FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 32, No 4 (2017), 485–502
https://doi.org/10.22190/FUMI1704485R

GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE VISCOELASTIC WAVE EQUATION WITH A CONSTANT DELAY TERM *

Melouka Remil and Ali Hakem

Abstract. In this paper, we investigate the following viscoelastic wave equation with a constant delay term

$$u''(x,t) - k_0\triangle u + \alpha \int_0^t g(t-s)\triangle u(x,s)ds + \mu_1(t)u'(x,t) + \mu_2(t)u'(x,t-\tau) = 0$$

in a bounded domain and under suitable assumptions. First, we prove the global existence by using Faedo-Galerkin procedure. Secondly, the multiplier method is used to establish a decay estimate for the energy, which depends on the behavior of α and g.

Keywords: Global existence, energy decay, Faedo-Galerkin method

1. Introduction

This paper is concerned with the following Cauchy problem of the form

$$(1.1)\qquad \begin{cases} u''(x,t) - k_0\triangle u + \alpha \displaystyle\int_0^t g(t-s)\triangle u(x,s)ds & \\ +\mu_1(t)u'(x,t) + \mu_2(t)u'(x,t-\tau) = 0, & \text{on } \Omega\times]0,+\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega\times]0,+\infty[\\ u(x,0) = u_0(x), u_t(x,t) = u_1(x), & \text{on } \Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{on } \Omega\times]0,t[\end{cases}$$

Where Ω is a bounded domain in $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$ with a smooth boundary $\partial\Omega$. The initial data u_0, u_1, f_0 belong to a suitable space. Moreover, $\tau > 0$ is the time delay term and $\mu_1,\ \mu_2$ are real functions that will be specified later. Furthermore, k_0 is a positive real number and g is a positive non-increasing function defined on $\mathbb{R}^+$.

Received June 21, 2016; accepted August 06, 2017
2010 *Mathematics Subject Classification.* Primary 35L15; 35L70; Secondary 93D15.
*The authors were supported in part by CNEPRU. COOL 03 UN 220 120 150001. ALGERIA.

In recent years, the PDEs with time delay effects have become an active area of research. Many authors have focused on this problem (see [1],[16],[17],[2],[3],[4],[5],[8],[9],[10]).

The presence of delay may lead to a source of instability. In [2] for example, R. Datko, J. Lagnese and M. P. Polis proved that a small delay may destabilize a system.
S. Nicaise, C. Pignotti studied in [8] the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_0 and $\mu_1 > 0$ in which the stability or alternatively instability takes place.

After that, they studied in [11] the stabilization problem by interior damping of the wave equation with boundary or internal time-varying delay feedback in a bounded and smooth domain. By introducing suitable Lyapunov functionals, exponential stability estimates are obtained if the time delay effect is appropriately compensated by the internal damping.

It is worth mentioning that recently Z. Y. Zhang et al. [14] have investigated global existence and uniform decay for wave equation with dissipative term and boundary damping under some assumptions on nonlinear feedback function. They have obtained the results by means of Galerkin method and the multiplier technique. More precisely, they introduced a new variables and transformed the boundary value problem into an equivalent one with zero initial data by argument of compacity and monotonicity. More details are present in [14]. Later on, Zhang et al. [21] studied the wellposedness and uniform stability of strong and weak solutions of the nonlinear generalized dissipative Klein-Gordon equation with nonlinear damped boundary conditions. Also, the authors proved the wellposedness by means of nonlinear semigroup method and obtain the uniform stabilization by using the perturbed energy functional method. In another works, Zai-Yun Zhang and al ([20],[14],[15]) considered a more general problem than (1.1). Their proof of the existence is based on the Galerkin approximation. For strong solutions, their approximation requires a change of variables to transform the main problem into an equivalent problem with initial value equals zero. Especially, they overcome some difficulties, that is, the presence of nonlinear terms and nonlinear boundary damping bringing up serious difficulties when passing to the limit, by combining arguments of compacity and monotonicity.

F. Tahamtani and A. Peyravi [12] investigated the nonlinear viscoelastic wave equation with dissipative boundary conditions:

$$u'' - k_0 \triangle u + \alpha \int g(t-s) div[a(s)\nabla u(s)] ds + (k_1 + b(x)|u'|^{m-2})u' = |u^{p-2}|u$$

They showed that the solutions blow up in finite time under some restrictions on initial data and for arbitrary initial energy in some case. In another case, they proved a nonexistence result when the initial energy is less than potential well depth.

