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BEST PROXIMITY POINT FOR GENERALIZED (α, ϕ, ψ)-PROXIMAL CONTRACTIONS ON SEMI-METRIC SPACES

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Abstract. In this paper, we introduce a class of generalized (α, ϕ, ψ)-proximal contraction non-self-maps in semi-metric spaces. For such maps, we provide sufficient conditions ensuring the existence and uniqueness of best proximity points by using the concept of α-proximal admissible mapping. As applications, we infer the best proximity point and fixed point results for mappings in partially ordered semi-metric spaces. The presented results generalize and improve various known results from the best proximity and fixed point theory.

Keywords: semi-metric space; best proximity point; fixed point; generalized (α, ϕ, ψ)-proximal maps

1. Introduction and preliminaries

Semi-metric spaces were considered by several authors as Fréchet, Menger [22], Chittenden [10] and Wilson [29] as a generalization of metric spaces. Since then, some fixed point results for this class of spaces have been investigated in [11]-[26]. On the other hand, the existence and approximation of best proximity points is an interesting topic in the optimization theory [13, 27]

Definition 1.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is said to be a symmetric on X if for any $x, y \in X$, the following conditions hold:

$(W1)$ $d(x, y) = 0$ if and only if $x = y$;

$(W2)$ $d(x, y) = d(y, x)$.

The pair (X, d) is then called a symmetric space.

Note that many topological notions in symmetric spaces can be defined similar to those in metric spaces. Recall that in each symmetric space (X, d) one can

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introduce a topology τ_d by defining the family of open sets as follows: a nonempty set $A \subseteq X$ is open (i.e. $A \in \tau_d$) if and only if for each $x \in A$, there is $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq A$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

Definition 1.2. [14] A symmetric d on X is said to be a semi-metric if for each $x \in X$ and $\varepsilon > 0$, the open ball $B_d(x, \varepsilon)$ is a neighborhood of x in the topology τ_d.

Proposition 1.1. *[3] Let (X, d) be a symmetric space. Then (X, d) is a semi-metric space if and only if the following conditions hold:*

(1) (X, τ_d) is first countable;

(2) For any sequence $\{x_n\}$ in X, $d(x_n, x) \to 0$ is equivalent to $x_n \to x$ in the topology τ_d.

Definition 1.3. [16, 14] Let (X, d) be a symmetric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is d-Cauchy sequence if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Furthermore, (X, d) is said to be d-Cauchy complete if every d-Cauchy sequence converges to some $x \in X$ in τ_d.

Definition 1.4. Let (X, d) be a symmetric space and $\{x_n\}$ be a sequence in X. We say that (X, d) satisfies the Fatou property if for all $x, y \in X$, we have

$$\lim_{n\to\infty} d(x_n, x) = 0 \Rightarrow d(x, y) \leq \liminf_{n\to\infty} d(x_n, y).$$

We introduce the concept of (W_C) property we will need in the sequel.

Definition 1.5. Let (X, d) be a symmetric space. We say that (X, d) satisfies the property (W_C) if for all sequences $\{x_n\}$, $\{y_n\}$ in X and all $x, y \in X$ such that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(y_n, y) = 0$, one has

$$d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n).$$

Remark 1.1. 1. If (X, d) be a symmetric space satisfying the property (W_C), then it is also satisfying the Fatou property.
2. Each metric space satisfies the property (W_C).

For A and B two nonempty subsets of a symmetric space (X, d), define

$$\begin{aligned} d(A, B) &= \inf\{d(a, b) : \ a \in A,\ b \in B\}, \\ A_0 &= \{a \in A : d(a, b) = d(A, B),\ \text{for some}\, b \in B\}, \\ B_0 &= \{b \in B : d(a, b) = d(A, B),\ \text{for some}\, a \in A\}. \end{aligned}$$

As in [17], we introduce in the setting of symmetric spaces the following.

Definition 1.6. Let A and B be nonempty subsets of a symmetric space (X, d) and $\alpha : X \times X \to [0, \infty)$. A mapping $T : A \to B$ is named α-proximal admissible if

$$\begin{cases} \alpha(x, y) \geq 1 \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{cases} \Rightarrow \alpha(u, v) \geq 1.$$

for all $x, y, u, v \in A$.

