular operation on $\alpha O(G)$. Then, the following conditions are satisfied:*

1. $0 \in \bigcap_{V \in B_0} V$.
2. *For any subsets U and V from B_0, there exists a subset $W \in B_0$ such that $W \subseteq U \cap V$.*
3. *For any subset $U \in B_0$, there exists a subset $V \in B_0$ such that $V + V \subseteq U$.*
4. *For any subset $U \in B_0$, there exists a subset $V \in B_0$ such that $-V \subseteq U$.*

Besides, if $a \in G$, then $B_a = \{a + V | V \in B_0\}$ is a basis of α_β-neighborhoods of the element a.

Proposition 2.2. [14] *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, B_0 be a basis of α_γ-neighborhoods of zero of an $\alpha_{(\beta,\gamma)}$-topological R-module M and γ be an α-regular operation on $\alpha O(M)$. Then conditions* (1) *to* (4) *of Proposition 2.1, are satisfied together with the following conditions:*

1. *For any subset $U \in B_0$, there exists a subset $V \in B_0$ and an α_β-neighborhood W of zero in R such that $W \cdot V \subseteq U$.*
2. *For any subset $U \in B_0$ and any element $r \in R$, there exists a subset $V \in B_0$ such that $r \cdot V \subseteq U$.*
3. *For any subset $U \in B_0$ and any element $a \in M$, there exists an α_β-neighborhood W of zero in R such that $W \cdot a \subseteq U$.*

Corollary 2.2. [13] *Let U and V be α_β-neighborhoods of zero of $\alpha_{(\beta,\beta)}$-topological abelian group G such that $V + V \subseteq U$, then $\alpha_\beta Cl(V) \subseteq U$.*

Proposition 2.3. [13] *Let G be $\alpha_{(\beta,\beta)}$-topological abelian group, G' be $\alpha_{(\beta',\beta')}$-topological abelian group and $f : G \to G'$ be a homomorphic mapping of G to G'. Then:*

1. *f is an $\alpha_{(\beta,\beta')}$-continuous if and only if $f^{-1}(U')$ is an α_β-neighborhood of zero in G for any $\alpha_{\beta'}$-neighborhood U' of zero in G'.*

2. *f is an $\alpha_{(\beta,\beta')}$-open if and only if $f(U)$ is an $\alpha_{\beta'}$-neighborhood of zero in G' for any α_β-neighborhood U of zero in G.*

Corollary 2.3. [14] *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, $a \in R$ and β an α-regular operation on $\alpha O(R)$. Let also $B_0(R)$ be a basis of α_β-neighborhoods of zero in R. Then, the element a has a basis of α_β-neighborhoods consisting of α_β-closed neighborhoods.*

Proposition 2.4. [14] *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, M an $\alpha_{(\beta,\gamma)}$-topological R-module, Q a subset in R and B a subset in M. Then $\alpha_\gamma Cl(Q \cdot B) \supseteq \alpha_\beta Cl(Q) \cdot \alpha_\gamma Cl(B)$.*

3. α_β-Bounded Sets

Definition 3.1. Let R be an $\alpha_{(\beta,\beta)}$-topological ring, M be an $\alpha_{(\beta,\gamma)}$-topological R-module. A subset $S \subseteq M$ is called α_β-bounded if for any α_γ-neighborhood U of zero in M, there exists an α_β-neighborhood V of zero in R such that $V \cdot S \subseteq U$. An $\alpha_{(\beta,\gamma)}$-topological R-module M is called α_β-bounded if M is an α_β-bounded subset of the module M.

Definition 3.2. A subset S of the $\alpha_{(\beta,\beta)}$-topological ring R is called α_β-bounded from left (right) if S is an α_β-bounded subset of the $\alpha_{(\beta,\beta)}$-topological left (right) R-module $R(+)$, that is, for any α_β-neighborhood U of zero in R, there exists an α_β-neighborhood V of zero in R such that $V \cdot S \subseteq U$ (respectively, $S \cdot V \subseteq U$). A subset S of the $\alpha_{(\beta,\beta)}$-topological ring R, α_β-bounded from left and from right, is called α_β-bounded. An $\alpha_{(\beta,\beta)}$-topological ring R is called α_β-bounded from left (α_β-bounded from right, α_β-bounded) if R is an α_β-bounded from left (respectively, α_β-bounded from right, α_β-bounded) subset of the ring R.