Wenjun Liu in [6] studied the weak viscoelastic equation with an internal time-varying delay term

$$u''(x,t) - k_0\triangle u + \alpha(t)\int g(t-s)\triangle u(x,s)ds + a_0 u'(x,t) + a_1 u'(x,t-\tau(t)) = 0$$

in a bounded domain. By introducing suitable energy and Lyapunov functionals, he establishes a general decay rate estimate for the energy under suitable assumptions. A. Benaissa, A. Benguessoum and S. A. Messaoudi [1] considered the wave equation with a weak internal constant delay term:

$$u''(x,t) - \triangle u + \mu_1(t)u'(x,t) + \mu_2(t)u'(x,t-\tau) = 0 \text{ on } [0,+\infty[$$

In a bounded domain. Under appropriate conditions on μ_1 and μ_2, they proved global existence of solutions by the Faedo–Galerkin method and establish a decay rate estimate for the energy by using the multiplier method.
However, according to our best knowledge, in the present paper, we have to treat Eq.(1.1) with a delay term and it is not considered in the literature. The proof of the existence is based on the Galerkin approximation.

The content of this paper is organized as follows. In Section 2, we provide assumptions that will be used later. We state and prove the existence result. In Section 3, we establish the energy decay result that is given in Theorem 4.1.

2. Main results

In the following, we will give sufficient conditions and assumptions which guarantee that the problem 1.1 has a global solution.

$(H1)$ g is a positive bounded function satisfying:

$$k_0 - \alpha\int_0^t g(s)ds = l > 0, \quad \alpha > 0, \tag{2.1}$$

and there exists a positive non-increasing function η such that for $t > 0$ we have

$$g'(t) \leq -\eta(t)g(t), \quad \eta(t) > 0 \tag{2.2}$$

$(H2)$ μ_1 is a positive function of class C^1 satisfying:

$$\mu_1(t) \leq M, \qquad M > 0 \tag{2.3}$$

$(H3)$ μ_2 is a real function of class C^1 such that:

$$\mu_2(t) \leq \beta\mu_1(t), \quad 0 < \beta < 1 \tag{2.4}$$

We also need the following technical Lemmas in the course of our investigation.

Lemma 2.1. *(Sobolev-Poincare's inequality). Let $2 \le p \le \frac{2n}{n-2}$. The inequality*

$$\|u\|_p \le C_s \|\nabla u\|_2 \quad for \quad u \in H_0^1(\Omega) \tag{2.5}$$

holds with some positive constant C_s.

Lemma 2.2. *[7] For any $g \in C^1$ and $\phi \in H_0^1(0,T)$ we have*

$$\begin{aligned}\int_0^t \int_\Omega g(t-s)\phi\phi_t dx ds = &-\frac{d}{dt}\left(\frac{1}{2}(g\circ\phi)(t) - \frac{1}{2}\int_0^t g(s)ds\|\phi\|_2^2\right) \\ &-\frac{1}{2}g(t)\|\phi\|_2^2 + \frac{1}{2}(g'\circ\phi)(t)\end{aligned} \tag{2.6}$$

where

$$(g\circ\phi)(t) = \int_0^t \int_\Omega g(t-s)|\phi(s)-\phi(t)|^2 dx\ ds$$

Lemma 2.3. *[7] Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0, \quad \phi(t) \to +\infty \quad when \quad t \to +\infty$$

Assume that there exist $p > 0$ and $\omega > 0$ such that

$$\forall S \ge 0, \quad \int_S^{+\infty} E^{p+1}(t)\phi'(t)dt \le \frac{1}{\omega}[E(0)]^p E(S),$$

then E has the following decay properties

$$\text{if } p = 0 \text{ then} \quad E(t) \le E(0)e^{1-\omega\phi(t)}, \quad \forall t \ge 0$$

$$\text{if } p > 0 \text{ then} \quad E(t) \le E(0)\left(\tfrac{1+p}{1+\omega\phi(t)}\right), \quad \forall t \ge 0$$

In order to prove the existence of solutions to the problem (1.1) we introduce as in [8] the unknown auxiliary

$$z(x,\rho,t) = u'(x, t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0$$

Then we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0$$

Therefore, the problem (1.1) takes the form

$$\begin{cases} u''(x,t) - k_0\Delta u(x,t) + \alpha \displaystyle\int_0^t g(t-s)\triangle u(x,s)ds & \\ +\mu_1(t)u'(x,t) + \mu_2(t)z(x,1,t) = 0, & \text{on } \Omega\times]0,+\infty[\\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & x \in \Omega, \rho \in (0,1), t > 0 \\ u(x,t) = 0, & \text{on } \partial\Omega\times]0,+\infty[\\ u(x,0) = u_0, u'(x,t) = u_1, & \text{on } \Omega \\ z(x,\rho,0) = f_0(x,-\tau\rho) & \text{on } \Omega\times]0,t[\end{cases} \tag{2.7}$$

Now, we are in the position to state our main result, namely the theorem of global existence.