Clearly, if $d(A, B) = 0$, T is α-proximal admissible implies that T is α-admissible [28].

We introduce the following notion.

Definition 1.7. Let A and B be nonempty subsets of a symmetric space (X, d) and $\alpha : X \times X \to [0, \infty)$. A mapping $T : A \to B$ is named triangular α-proximal admissible if

(T_1) T is α-proximal admissible,

(T_2) $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1,\ x, y, z \in A$.

Definition 1.8. Let A and B be nonempty subsets of a symmetric space (X, d), $\alpha : X \times X \to [0, \infty)$ and $T : A \to B$ be non-self-map. We say that A_0 is α-proximal T-orbitally d-Cauchy complete if every d-Cauchy sequence $\{x_n\}$ in A_0 with $\alpha(x_n, x_{n+1}) \geq 1$ and $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \geq 0$, converges to some element in A_0 in the topology τ_d.

On the other hand, the definition of the best proximity point is as follows.

Definition 1.9. Let (X, d) be a symmetric space. Consider A and B two nonempty subsets of X. An element $a \in X$ is said to be a best proximity point of $T : A \to B$ if

$$d(a, Ta) = d(A, B).$$

It is clear that a fixed point coincides with a best proximity point if $d(A, B) = 0$. For some results on above concept, see for example [18]-[30].

Denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying
(ψ_1) ψ is nondecreasing;
(ψ_2) $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$, where ψ^n is the n−th iterate of ψ.

Also, denote by Φ the set of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying
(ϕ_1) ϕ is nondecreasing;
(ϕ_2) $\phi^{-1}(\{0\}) = \{0\}$ and $\lim_{x\to 0^+} \phi(x) = 0$.

Lemma 1.1. *If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$, ψ is continuous at 0 and $\psi(0) = 0$.*

Lemma 1.2. *Let $\phi \in \Phi$ and $\{a_n\} \subseteq [0,\infty)$. Then*

$$\lim_{n\to\infty} \phi(a_n) = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} a_n = 0.$$

Proof. Let $\{a_n\} \subseteq [0,\infty)$. Suppose that $\lim_{n\to\infty} a_n = 0$. From (ϕ_2), we get $\lim\limits_{n\to\infty} \phi(a_n) = 0$. Now, suppose that $\lim\limits_{n\to\infty} \phi(a_n) = 0$ and $\lim_{n\to\infty} a_n \neq 0$. It follows that there exist a constant $c > 0$ ad a subsequence $\{a_{n(k)}\}$ of $\{a_n\}$ such that $a_{n(k)} \geq c$ for all $k \geq 0$. Since ϕ is nondecreasing, then $\phi(a_{n(k)}) \geq \phi(c) > 0$ for all $k \geq 0$. Thus, by letting $k \to \infty$, we get $0 \geq \phi(c)$, which is a contradiction. Hence $\lim_{n\to\infty} a_n = 0$. □

Lemma 1.3. *Let (X,d) be a symmetric space and $\phi \in \Phi$. Consider the function $\phi od : X \times X \to [0,\infty)$ defined as follows:*

$$\phi od(x,y) = \phi(d(x,y)) \quad \textit{for all } x,y \in X.$$

Then $(X, \phi od)$ is also a symmetric space.

Proof. $(W1)$ From (ϕ_2), we have $\phi od(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$.
$(W2)$ Since $d(x,y) = d(y,x)$, then $\phi od(x,y) = \phi od(y,x)$.
□

Definition 1.10. Let A and B two nonempty subsets of a symmetric space (X,d), $\phi \in \Phi$, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$. Consider a non-self map $T : A \to B$. We say that T is a generalized (α,ϕ,ψ)-proximal contraction if

(1.1)
$$\begin{cases} \alpha(x,y) \geq 1 \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{cases}$$
$$\Rightarrow \phi(d(u,v)) \leq \psi(\max\{\phi(d(x,y)), \phi(d(x,u)), \phi(d(y,v)), \phi(d(x,v)), \phi(d(y,u))\}),$$

where $x,y,u,v \in A$.