Corollary 3.1. *Any finite subset Q of an $\alpha_{(\beta,\beta)}$-topological ring R is α_β-bounded.*

Proof. Considering R as left and right $\alpha_{(\beta,\beta)}$-topological R-modules, we get that Q is α_β-bounded from left and from right in R. □

Proposition 3.1. *Let S be a subring of an $\alpha_{(\beta,\beta)}$-topological ring R and A be an α_β-bounded subset of an $\alpha_{(\beta,\gamma)}$-topological R-module M. Let N be some S-submodule in M containing A, then A is an α_β-bounded subset of S-module N.*

Proof. Let U be an α_γ-neighborhood of zero in N. Then $U = V \cap N$ for a certain α_γ-neighborhood V of zero in M. Since A is an α_β-bounded subset in M, then there exists an α_β-neighborhood W of zero in R such that $W \cdot A \subseteq V$. Then $W \cap S$ is an α_β-neighborhood of zero in S and $(W \cap S) \cdot A \subseteq W \cdot A \cap S \cdot A \subseteq V \cap N = U$. □

Corollary 3.2. *Let A be an α_β-bounded from left (α_β-bounded from right, α_β-bounded) subset of an $\alpha_{(\beta,\beta)}$-topological ring R and Q be a subring of R, which contains A. Then A is an α_β-bounded from left (α_β-bounded from right, α_β-bounded) subset of the ring Q.*

Proof. The statement follows from Proposition 3.1, if we consider R as a left or right $\alpha_{(\beta,\beta)}$-topological R-module and Q as a left or right its topological Q-submodule. □

Remark 3.1. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, S be an α_β-bounded subset of the $\alpha_{(\beta,\gamma)}$-topological R-module M. If $S_1 \subseteq S$, then S_1 is an α_β-bounded subset of the $\alpha_{(\beta,\gamma)}$-topological R-module M.*

Remark 3.2. *Let S be an α_β-bounded from left (α_β-bounded from right, α_β-bounded) subset of an $\alpha_{(\beta,\beta)}$-topological ring R and $S_1 \subseteq S$. Then S_1 is an α_β-bounded from left (α_β-bounded from right, α_β-bounded) subset of the $\alpha_{(\beta,\beta)}$-topological ring R.*

Proposition 3.2. *Let M be an $\alpha_{(\beta,\gamma)}$-topological module over an $\alpha_{(\beta,\beta)}$-topological ring R, B be α_β-bounded subsets of M and γ be an α-regular operation on $\alpha O(M)$. Then $\alpha_\gamma Cl(B)$ is α_β-bounded.*

Proof. Let U be an α_γ-neighborhood of zero in M, then by Proposition 2.2, there is an α_γ-neighborhood V of zero in M such that $V + V \subseteq U$, and so by Corollary 2.2, we have $\alpha_\gamma Cl(V) \subseteq U$ and $\alpha_\gamma Cl(V)$ be an α_γ-closed α_γ-neighborhood of zero in M. Since B is an α_β-bounded subset of R-module M, then there exists an α_β-neighborhood W of zero in R such that $W \cdot B \subseteq V$. Then,

$$W \cdot \alpha_\gamma Cl(B) \subseteq \alpha_\beta Cl(W) \cdot \alpha_\gamma Cl(B) \subseteq \alpha_\gamma Cl(W \cdot B) \subseteq \alpha_\gamma Cl(V) \subseteq U,$$

that is, $\alpha_\gamma Cl(B)$ is α_β-bounded. □

Corollary 3.3. *The α_β-closure of an α_β-bounded from left (α_β-bounded from right, α_β-bounded) subset of an $\alpha_{(\beta,\beta)}$-topological ring is a subset α_β-bounded from left (α_β-bounded from right, α_β-bounded), where β is α-regular operation on $\alpha O(R)$.*

Proof. The proof results from Proposition 3.2. □

Proposition 3.3. *Let Q be an α_β-bounded from left subset of an $\alpha_{(\beta,\beta)}$-topological ring R and S be an α_β-bounded subset of an $\alpha_{(\beta,\gamma)}$-topological R-module M, then $Q \cdot S$ is an α_β-bounded subset of the R-module M.*