Theorem 2.1. *Let* $(u_0, u_1, f_0) \in H_0^1(\Omega)\times\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega\times(0,1))$ *be given. Assume that assumptions (H1) -(H3) are fulfilled. Then the problem (2.7) admits a unique global weak solution* (u, z) *satisfying*

$$u \in C([0,T]; H_0^1(\Omega)),\;\; u' \in C([0,T]; H_0^1(\Omega)),\;\; z \in C([0,T]; \mathbb{L}^2(\Omega \times (0,1))$$

To prove this theorem, we need the following lemma. First, we define the energy associated to the solution of the problem (2.7) by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u'\|_2^2 + \left(\frac{k_0}{2} - \frac{\alpha}{2}\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ &+ \frac{\alpha}{2}(g\circ\nabla u)(t) + \frac{1}{2}\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \end{aligned} \tag{2.8}$$

Where ξ is non-increasing function such that

$$\tau\beta < \zeta < \tau(2-\beta), \quad t > 0 \tag{2.9}$$

Where $\quad \xi(t) = \zeta\mu_1(t)$.

Lemma 2.4. *Let* (u, z) *be a regular solution of problem (2.7). Then the energy functional defined by (2.8) satisfies*

$$\begin{aligned} E'(t) &\leq -\Big(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|u'(x,t)\|^2 \\ &- \Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2 \leq 0 \end{aligned} \tag{2.10}$$

Proof. Multiplying the first equation in (2.7) by $u'(x,t)$, integrating over Ω and using Green's identity we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u'\|_2^2 + k_0\|\nabla u\|_2^2\Big) + \mu_1(t)\|u'\|_2^2 + \mu_2(t)\int_\Omega u'z(x,1,t)dx \\ &- \alpha\int_0^t g(t-s)\int_\Omega \nabla u(x,s)\nabla u'(x,t)dxds = 0 \end{aligned} \tag{2.11}$$

We simplify the last term in (2.11) by applying the lemma 2.2, we get

$$\begin{aligned} &- \alpha\int_0^t g(t-s)\int_\Omega \nabla u(x,s)\nabla u'(x,t)dxds = \frac{\alpha}{2}\frac{d}{dt}(g\circ\nabla u) \\ &- \frac{\alpha}{2}(g'\circ\nabla u) + \frac{\alpha}{2}g(t)\|\nabla u\|^2 - \frac{\alpha}{2}\frac{d}{dt}\int_0^t g(s)ds\|\nabla u\|^2 \end{aligned} \tag{2.12}$$

Replacing (2.12) in (2.11) we arrive at

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}\|u'\|_2^2+\Big(\frac{k_0}{2}-\frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u\|_2^2-\frac{\alpha}{2}(g\circ\nabla u)\Big)=\frac{\alpha}{2}(g'\circ\nabla u)(t)\\&-\frac{1}{2}g(t)\|\nabla u\|_2^2-\mu_1(t)\|u'\|_2^2-\mu_2(t)\int_\Omega z(x,1,s)u'dx\end{aligned}\tag{2.13}$$

Multiplying the second equation in (2.7) by $\frac{\xi(t)z}{\tau}$, where $\xi(t)$ satisfies (2.9) and integrating over $\Omega\times(0,1)$, we obtain

$$\frac{\xi(t)}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx+\frac{\xi(t)}{2\tau}\int_0^1\int_\Omega\frac{d}{d\rho}z^2(x,\rho,t)dxd\rho=0\tag{2.14}$$

which gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx=\frac{\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\\&-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx+\frac{\xi(t)}{2\tau}\int_\Omega u'^2(x,t)dx=0\end{aligned}\tag{2.15}$$

A combination of (2.13) and (2.15) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u'\|_2^2+(k_0-\alpha\int_0^t g(s)ds)\|\nabla u\|_2^2\Big)\\&+\frac{1}{2}\frac{d}{dt}\Big(\alpha(g\circ\nabla u)+\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\Big)\\&=\frac{\alpha}{2}(g'\circ\nabla u)-\frac{1}{2}g(s)\|\nabla u\|_2^2ds-\mu_1(t)\|u_n'\|_2^2ds-\mu_2(t)\int_\Omega z(x,1,s)u'dx\\&+\frac{\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx+\frac{\xi(t)}{2\tau}\int_\Omega u'^2(x,t)dx\end{aligned}\tag{2.16}$$