This paper is devoted to the proof of the existence and uniqueness of best proximity points for generalized (α,ϕ,ψ)-proximal contraction non-self-maps in semimetric spaces by using the concept of α-proximal admissible mapping. Some nice consequences are provided.

2. Main results

The first main result is

Theorem 2.1. *Let A and B be nonempty subsets of a semi-metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-map. Suppose that the following conditions hold:*

- (i) *A_0 is α-proximal T-orbitally d-Cauchy complete;*
- (ii) *$T(A_0) \subseteq B_0$;*
- (iii) *d is bounded, that is, $\sup_{x,y\in X} d(x, y) < \infty$;*
- (iv) *T is a generalized (α, ϕ, ψ)-proximal contraction;*
- (v) *T is triangular α-proximal admissible;*
- (vi) *There exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1;$$

- (vii) *If $\{x_n\}$ is a sequence in A_0 such that $\alpha(x_n, x_{n+1}) \geq 1$, $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \geq 0$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $\alpha(x_n, x) \geq 1$ for all $n \geq 0$;*
- $(viii)$ *$(A_0, \phi od)$ satisfies the Fatou property.*

Then, T has a best proximity point, that is, there exists $z \in A$ such that $d(z, Tz) = d(A, B)$.

Proof. By assumption (vi), there exist x_0 and $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1. \tag{2.1}$$

From condition (ii), we have $T(A_0) \subseteq B_0$, so there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{2.2}$$

By (2.1), (2.2) and the fact that T is α-proximal admissible, we have

$$\alpha(x_1, x_2) \geq 1.$$

Repeating the above strategy, by induction, we arrive to construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0. \tag{2.3}$$

Since T is triangular α-proximal admissible, then

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ and } \alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}) \geq 1.$$

Thus by induction, we get

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all } m > n \geq 0. \tag{2.4}$$

For all $n = 0, 1, \cdots$, we denote

$$\delta_n = \sup_{j,k\in\mathbb{N}} \phi(d(x_{n+j}, x_{n+k})).$$

Note that by condition (ii) and the fact that ϕ is nondecreasing function, we have $\delta_n < \infty$, for all $n = 0, 1, \cdots$

On the other hand, from (2.3), we have

$$d(x_{n+j}, Tx_{n+j-1}) = d(A, B), \quad d(x_{n+k}, Tx_{n+k-1}) = d(A, B) \text{ for all } n, j, k \in \mathbb{N}.$$

It follows from (2.4) and (1.1)

$$\phi(d(x_{n+j}, x_{n+k})) \leq \psi(max\{\phi(d(x_{n+j-1}, x_{n+k-1})), \phi(d(x_{n+j}, x_{n+j-1})), \phi(d(x_{n+k}, x_{n+k-1})), \phi(d(x_{n+j-1}, x_{n+k})), \phi(d(x_{n+j}, x_{n+k-1}))\})$$

for all $j < k$. Since ψ is nondecreasing function, then

$$\phi(d(x_{n+j}, x_{n+k})) \leq \psi(\delta_{n-1}), \quad \text{for all } j < k.$$

By symmetry of d, we get

$$\phi(d(x_{n+j}, x_{n+k})) \leq \psi(\delta_{n-1}) \quad \text{for all } j > k.$$

Also, for $j = k$, we have $\phi(d(x_{n+j}, x_{n+k})) = \phi(0) = 0 \leq \psi(\delta_{n-1})$. Thus

$$\phi(d(x_{n+j}, x_{n+k})) \leq \psi(\delta_{n-1}) \quad \text{for all } j, k \in \mathbb{N}.$$

So, we have

$$\delta_n \leq \psi(\delta_{n-1}) \quad \text{for all } n \in \mathbb{N}.$$

By induction, we get

$$\delta_n \leq \psi^n(\delta_0) \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

We have

$$\phi(d(x_n, x_{n+m})) \leq \delta_{n-1} \leq \psi^{n-1}(\delta_0) \quad \text{for all } n, m \geq 1. \tag{2.6}$$

This implies that

$$\lim_{n\to\infty} \phi(d(x_n, x_{n+m})) = 0.$$

It follows from Lemma 1.2 that

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0,$$

which implies that $\{x_n\}$ is a d-Cauchy sequence in A_0. Since A_0 is α-proximal T-orbitally d-Cauchy complete, there is $z \in A_0$ such that $\lim_{n\to\infty} x_n = z$ in the topology τ_d and so $\lim_{n\to\infty} d(x_n, z) = 0$.