Proof. Let U be an α_γ-neighborhood of zero in M and V be an α_β-neighborhood of zero in R such that $V \cdot S \subseteq U$. We can choose an α_β-neighborhood W of zero in R such that $W \cdot Q \subseteq V$. Then,

$$W \cdot (Q \cdot S) = (W \cdot Q) \cdot S \subseteq V \cdot S \subseteq U,$$

that is, $Q \cdot S$ is an α_β-bounded subset of the $\alpha_{(\beta,\gamma)}$-topological R-module M. $\square$

Proposition 3.4. *Let M be an $\alpha_{(\beta,\gamma)}$-topological module over an $\alpha_{(\beta,\beta)}$-topological ring R, B_1 and B_2 be α_β-bounded subsets of M, β be an α-regular operation on $\alpha O(R)$ and γ be an α-regular operation on $\alpha O(M)$. Then*

1. *$B_1 + B_2$ is α_β-bounded.*
2. *$B_1 \cup B_2$ is α_β-bounded.*

Proof. Let W be an α_γ-neighborhood of zero such that $W + W \subseteq U$, where U is α_γ-neighborhood of zero in M. Let V_1 and V_2 be α_β-neighborhoods of zero in R such that $V_1 \cdot B_1 \subseteq W$ and $V_2 \cdot B_2 \subseteq W$. Then

1. $(V_1 \cap V_2) \cdot (B_1 \cup B_2) \subseteq V_1 \cdot B_1 \cup V_2 \cdot B_2 \subseteq W \subseteq U$.
2. $(V_1 \cap V_2) \cdot (B_1 + B_2) \subseteq V_1 \cdot B_1 + V_2 \cdot B_2 \subseteq W + W \subseteq U$.

Consequently, the union or the sum of finitely many α_β-bounded subsets of an $\alpha_{(\beta,\gamma)}$-topological module is α_β-bounded. $\square$

Remark 3.3. *If B and C are left (right) α_β-bounded subsets of an $\alpha_{(\beta,\beta)}$-topological ring R and β be an α-regular operation on $\alpha O(R)$, then $B \cup C$ and $B + C$ are left (right) α_β-bounded.*

Corollary 3.4. *Let each of the subsets Q_i for $i = 1, 2, ..., n$ of an $\alpha_{(\beta,\beta)}$-topological ring R be α_β-bounded from left (α_β-bounded from right, α_β-bounded), then the subset $Q_1 \cdot Q_2 \cdot ... \cdot Q_n$ is α_β-bounded from left (α_β-bounded from right, α_β-bounded).*

Proof. The proof is clear. $\square$

Proposition 3.5. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, M be $\alpha_{(\beta,\gamma)}$-topological R-module and M' be $\alpha_{(\beta,\gamma')}$-topological R-module. Let f be an $\alpha_{(\gamma,\gamma')}$-continuous homomorphism from the module M to the module M' and a subset N is α_β-bounded in M. Then the subset $f(N)$ is α_β-bounded in M'.*

Proof. Let U' be an $\alpha_{\gamma'}$-neighborhood of zero in the R-module M'. Due to Proposition 2.3, $f^{-1}(U')$ is an α_γ-neighborhood of zero in the R-module M. Since N is an α_β-bounded subset, then there exists an α_β-neighborhood V of zero in the ring R such that $V \cdot N \subseteq f^{-1}(U')$. Then,

$$V \cdot f(N) = f(V \cdot N) \subseteq f(f^{-1}(U')) \subseteq U',$$

that is, $f(N)$ is an α_β-bounded subset in M'. $\square$

Proposition 3.6. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring and $R^{'}$ be an $\alpha_{(\beta^{'},\beta^{'})}$-topological ring, $f : R \to R^{'}$ be $\alpha_{(\beta,\beta^{'})}$-continuous and $\alpha_{(\beta,\beta^{'})}$-open homomorphism from the ring R to the ring $R^{'}$. Let a subset S be α_β-bounded from left (α_β-bounded from right, α_β-bounded) in the ring R, then the subset $f(S)$ is α_β-bounded from left (α_β-bounded from right, α_β-bounded) in the ring $R^{'}$.*

Proof. Let $U^{'}$ be an $\alpha_{\beta^{'}}$-neighborhood of zero in the ring $R^{'}$. Then, due to Proposition 2.3, $f^{-1}(U^{'})$ is an α_β-neighborhood of zero in the ring R. Since the subset S of the ring R is α_β-bounded from left, then there exists an α_β-neighborhood V of zero in R such that $V \cdot S \subseteq f^{-1}(U^{'})$. Due to Proposition 2.3, $f(V)$ is an $\alpha_{\beta^{'}}$-neighborhood of zero in the ring $R^{'}$. Then,

$$f(V) \cdot f(S) = f(V \cdot S) \subseteq f(f^{-1}(U^{'})) \subseteq U^{'},$$

that is, the subset $f(S)$ is α_β-bounded from left in $R^{'}$.