Using the definition (2.8) of $E(t)$, we deduce that

$$\begin{aligned}E'(t)=&\frac{\alpha}{2}(g'\circ\nabla u)-\frac{1}{2}g(t)\|\nabla u\|_2^2-\mu_1(t)\|u_n'\|_2^2ds\\&-\mu_2(t)\int_\Omega z(x,1,s)u'dx+\frac{\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\\&-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx+\frac{\xi(t)}{2\tau}\int_\Omega u'^2(x,t)dx\end{aligned}\tag{2.17}$$

Due to Young's inequality we have

$$\begin{aligned}E'(t)\le&\frac{\alpha}{2}(g'\circ\nabla u)-\frac{1}{2}g(t)\|\nabla u\|_2^2ds-\Big(\mu_1(t)-\frac{\xi(t)}{2\tau}-\frac{\mu_2(t)}{2}\Big)\|u'(x,t)\|^2\\&-\Big(\frac{\xi(t)}{2\tau}-\frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2\end{aligned}\tag{2.18}$$

Using the assumption (2.9) for $\xi(t)$ we see that

$$C_1 = \mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} > 0, \quad C_2 = \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} > 0,$$

then we easily deduce that

$$\begin{aligned} E'(t) \le & -\Big(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|u'(x,t)\|^2 \\ & - \Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2 \le 0 \end{aligned} \tag{2.19}$$

This completes the proof of the lemma. □

3. Global existence

We will use the Faedo-Galerkin method to prove the global existence of solutions. Let $(w_n)_{n\in\mathbb{N}}$ be a basis in $H_0^1(\Omega)$ and W_n be the space generated by $w_1, ..., w_n,\ n \in \mathbb{N}$. Now, we define for $1 \le i \le n$ the sequence $\varphi_i(x,\rho)$ as follows $\varphi_i(x,0) = w_i(x)$.Then, we may extend $\varphi_i(x,0)$ by $\varphi_i(x,\rho)$ over $(\mathbb{L}^2 \times [0,1])$ and denote V_n to be the space generated by $\varphi_1, ..., \varphi_n,\ n \in \mathbb{N}$.

We consider the approximate solution $(u_n(t), z_n(t))$ as follow for any given i

$$u_n(t) = \sum_{i=0}^{n} c_{in}(t)w_i; \quad z_n(t) = \sum_{i=0}^{n} r_{in}(t)\varphi_i$$

which satisfies

$$\begin{aligned} \int_\Omega u_n''(t)w_i dx - k_0 \int_\Omega \triangle u_n(t) w_i dx + \alpha \int_0^t g(t-s) \int_\Omega \triangle u_n(s) w_i dx ds \\ +\mu_1(t) \int_\Omega u_n'(t) w_i dx + \mu_2(t) \int_\Omega z_n(x,1,t) w_i dx = 0, \end{aligned} \tag{3.1}$$

and

$$\int_\Omega (\tau z_{nt}(x,\rho,t) + z_{n\rho}(x,\rho,t))\varphi_i dx = 0. \tag{3.2}$$

The system is completed by the initial conditions:

$$u_n(0) = \sum_{i=0}^{n} c_{in}(0) w_i \to u_0 \quad \text{in} \quad H_0^1(\Omega) \quad \text{when} \quad n \to \infty$$

$$u_n'(0) = \sum_{i=0}^{n} c_{in}'(0) w_i \to u_1 \quad \text{in} \quad H_0^1(\Omega) \quad \text{when} \quad n \to \infty$$

$$z_n(0) = \sum_{i=0}^{n} r_{in}(0)\varphi_i \to f_0 \quad \text{in} \quad \mathbb{L}^2\left(\Omega \times (0,1)\right) \quad \text{when} \qquad n \to \infty$$

Then the problem (2.7) can be reduced to a second-order ODE system and we infer that this problem admits a unique local solution $(u_n(t), z_n(t))$ in $[0, t_n[$ where $0 < t_n < T$. This solution can be extended to $[0; T[,\ 0 < T \leq +\infty$ by Zorn lemma. In the next step we shall prove that this solution is global.