From (2.6), as $(A_0, \phi od)$ satisfies the Fatou property, by letting $m \to \infty$, we get

$$\phi(d(x_n, z)) \leq \psi^{n-1}(\delta_0) \quad \text{for all } n \geq 1. \tag{2.7}$$

As $z \in A_0$, there is $w \in A_0$ such that

$$d(w, Tz) = d(A, B). \tag{2.8}$$

Further, from (2.3), we have

$$d(x_2, Tx_1) = d(A, B).$$

By condition (vii), (1.1), (2.6) and (2.7), we get

(2.9)

$$\begin{aligned}
\phi(d(w, x_2)) &\leq \psi(max\{\phi(d(x_1, z), \phi(d(z, w)), \phi(d(x_1, x_2)), \phi(d(z, x_)), \phi(d(x_1, w))\}) \\
&\leq \psi(max\{\delta_0, \phi(d(z, w)), \psi(\delta_0), \phi(d(x_1, w))\}) \\
&= \max\{\psi(\delta_0), \psi^2(\delta_0), \psi(\phi(d(z, w))), \psi(\phi(d(x_1, w)))\}.
\end{aligned}$$

Again, from (2.3), we have

$$d(x_3, Tx_2) = d(A, B).$$

Then, by (vii), (1.1), (2.6), (2.7) and (2.9), we get

$$\begin{aligned}
\phi(d(w, x_3)) &\leq \psi(max\{\phi(d(x_2, z)), \phi(d(z, w)), \phi(d(x_2, x_3)), \phi(d(z, x_3)), \phi(d(x_2, w))\}) \\
&\leq \psi(\max\{\psi(\delta_0), \phi(d(z, w)), \psi^2(\delta_0), \phi(d(x_2, w))\}) \\
&\leq \psi(\max\{\psi(\delta_0), \psi^2(\delta_0), \phi(d(z, w)), \psi(d(z, w))\}) \\
&= \max\{\psi^2(\delta_0), \psi^3(\delta_0), \psi(\phi(d(z, w))), \psi^2(\phi(d(z, w))), \psi^2(\phi(d(x_1, w)))\}.
\end{aligned}$$

Continuing in this fashion, by induction, we get

(2.10)

$$\phi(d(w, x_n)) \leq \max\{\psi^{n-1}(\delta_0), \psi^n(\delta_0), \psi(\phi(d(z, w))), \psi^{n-1}(\phi(d(z, w))), \psi^n(\phi(d(x_1, w)))\}.$$

Using the Fatou property, we get from (2.10)

$$\begin{aligned}
&\phi(d(w, z)) \leq \liminf_{n \to \infty} \phi(d(w, x_n)) \\
&\leq \limsup_{n \to \infty} \phi(d(w, x_n)) \\
&\leq \limsup_{n \to \infty} \max\{\psi^{n-1}(\delta_0), \psi^n(\delta_0), \psi(\phi(d(z, w))), \psi^{n-1}(\phi(d(z, w))), \psi^n(\phi(d(x_1, w)))\} \\
&= \max\{\psi(\phi(d(z, w))), 0\} = \psi(\phi(d(z, w))).
\end{aligned}$$

Then

$$\phi(d(z, w)) \leq \psi(\phi(d(z, w))),$$

which implies that $\phi od(w, z) = 0$ and so $w = z$. From (2.8), we obtain $d(z, Tz) = d(A, B)$, that is z is a best proximity point of T. $\square$

Theorem 2.2. *Let A and B be nonempty subsets of a semi-metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-map. Suppose that the following conditions hold:*

(i) *A_0 is α-proximal T-orbitally d-Cauchy complete;*

(ii) *$T(A_0) \subseteq B_0$;*

(iii) *d is bounded, that is, $\sup_{x,y\in X} d(x, y) < \infty$;*

(iv) *T is a generalized (α, ϕ, ψ)-proximal contraction;*

(v) *T is triangular α-proximal admissible;*

(vi) *There exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1;$$

(vii) *T is τ_d-continuous;*

(viii) *(X, d) satisfies the property (W_c).*

Then, T has a best proximity point.