When the subset S of the ring R is α_β-bounded from right or α_β-bounded, the proof is analogous. □

Definition 3.3. An element a of an $\alpha_{(\beta,\beta)}$-topological ring R is called a left (right) α_β-topological divisor of zero if there exists a subset $S \subseteq R$ such that:

1. $0 \notin \alpha_\beta Cl(S)$.
2. $0 \in \alpha_\beta Cl(a \cdot S)$ (respectively, $0 \in \alpha_\beta Cl(S \cdot a)$).

An element a is called an α_β-topological divisor of zero if it is a left and right α_β-topological divisor of zero, that is, there exist subsets $S_1 \subseteq R$ and $S_2 \subseteq R$ such that $0 \notin \alpha_\beta Cl(S_1)$ and $0 \notin \alpha_\beta Cl(S_2)$, but as well $0 \in \alpha_\beta Cl(a \cdot S_1)$ and $0 \in \alpha_\beta Cl(S_2 \cdot a)$.

Remark 3.4. *In an $\alpha_\beta T_2$ $\alpha_{(\beta,\beta)}$-topological ring R any left (right) divisor of zero is a left (right) α_β-topological divisor of zero.*

Indeed, if a is a left divisor of zero in R and $0 \neq b$ is such that $a \cdot b = 0$, then, the subset $\{b\}$ is α_β-closed in R and $0 \notin \{b\} = \alpha_\beta Cl(\{b\})$. It is evident that $0 \in \alpha_\beta Cl(a \cdot \{b\}) = \{0\}$, that is, a is a left α_β-topological divisor of zero in R.

The following example shows that the condition that R is $\alpha_\beta T_2$ is necessary for the above remark.

Example 3.1. Consider the ring Z_4. Let τ be the discrete topology on Z_4. For each $A \in \alpha O(Z_4, \tau)$, we define β on $\alpha O(Z_4, \tau)$ by $A^\beta = Z_4$. Since Z_4 is not $\alpha_\beta T_2$, so the element 2 is divisor of zero in Z_4, but it is not α_β-topological divisor of zero because $0 \in \alpha_\beta Cl(S) = Z_4$ for any subset S of Z_4.

Proposition 3.7. *Let a be a left (right) α_β-topological divisor of zero in an $\alpha_{(\beta,\beta)}$-topological ring R. Then for any $b \in R$ the element $b \cdot a$ is a left α_β-topological divisor of zero in R (respectively, the element $a \cdot b$ is a right α_β-topological divisor of zero in R).*

Proof. Let $S \subseteq R$, $0 \notin \alpha_\beta Cl(S)$ and $0 \in \alpha_\beta Cl(a \cdot S)$. Then $0 \in b \cdot \alpha_\beta Cl(a \cdot S) \subseteq \alpha_\beta Cl(b \cdot a \cdot S)$, that is, $b \cdot a$ is a left α_β-topological divisor of zero in R. Analogously is considered the case when a is a right α_β-topological divisor of zero in R. □

Proposition 3.8. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring and $a, b \in R$. If $a \cdot b$ is a left (right) α_β-topological divisor of zero in R, then either a or b is a left (right) α_β-topological divisor of zero.*

Proof. Let $a \cdot b$ be a left α_β-topological divisor of zero in R and $S \subseteq R$ be such that $0 \notin \alpha_\beta Cl(S)$, and $0 \in \alpha_\beta Cl((a \cdot b) \cdot S)$. If $0 \in \alpha_\beta Cl(b \cdot S)$, then b is a left α_β-topological divisor of zero. If $0 \notin \alpha_\beta Cl(b \cdot S)$, then, taking into account that $0 \in \alpha_\beta Cl((a \cdot b) \cdot S) = \alpha_\beta Cl(a \cdot (b \cdot S))$, we get that a is a left α_β-topological divisor of zero. Analogously is considered the case when $a \cdot b$ is a right α_β-topological divisor of zero in R. □