1. **First estimate.** Multiplying the equation in (3.1) by $c'_{in}(t)$ and summing with respect to i we obtain

$$\int_\Omega u''_n(t)u'_n(t)dx + k_0 \int_\Omega \nabla u_n(t)\nabla u'_n(t)dx - \alpha \int_0^t g(t-s) \int_\Omega \nabla u_n(s)\nabla u'_n(t)dxds$$
$$+ \mu_1(t) \int_\Omega u'^2_n(t)dx + \mu_2(t) \int_\Omega z_n(x,1,t)u'_n(t)dx = 0,$$

then

$$\tag{3.3} \begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u'_n\|_2^2 + \frac{k_0}{2}\|\nabla u_n\|_2^2\right) + \mu_1(t)\|u'_n\|_2^2 + \mu_2(t)\int_\Omega z_n(x,1,t)u'_n(t)dx \\ &- \alpha \int_0^t g(t-s) \int_\Omega u_n(t)\nabla u'_n(t)dxds = 0. \end{aligned}$$

We use the lemma 2.2 to simplify the last term in (3.3)

$$\tag{3.4} \begin{aligned} &- \alpha \int_0^t g(t-s) \int_\Omega u_n(t)\nabla u'_n(t)dxds = \frac{\alpha}{2}\frac{d}{dt}(g \circ \nabla u_n)(t) \\ &- \frac{\alpha}{2}(g' \circ \nabla u_n)(t) + \frac{\alpha}{2}g(t)\|\nabla u_n(t)\|^2 - \frac{\alpha}{2}\frac{d}{dt}\int_0^t g(s)ds\|\nabla u_n(t)\|^2 \end{aligned}$$

Replacing (3.4) in (3.3) and integrating over $(0,t)$ we arrive at

$$\tag{3.5} \begin{aligned} &\frac{1}{2}\|u'_n\|_2^2 + \Big(\frac{k_0}{2} - \frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u_n\|_2^2 - \frac{\alpha}{2}(g \circ \nabla u_n)(t) \\ &- \frac{\alpha}{2}\int_0^t (g' \circ \nabla u_n)(s)ds + \frac{1}{2}\int_0^t g(s)\|\nabla u_n\|_2^2 ds + \int_0^t \mu_1(s)\|u'_n\|_2^2 ds \\ &+ \int_0^t \mu_2(s)\int_\Omega z(x,1,s)u'_n(t)dxds = \frac{1}{2}\|u_{1n}\|^2 + \frac{k_0}{2}\|\nabla u_{0n}\|^2 \end{aligned}$$

Multiplying the equation (3.2) by $r_{in}(t)$, summing with respect to i and integrating over $\Omega \times (0,1)$, we obtain

$$\tag{3.6} \frac{\xi(t)}{2}\frac{d}{dt}\int_\Omega \int_0^1 z_n^2(x,\rho,t)d\rho dx + \frac{\xi(t)}{2\tau}\int_0^1 \int_\Omega \frac{d}{d\rho}z_n^2(x,\rho,t)dxd\rho = 0,$$

which gives

$$\frac{1}{2}\Big[\frac{d}{dt}\xi(t)\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx-\xi'(t)\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx\Big] +\frac{\xi(t)}{2\tau}\int_\Omega z_n^2(x,1,t)dx-\frac{\xi(t)}{2\tau}\int_\Omega u_n'^2(x,t)dx=0. \tag{3.7}$$

Integrating (3.7) over $(0,t)$ we get

$$\begin{aligned}&\frac{1}{2}\Big[\xi(t)\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx-\int_0^t\int_\Omega\int_0^1\xi'(s)z_n^2(x,\rho,s)d\rho dxds\Big]\\&+\frac{1}{2\tau}\int_0^t\int_\Omega\xi(s)z_n^2(x,1,s)dxds\\&-\frac{1}{2\tau}\int_0^t\int_\Omega\xi(t)u_n'^2(x,t)dxds=\frac{\xi(0)}{2}\|f_0\|^2\end{aligned} \tag{3.8}$$

Combining (3.5) and (3.8) we find

$$\begin{aligned}&\frac{1}{2}\|u_n'\|_2^2+\Big(\frac{k_0}{2}-\frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u_n\|_2^2+\frac{1}{2}\xi(t)\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx\\&+\frac{\alpha}{2}(g\circ\nabla u_n)(t)-\frac{\alpha}{2}\int_0^t(g'\circ\nabla u_n)(s)ds+\frac{\alpha}{2}\int_0^t g(s)\|\nabla u_n\|_2^2ds\\&+\int_0^t\mu_1(s)\|u_n'\|_2^2ds+\int_0^t\mu_2(s)\int_\Omega z(x,1,s)u_n'(t)dxds\\&-\frac{1}{2}\int_\Omega\int_0^t\int_0^1\xi'(s)z_n^2(x,\rho,s)d\rho dxds+\frac{1}{2\tau}\int_0^t\int_\Omega\xi(s)z_n^2(x,1,s)dxds\\&-\frac{1}{2\tau}\int_0^t\int_\Omega\xi(t)u_n'^2(x,t)dxds=\frac{1}{2}\|u_{1n}\|^2+\frac{k_0}{2}\|\nabla u_{0n}\|^2+\frac{\xi(0)}{2}\|f_0\|^2\end{aligned} \tag{3.9}$$