Proof. Following the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ in A_0 such that (2.3) and (2.4) hold. Also, $\{x_n\}$ is d-Cauchy in the subset A_0, which is α-proximal T-orbitally d-Cauchy complete, then there exists $z \in A_0$ such that $x_n \to z$ as $n \to \infty$ in the topology τ_d. We shall prove that z is a best proximity point of T. Since T is τ_d-continuous, then $\lim_{n\to\infty} Tx_n = Tz$ in τ_d and so $\lim_{n\to\infty} d(Tx_n, Tz) = 0$. From (2.3) and as (X, d) satisfies the property (W_c), we have

$$d(A, B) \leq d(z, Tz) \leq \liminf_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B),$$

which implies that $d(z, Tz) = d(A, B)$, i.e., z is a best proximity point of T. □

Now, we prove the uniqueness of such best proximity point. For this, we need the following additional condition.

(U): For all $x, y \in B(T)$, we have $\alpha(x, y) \geq 1$, where $B(T)$, denotes the set of best proximity points of T.

Theorem 2.3. *Adding condition (U) to the hypotheses of Theorem 2.1 (resp. Theorem Theorem 2.2), we obtain that z is the unique best proximity point of T.*

Proof. Suppose there exist $z, w \in A$ such that $d(A, B) = d(z, Tz) = d(w, Tw)$. By assumption (U), we have $\alpha(z, w) \geq 1$, it follows from (1.1),

$$\begin{aligned}
\phi(d(z, w)) &\leq \psi(\max\{\phi(d(z, w)), \phi(d(z, z)), \phi(d(w, w)), \phi(d(z, w)), \phi(d(w, z))\}) \\
&= \psi(\max\{\phi(d(z, w)), \phi(0)\}) \\
&= \psi(\phi(d(z, w))),
\end{aligned}$$

which implies that $\phi o d(z, w) = 0$ and so $z = w$. □

Example 2.1. Let $X = [0, \infty)\times[0, \infty)$ endowed with the semi-metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Take $A = \{1\} \times [0, \infty)$ and $B = \{0\} \times [0, \infty)$. Mention that $d(A, B) = 1$, $A_0 = A$ and $B_0 = B$. Consider the mapping $T : A \to B$ as

$$T(1, x) = \begin{cases} (0, \frac{x^2+1}{4}) & \text{if } 0 \leq x \leq 1 \\ (0, x - \frac{1}{2}) & \text{if } x > 1. \end{cases}$$

We have $T(A_0) \subseteq B_0$. Take $\psi(t) = \frac{1}{4}t$, $\phi(t) = t^2$ for all $t \geq 0$. Define $\alpha : X \times X \to [0, \infty)$ as follows

$$\begin{cases} \alpha((x, y), (s, t)) = 1 & \text{if} \quad (x, y), (s, t) \in [0, 1] \times [0, 1] \\ \alpha((x, y), (s, t)) = 0 & \text{if not.} \end{cases}$$

Let $(1, x_1), (1, x_2), (1, u_1)$ and $(1, u_2)$ in A such that

$$\begin{cases} \alpha((1, x_1), (1, x_2)) \geq 1 \\ d((1, u_1), T(1, x_1)) = d(A, B) = 1, \\ d((1, u_2), T(1, x_2)) = d(A, B) = 1. \end{cases}$$

Then, necessarily, $(x_1, x_2) \in [0, 1] \times [0, 1]$. Also, we have ($u_1 = \frac{1+x_1^2}{4}$ and $u_2 = \frac{1+x_2^2}{4}$). So

$$\alpha((1, u_1), (1, u_2)) \geq 1,$$

that is, T is an α-proximal admissible. Moreover, T is triangular α-proximal admissible. Therefore,

$$\begin{aligned} d((1, u_1), (1, u_2)) &= d((1, \frac{1 + x_1^2}{4}), (1, \frac{1 + x_2^2}{4})) \\ &= |\frac{1 + x_1^2}{4} - \frac{1 + x_2^2}{4}| = |\frac{x_1^2}{4} - \frac{x_2^2}{4}| = \frac{1}{4}(x_1 + x_2)|x_1 - x_2| \\ &\leq \frac{1}{2}|x_1 - x_2| = \frac{1}{2}d((1, x_1), (1, x_2)). \end{aligned}$$