4. α_β-Topologically Nilpotent Elements

Definition 4.1. A subset S of an $\alpha_{(\beta,\beta)}$-topological ring R is called α_β-topologically nilpotent if for any α_β-neighborhood U of zero in R there exists a natural number n_0 such that $S^{(n)} \subseteq U$ for all $n \geq n_0$. An element $a \in R$ is called α_β-topologically nilpotent if the one-element set $\{a\}$ of the ring R is α_β-topologically nilpotent, that is, if for any α_β-neighborhood U of zero in R there exists a natural number n_0 such that $a^n \in U$ for all $n \geq n_0$.

Remark 4.1. *Since for any subsets S_1 and S of a ring R from the inclusion $S_1 \subseteq S$ follows that $S_1^{(n)} \subseteq S^{(n)}$ for any $n \in \mathbb{N}$, then any subset of an α_β-topologically nilpotent subset of an $\alpha_{(\beta,\beta)}$-topological ring is an α_β-topologically nilpotent subset, too. In particular, any element of an α_β-topologically nilpotent subset is α_β-topologically nilpotent.*

Proposition 4.1. *Let K be a skew field endowed with $\alpha_\beta T_2$ ring $\alpha_{(\beta,\beta)}$-topology, β be an α-regular operation on $\alpha O(K)$ and a be a non-zero α_β-topologically nilpotent element in K. Then the element a^{-1} is not α_β-topologically nilpotent.*

Proof. Assume the contrary, that is, that a^{-1} is an α_β-topologically nilpotent element. Since the skew field K is $\alpha_\beta T_2$, there exists an α_β-neighborhood U of zero in K such that $1 \notin U$. We can choose an α_β-neighborhood V of zero in K such that $V \cdot V \subseteq U$. It is clear that $1 \notin V$. Since the elements a and a^{-1} are α_β-topologically nilpotent, then there exist natural numbers n_1 and n_2 such that $a^n \in V$ for $n \geq n_1$ and $a^{-n} \in V$ for $n \geq n_2$. Putting $n_0 = max\{n_1, n_2\}$, we get that $1 = a^{n_0} \cdot a^{-n_0} \in V \cdot V \subseteq U$, which contradicts the choice of the α_β-neighborhood U. □

Proposition 4.2. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring with the unitary element and a be an invertible element in R. Then, the element $x \in R$ is α_β-topologically nilpotent if and only if the element $a \cdot x \cdot a^{-1}$ is α_β-topologically nilpotent.*

Proof. Let x be an α_β-topologically nilpotent element and U be an α_β-neighborhood of zero in R. We can choose an α_β-neighborhood V of zero in R such that $a \cdot V \cdot a^{-1} \subseteq U$. Let n_0 be a natural number such that $x^n \in V$ for $n \geq n_0$. Then $(a \cdot x \cdot a^{-1})^n = a \cdot x^n \cdot a^{-1} \in a \cdot V \cdot a^{-1} \subseteq U$ for $n \geq n_0$, that is, $a \cdot x \cdot a^{-1}$ is an α_β-topologically nilpotent element.

Conversely, let $a \cdot x \cdot a^{-1}$ be an α_β-topologically nilpotent element. Then, as it was shown above, the element $x = a^{-1} \cdot (a \cdot x \cdot a^{-1}) \cdot a = a^{-1} \cdot (a \cdot x \cdot a^{-1}) \cdot (a^{-1})^{-1}$ is α_β-topologically nilpotent. □

Proposition 4.3. *Let S be an α_β-topologically nilpotent subset of an $\alpha_{(\beta,\beta)}$-topological ring R and β an α-regular operation on $\alpha O(R)$. Then, the subset $\alpha_\beta Cl(S)$ is α_β-topologically nilpotent.*