Using Hölder's and Young's inequalities on the eighth term of (3.9) we obtain

$$\begin{aligned}\int_0^t\mu_2(s)\int_\Omega z(x,1,s)u_n'(t)dxds\le&\frac{1}{2}\int_0^t\mu_2(s)\int_\Omega z^2(x,1,s)dxds\\&+\frac{1}{2}\int_0^t\mu_2(s)\int_\Omega u_n'^2(t)dxds\end{aligned} \tag{3.10}$$

Then the equation (3.9) takes the form

$$\begin{aligned}&E_n(t)-\frac{\alpha}{2}\int_0^t(g'\circ\nabla u_n)(s)ds+\frac{1}{2}\int_0^t g(s)\|\nabla u_n\|_2^2ds\\&+\int_0^t\Big(\mu_1(s)-\frac{\xi(s)}{2\tau}-\frac{\mu_2(s)}{2}\Big)\|u_n'\|_2^2ds\\&+\frac{1}{2}\int_\Omega\int_0^t\int_0^1\xi'(s)z_n^2(x,\rho,s)d\rho dxds\\&+\int_0^t\Big(\frac{\xi(s)}{2\tau}-\frac{\mu_2(s)}{2}\Big)\int_\Omega z_n^2(x,1,s)dxds\le E_n(0),\end{aligned} \tag{3.11}$$

where

$$E_n(t) = \frac{1}{2}\|u_n'\|_2^2 + \Big(\frac{k_0}{2} - \frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u_n\|_2^2 + \frac{\alpha}{2}(g\circ\nabla u_n)(t) + \frac{1}{2}\xi(t)\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx, \tag{3.12}$$

and

$$E_n(0) = \frac{1}{2}\|u_{1n}\|^2 + \frac{k_0}{2}\|\nabla u_{0n}\|^2 + \frac{\xi(0)}{2}\|f_0\|^2. \tag{3.13}$$

Since u_{0n}, u_{1n}, f_0 converge, we can find a constant $L_1 > 0$ independent of n such that

$$\frac{1}{2}\|u_n'\|_2^2 + \Big(\frac{k_0}{2} - \frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u_n\|_2^2 + \frac{\alpha}{2}(g\circ\nabla u_n)(t) + \frac{1}{2}\xi(t)\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx \le L_1. \tag{3.14}$$

So this estimate gives

u_n is bounded in $\mathbb{L}^2(0,\infty;H_0^1(\Omega))$

u_n' is bounded in $\mathbb{L}^\infty(0,\infty;H_0^1(\Omega))$

z_n is bounded in $\mathbb{L}^\infty(0,\infty;L^2(\Omega)\times(0,1))$.

2. **Second estimate.** Multiplying the first equation in (2.7) by $u_n''(t)$ and summing with respect to i we obtain

$$\begin{aligned}&\|u_n''(t)\|^2 + k_0\int_\Omega \nabla u_n(t)\nabla u_n''(t)dx \\ &-\alpha\int_0^t g(t-s)\int_\Omega \nabla u_n(s)\nabla u_n''(t)dxds \\ &+\mu_1(t)\|u_n'(t)\|^2 + \mu_2(t)\int_\Omega z_n(x,1,t)u_n''(t)dx = 0\end{aligned} \tag{3.15}$$

Exploiting the Hölder's, Young's and Poincaré's inequalities and the assumptions $(H1),(H2)$ we have the following estimates

$$\Big|k_0\int_\Omega \nabla u_n(t)\nabla u_n''(t)dx\Big| \le k_0\eta\|\nabla u_n''(t)\|_2^2 + \frac{k_0}{4\eta}\|\nabla u_n(t)\|_2^2 \tag{3.16}$$

$$\Big|-\alpha\int_0^t g(t-s)\int_\Omega \nabla u_n(s)\nabla u_n''(t)dxds\Big| \le \alpha\eta\|\nabla u_n''(t)\|_2^2 + \frac{(k_0-l)}{4\eta}\int_0^t \|\nabla u_n(s)\|_2^2 ds \tag{3.17}$$

$$\left|\mu_2(t)\int_\Omega z_n(x,1,t)u_n''(t)dx\right| \le C_s\eta\|\nabla u_n''(t)\|_2^2 + \frac{\beta M}{4\eta}\int_\Omega z_n^2(x,1,t)dx \tag{3.18}$$