Then

$$\begin{aligned} d^2((1, u_1), (1, u_2)) &\leq \frac{1}{4}d^2((1, x_1), (1, x_2)) = \psi(\phi(d((1, x_1), (1, x_2)))) \\ &\leq \psi(\max\{\phi(d((1, x_1), (1, x_2))), \phi(d((1, x_1), (1, u_1))), \phi(d((1, x_1), (1, u_2))), \\ &\phi(d((1, x_2), (1, u_1))), \phi(d((1, x_2), (1, u_2)))\}). \end{aligned}$$

So the condition contraction (1.1) holds. Also, A_0 is α-proximal T-orbitally d-Cauchy complete. Furthermore, T is τ_d-continuous. Moreover, the condition (vi) of Theorem 2.2 is verified. Indeed, for $x_0 = (1, 1)$ and $x_1 = (1, \frac{1}{2})$, we have

$$d(x_1, Tx_0) = d((1, \frac{1}{2}), (0, \frac{1}{2})) = 1 = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1.$$

Hence, all hypotheses of Theorem 2.2 are verified. So T has a best proximity point which is $u = (1, 2 - \sqrt{3})$. It is also unique.

3. Consequences

In this paragraph, we present some consequences on our obtained results.

3.1. Some classical best proximity point results

Denote by Λ the set of Lebesgue integrable mappings $\lambda : [0,\infty) \to [0,\infty)$, summable on each compact of $[0,\infty)$ and satisfying: $\int_0^{\varepsilon} \lambda(s)ds > 0 \quad \text{for each } \varepsilon > 0.$

Corollary 3.1. *Let A and B be nonempty subsets of a semi-metric space (X,d). Let $T : A \to B$ be a given non-self-map, $k \in [0,1)$, $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$ such that*

$$\begin{cases} \alpha(x,y) \geq 1 \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{cases}$$
$$\Rightarrow \int_0^{d(u,v)} \lambda(t)dt \leq k \max\{ \int_0^{d(x,y)} \lambda(t)dt, \int_0^{d(x,u)} \lambda(t)dt, \int_0^{d(y,v)} \lambda(t)dt, \int_0^{d(x,v)} \lambda(t)dt, \int_0^{d(y,u)} \lambda(t)dt\},$$

where $x,y,u,v \in A$. Suppose that the following conditions hold:

(*i*) *A_0 is α-proximal T-orbitally d-Cauchy complete;*

(*ii*) *$T(A_0) \subseteq B_0$;*

(*iii*) *d is bounded, that is, $\sup_{x,y\in X} d(x,y) < \infty$;*

(*iv*) *T is triangular α-proximal admissible;*

(*v*) *There exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0,x_1) \geq 1;$$

(*vi*) *T is τ_d-continuous;*

(*vii*) *(X,d) satisfies the property (W_c).*

Then, T has a best proximity point.

Proof. It suffices to take $\alpha(x,y) = 1$, $\phi(t) = \int_0^t \lambda(s)ds$ and $\psi(t) = kt$ in Theorem 2.2. It is clear that $\phi \in \Phi$ and $\psi \in \Psi$. □

Corollary 3.2. *Let A and B be nonempty subsets of a semi-metric space (X,d). Let $T : A \to B$ be a given non-self-map, $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{cases} d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{cases}$$
$$\Rightarrow \phi(d(u,v)) \leq \psi(\max\{\phi(d(x,y)), \phi(d(x,u)), \phi(d(y,v)), \phi(d(x,v)), \phi(d(y,u))\}),$$

where $x,y,u,v \in A$. Suppose that the following conditions hold:

(*i*) *Every d-Cauchy sequence* $\{x_n\}$ *in* A_0 *with* $d(x_{n+1}, Tx_n) = d(A, B)$ *for all* $n \geq 0$, *converges to some element in* A_0 *in the topology* τ_d;

(*ii*) $T(A_0) \subseteq B_0$;

(*iii*) *d is bounded, that is,* $\sup_{x,y\in X} d(x, y) < \infty$;

(*iv*) $(A_0, \phi od)$ *satisfies the Fatou property.*

Then, T has a unique best proximity point.