Proof. Let S be an α_β-topologically nilpotent subset and U be an α_β-neighborhood of zero in R. Due to Corollary 2.3, there exists an α_β-closed α_β-neighborhood V of zero in R such that $V \subseteq U$. Let n_0 be a natural number such that $S^{(n)} \subseteq V$ for all $n \geq n_0$. Then

$$(\alpha_\beta Cl(S))^{(n)} \subseteq \alpha_\beta Cl(S^{(n)}) \subseteq \alpha_\beta Cl(V) = V \subseteq U \text{ (see Proposition 2.4)},$$

hence, $\alpha_\beta Cl(S)$ is an α_β-topologically nilpotent subset. □

Proposition 4.4. *Let f be an $\alpha_{(\beta,\beta)}$-continuous homomorphism of an $\alpha_{(\beta,\beta)}$-topological ring R onto an $\alpha_{(\beta',\beta')}$-topological ring R' and let S be an α_β-topologically nilpotent subset of R. Then $f(S)$ is an $\alpha_{\beta'}$-topologically nilpotent subset of the ring R'.*

Proof. Let U' be an $\alpha_{\beta'}$-neighborhood of zero in R', then $f^{-1}(U')$ is an α_β-neighborhood of zero in R. There exists a natural number n_0 such that $S^{(n)} \subseteq f^{-1}(U')$ for all $n \geq n_0$. Then $(f(S))^{(n)} = f(S^{(n)}) \subseteq f(f^{-1}(U')) = U'$ for all $n \geq n_0$, that is, the subset $f(S)$ is $\alpha_{\beta'}$-topologically nilpotent in R'. □

Proposition 4.5. *Let S be an α_β-bounded from left (right) subset of an $\alpha_{(\beta,\beta)}$-topological ring R, then the following statements are equivalent:*

1. *S is α_β-topologically nilpotent subset.*
2. *$S^{(k)}$ is α_β-topologically nilpotent subset for any natural number k.*
3. *There exists a natural number k_0 such that $S^{(k_0)}$ is an α_β-topologically nilpotent subset.*

Proof. It is evident that (1) ⇒ (2) ⇒ (3).

Let us show that (3) ⇒ (1). Let $S^{(k_0)}$ be an α_β-topologically nilpotent subset in R and U be an α_β-neighborhood of zero in R. Due to Corollary 3.4, the subsets

$S^{(2)}, S^{(3)}, ..., S^{(k_0-1)}$ are α_β-bounded from left. Therefore, we can choose an α_β-neighborhood V of zero in R such that $V \cdot S^{(i)} \subseteq U$ for $i = 1, 2, ..., k_0 - 1$. Since the subset $S^{(k_0)}$ is α_β-topologically nilpotent, there exists a natural number n_0 such that $(S^{(k_0)})^{(n)} \subseteq V$ for all $n \geq n_0$. Let $m \geq n_0 \cdot k_0$. Then $m = k_0 \cdot q + r$, where $q \geq n_0$ and $0 \leq r < k_0$. Thus,

$$S^{(m)} = S^{(k_0 \cdot q + r)} = (S^{(k_0)})^{(q)} \cdot S^{(r)} \subseteq V \cdot S^{(r)} \subseteq U,$$

that is, S is an α_β-topologically nilpotent subset. □

Corollary 4.1. *Let a be an element of an $\alpha_{(\beta,\beta)}$-topological ring R. Then the following conditions are equivalent:*

1. *a is an α_β-topologically nilpotent element.*
2. *a^k is an α_β-topologically nilpotent element for any natural number k.*
3. *a^{k_0} is an α_β-topologically nilpotent element for a certain natural number k_0.*

Proof. The statement results from Proposition 4.5 and from the α_β-boundedness of the one-element subset $\{a\}$ (see Corollary 3.1). □

Proposition 4.6. *Let T be the subset of all α_β-topologically nilpotent elements of an $\alpha_{(\beta,\beta)}$-topological ring R and β an α-regular operation on $\alpha O(R)$. Then, the following statements are equivalent:*

1. *T is an α_β-open subset.*
2. *There exists an α_β-open α_β-neighborhood U of zero in R consisting of α_β-topologically nilpotent elements.*

Proof. It is evident that (1) ⇒ (2).

Let us show that (2) ⇒ (1). Let $t \in T$ and n_0 be a natural number such that $t^{n_0} \in U$, where U is α_β-open α_β-neighborhood of zero in R. We can choose an α_β-neighborhood V of the element t such that

$$V^{(n_0)} = \underbrace{V \cdot V \cdot ... \cdot V}_{n_0-times} \subseteq U.$$

Then v^{n_0} is an α_β-topologically nilpotent element, for any $v \in V$. Due to Corollary 4.1, any element from V is α_β-topologically nilpotent, that is, $V \subseteq T$, which means that the subset T is α_β-open. □

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