Substituting these three estimates into (3.15) and using (3.14) we deduce that

$$\begin{aligned}\|u_n''(t)\|^2 + \mu_1(t)\int_0^t \|u_n'(s)\|^2 ds &\le \eta(k_0+\alpha+C_s)\|\nabla u_n''(t)\|_2^2 \\ &\quad + \frac{(2k_0-l)}{4\eta}L_1 + \frac{\beta M}{4\eta}\int_\Omega z_n^2(x,1,t)dx\end{aligned} \tag{3.19}$$

We easily get the estimate

$$\begin{aligned}\|u_n''(t)\|^2 ds &\le \left(\frac{2k_0-l}{4\eta} + M + \frac{\beta M}{4\eta}\right)L_1 \\ &\quad + \eta(k_0+\alpha+C_s)\|\nabla u_n''(t)\|_2^2\end{aligned} \tag{3.20}$$

Choosing $\eta > 0$ small enough in (3.20), we obtain the second estimate below

$$\|u_n''(t)\|_2^2 \le L_2, \tag{3.21}$$

where L_2 is a positive constant independent of $n \in \mathbb{N}$ and $t \in [0,T)$. We observe for estimates (3.14) and (3.21) that

u_n is bounded in $\mathbb{L}^2(0,\infty;H_0^1(\Omega))$

u_n' is bounded in $\mathbb{L}^\infty(0,\infty;H_0^1(\Omega))$

u_n'' is bounded in $\mathbb{L}^\infty(0,\infty;H_0^1(\Omega))$

z_n is bounded in $\mathbb{L}^\infty(0,\infty;\mathbb{L}^2(\Omega)\times(0,1))$.

Applying Dunford Pettis theorem, we deduce that there exists a subsequence (u_i,z_i) of (u_n,z_n) and we can replace the subsequence (u_i,z_i) with the sequence (u_n,z_n) such that

$u_n \rightharpoonup u$ weak star in $\mathbb{L}^2(0,T;H_0^1(\Omega))$

$u_n' \rightharpoonup u'$ weak star in $\mathbb{L}^\infty(0,T;H_0^1(\Omega)$

$u_n'' \rightharpoonup u''$ weakly in $\mathbb{L}^\infty(0,T;H_0^1(\Omega))$

$z_n \rightharpoonup z$ weak star in $\mathbb{L}^\infty(0,T;\mathbb{L}^2(\Omega)\times(0,1))$

Moreover u_n'' is bounded in $\mathbb{L}^\infty(0,T;H_0^1(\Omega))$ then u_n'' is bounded in $\mathbb{L}^2(0,T;H_0^1(\Omega))$. The same method is used to prove that u_n' is bounded in $\mathbb{L}^2(0,T;H_0^1(\Omega))$.

Consequently u_n' is bounded in $H^1(0,T;H_0^1(\Omega))$. Furthermore, by Aubins-Lions theorem [5] there exists a subsequence (u_j) still represented by the same notation such that

$$u_j' \rightharpoonup u' \text{ strongly in } \mathbb{L}^2(0,T;H_0^1(\Omega))$$

which implies

$$\begin{aligned} u_j' &\rightharpoonup u' \text{ a.e. on } \Omega \times (0,T). \\ z_j &\rightharpoonup z \text{ a.e. on } \Omega \times (0,T). \end{aligned}$$

And we have for each $w_i \in \mathbb{L}^2(\Omega), v_i \in \mathbb{L}^2(\Omega)$

$$\int_\Omega \Big(u_j'' - k_0\Delta u_j + \alpha \int_\Omega g(t-s)\triangle u_j ds + \mu_1 u_j' + \mu_2 z_j\Big) w_i dx$$

$$\to \int_\Omega \Big(u'' - k_0\Delta u + \alpha \int_\Omega g(t-s)\triangle u ds + a_1 u' + a_2 z\Big) w_i,$$

and

$$\int_\Omega \tau(z_{jt} + z_{j\rho}) v_i dx \to \int_\Omega \tau(z_t + z_\rho) v_i dx.$$

When $j \to \infty$. Then, problem (1.1) admits a global weak solution u.

4. Asymptotic behavior

In this section, we shall investigate the asymptotic behavior of our problem. Our stability result, namely the exponential decay of the energy is obtained by the following theorem.