Proof. It suffices to take $\alpha(x, y) = 1$ in Theorem 2.1. The uniqueness of z holds since (U) is satisfied. □

Corollary 3.3. *Let A and B be nonempty subsets of a semi-metric space* (X, d). *Let* $T : A \to B$ *be a given non-self-map and* $\psi \in \Psi$ *such that*

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{cases} \Rightarrow d(u, v) \leq \psi(\max\{d(x, y), d(x, u), d(y, v), d(x, v), d(y, u)\}),$$

where $x, y, u, v \in A$. *Suppose that the following conditions hold:*

(*i*) *Every d-Cauchy sequence* $\{x_n\}$ *in* A_0 *with* $d(x_{n+1}, Tx_n) = d(A, B)$ *for all* $n \geq 0$, *converges to some element in* A_0 *in the topology* τ_d;

(*ii*) $T(A_0) \subseteq B_0$;

(*iii*) *d is bounded, that is,* $\sup_{x,y\in X} d(x, y) < \infty$;

(*iv*) (A_0, d) *satisfies the Fatou property.*

Then, T has a unique best proximity point.

Proof. It suffices to take $\phi(t) = t$ in Corollary 3.2. □

Corollary 3.4. *Let A and B be nonempty subsets of a semi-metric space* (X, d). *Let* $T : A \to B$ *be a given non-self-map,* $\phi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{cases}$$
$$\Rightarrow \phi(d(u, v)) \leq \psi(\max\{\phi(d(x, y)), \phi(d(x, u)), \phi(d(y, v)), \phi(d(x, v)), \phi(d(y, u))\}),$$

where $x, y, u, v \in A$. *Suppose that the following conditions hold:*

(*i*) *Every d-Cauchy sequence* $\{x_n\}$ *in* A_0 *with* $d(x_{n+1}, Tx_n) = d(A, B)$ *for all* $n \geq 0$, *converges to some element in* A_0 *in the topology* τ_d;

(*ii*) $T(A_0) \subseteq B_0$;

(*iii*) *d is bounded, that is,* $\sup_{x,y\in X} d(x,y) < \infty$*;*

(*iv*) *T is* τ_d*-continuous;*

(*v*) (X, d) *satisfies the property* (W_c)*.*

Then, T has a unique best proximity point.

3.2. Some classical fixed point results

If we take $A = B$ in the previous results, we have the following fixed point results.

Corollary 3.5. *Let A be nonempty subset of a semi-metric space* (X, d)*. Let* $T : A \to A$ *be a given self-map,* $\phi \in \Phi$, $\psi \in \Psi$ *and* $\alpha : A \times A \to [0, \infty)$ *such that*

$$\phi o d(Tx, Ty) \le \psi(\max\{\phi o d(x, y), \phi o d(x, Tx), \phi o d(y, Ty), \phi o d(x, Ty), \phi o d(y, Tx)\})$$

for all $x, y \in A$ *satisfying* $\alpha(x, y) \ge 1$*. Suppose that the following conditions hold:*

(*i*) *Every d-Cauchy sequence* $\{x_n\}$ *in A with* $x_{n+1} = Tx_n$ *for all* $n \ge 0$*, converges to some element in A in the topology* τ_d*;*

(*ii*) *d is bounded, that is,* $\sup_{x,y\in X} d(x,y) < \infty$*;*

(*iii*) *T is triangular* α*-proximal admissible;*

(*iv*) *There exist elements* $x_0 \in A$ *such that* $\alpha(x_0, Tx_0) \ge 1$*;*

(*v*) *If* $\{x_n\}$ *is a sequence in A such that* $\alpha(x_n, x_{n+1}) \ge 1$ *for all* $n \ge 0$ *and* $\lim_{n\to\infty} d(x_n, x) = 0$*, then* $\alpha(x_n, x) \ge 1$ *for all* $n \ge 0$*;*

(*vi*) $(A, \phi o d)$ *satisfies the Fatou property.*

Then, T has a fixed point in A.

Corollary 3.6. *Let A be nonempty subset of a semi-metric space* (X, d)*. Let* $T : A \to A$ *be a given self-map,* $\phi \in \Phi$, $\psi \in \Psi$ *and* $\alpha : A \times A \to [0, \infty)$ *such that*

$$\phi o d(Tx, Ty) \le \psi(\max\{\phi o d(x, y), \phi o d(x, Tx), \phi o d(y, Ty), \phi o d(x, Ty), \phi o d(y, Tx)\})$$

for all $x, y \in A$ *satisfying* $\alpha(x, y) \ge 1$*. Suppose that the following conditions hold:*

(*i*) *Every d-Cauchy sequence* $\{x_n\}$ *in A with* $x_{n+1} = Tx_n$ *for all* $n \ge 0$*, converges to some element in A in the topology* τ_d*;*

(*ii*) *d is bounded, that is,* $\sup_{x,y\in X} d(x,y) < \infty$*;*

(*iii*) *T is triangular* α*-proximal admissible;*

(*iv*) *There exist elements* $x_0 \in A$ *such that* $\alpha(x_0, Tx_0) \ge 1$*;*

(*v*) *T is* τ_d*-continuous;*

(*vi*) (X, d) *satisfies the property* (W_c)*.*

Then, T has a fixed point in A.

3.3. Some best proximity results on a semi-metric space endowed with a partial order

Let (X, d) a symmetric space endowed with a partial order $\leq$. We introduce the following definition.

Definition 3.1. Let A and B be nonempty subsets of a symmetric space (X, d) and $\leq$ a partial order on X. $T : A \to B$ is named a proximal nondecreasing map if

$$\begin{cases} x \leq y \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{cases} \Rightarrow u \leq v$$

for all $x, y, u, v \in A$.

Wa also need the following hypothesis.

(H) if $\{x_n\}$ is a sequence in A such that $x_n \leq x_{n+1},\ d(x_{n+1}, Tx_n) = d(A, B)$ for all n and $x_n \to x \in A$ as $n \to \infty$, then $x_n \leq x$ for all n.

We state the following.

Corollary 3.7. *Let A and B be nonempty subsets of a semi-metric space (X, d). Let $T : A \to B$ be a given non-self-map and $\psi \in \Psi$ such that*

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{cases}$$
$$\Rightarrow \phi(d(u, v)) \leq \psi(\max\{\phi(d(x, y)), \phi(d(x, u)), \phi(d(y, v)), \phi(d(x, v)), \phi(d(y, u))\}),$$

for all $x, y \in A$ such that $x \leq y$. Suppose that

(i) *Every d-Cauchy sequence $\{x_n\}$ in A_0 with $x_n \leq x_{n+1},\ d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \geq 0$, converges to some element in A_0 in the topology τ_d;*

(ii) *$T(A_0) \subseteq B_0$;*

(iii) *T is a proximal nondecreasing map;*

(iv) *There exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad x_0 \leq x_1;$$

(v) *$(A_0, \phi o d)$ satisfies the Fatou property;*

(vi) *(H) holds.*

Then T has a best proximity point.

Proof. It suffices to consider $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{if not.} \end{cases}$$

All hypotheses of Theorem 2.1 are satisfied. This completes the proof. □

Corollary 3.8. *Let A and B be nonempty subsets of a semi-metric space (X, d). Let $T : A \to B$ be a given non-self-map and $\psi \in \Psi$ such that*

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{cases}$$
$$\Rightarrow \phi(d(u, v)) \leq \psi(\max\{\phi(d(x, y)), \phi(d(x, u)), \phi(d(y, v)), \phi(d(x, v)), \phi(d(y, u))\}),$$

for all $x, y \in A$ such that $x \leq y$. Suppose that

(i) *Every d-Cauchy sequence $\{x_n\}$ in A_0 with $x_n \leq x_{n+1}$, $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \geq 0$, converges to some element in A_0 in the topology τ_d;*

(ii) *$T(A_0) \subseteq B_0$;*

(iii) *T is a proximal nondecreasing map;*

(iv) *There exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad x_0 \leq x_1;$$

(v) *(X, d) satisfies the property (W_c);*

(vi) *T is τ_d-continuous.*

Then T has a best proximity point.

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