Theorem 4.1. *Let $(u_0, u_1, f_0) \in (H_0^1(\Omega) \times \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega \times (0,1)))$ be given. Assume that the assumptions (H1)-(H3) are fulfilled. Then for some positive constants K, k we obtain the following decay property*

$$E(t) \le E(0)e^{1-k\phi(t)}$$

Proof. Given $0 \le S < T < \infty$ arbitrarily. We multiply the first equation of (2.7) by $E^p\phi' u$, $p \in \mathbb{R}$ where ϕ is a function will be chosen later satisfying all the hypotheses of Lemma 2.1 and we integrate over $(S,T)\times\Omega$ we obtain

$$
\begin{aligned}
0 &= \int_S^T E^p\phi' \int_\Omega uu''(x,t)dxdt - k_0\int_S^T E^p\phi' \int_\Omega u\Delta u(x,t)dxdt \\
&+ \alpha\int_S^T E^p\phi' \int_\Omega\int_0^t g(t-s)\triangle u(x,s)udsdxdt \\
&+ \int_S^T E^p\phi'\mu_1(t)\int_\Omega uu'(x,t) + E^p\phi'\mu_2(t)\int_\Omega uu'(x,t-\tau)dxdt \\
&= \Big[E^p\phi'\int_\Omega uu'dx\Big]_S^T - \int_S^T (E^p\phi')\int_\Omega uu'dxdt - \int_S^T E^p\phi'\int_\Omega u'^2dxdt \\
&+ k_0\int_S^T E^p\phi'\int_\Omega |\nabla u|^2dxdt + \alpha\int_S^T E^p\phi'(g\circ\nabla u(x,t))dt \\
&- \frac{\alpha}{2}\int_S^T E^p\phi'\|\nabla u\|^2\int_0^t g(s)dsdt - \frac{\alpha}{2}\int_S^T E^p\phi'\int_0^t g(s)\|\nabla u\|^2dsdt \\
&+ \int_S^T E^p\phi'\mu_1(t)\int_\Omega uu'(x,t)dxdt + \int_S^T E^p\phi'\mu_2(t)\int_\Omega uu'(x,t-\tau)dx
\end{aligned}
\tag{4.1}
$$

Multiplying the second equation of (2.7) by $E^p\phi'\xi(t)e^{-2\tau\rho}z$ and integrating over $(S,T)\times\Omega\times(0,1)$ we find

$$
\begin{aligned}
0 &= \int_S^T\int_0^1 \tau E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega zz'dxd\rho dt \\
&+ \int_S^T E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega\int_0^1 zz_\rho d\rho dxdt \\
&= \frac{\tau}{2}\Big[\int_0^1 E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho\Big]_S^T \\
&- \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega (E^p\phi'\xi(t)e^{-2\tau\rho})'z^2dxd\rho dt \\
&+ \int_S^T E^p\phi'\int_\Omega\int_0^1 \xi(t)\Big(\frac{1}{2}\frac{d}{d\rho}(e^{-2\tau\rho}z^2) + \tau e^{-2\tau\rho}z^2\Big)d\rho dxdt \\
&= \frac{\tau}{2}\Big[\int_0^1 E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho\Big]_S^T \\
&- \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega (E^p\phi'\xi(t)e^{-2\tau\rho})'z^2dxd\rho dt \\
&+ \tau\int_S^T E^p\phi'\xi(t)\int_\Omega\int_0^1 e^{-2\tau\rho}z^2dxd\rho \\
&+ \frac{1}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega (e^{-2\tau}z^2(x,1,t) - z^2(x,0,t))dxdt.
\end{aligned}
\tag{4.2}
$$

Summing (4.1) and (4.2) and taking $A = \min(1, \tau e^{-2\tau})$ we get

$$
\begin{aligned}
A\int_S^T E^{p+1}\phi' dt \leq & -\Big[E^p\phi'\int_\Omega uu'\Big]_S^T + \int_S^T (E^p\phi')'\int_\Omega uu'dxdt \\
& + \frac{\alpha}{2}\int_S^T E^p\phi'\int_0^t g(s)\|\nabla u\|^2 dsdt - \int_S^T E^p\phi'\mu_1(t)\int_\Omega uu'(x,t)dxdt \\
& - \int_S^T E^p\phi'\mu_2(t)\int_\Omega uz(x,1,t)dx \\
& - \frac{\tau}{2}\Big[\int_0^1 E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2 dxd\rho\Big]_S^T \\
& + \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega e^{-2\tau\rho}(E^p\phi'\xi(t))'z^2 dxd\rho dt \\
& - \frac{1}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega (e^{-2\tau}z^2(x,1,t) - z^2(x,0,t))dxdt \\
& + \frac{3}{2}\int_S^T E^p\phi'\int_\Omega u'^2 dxdt
\end{aligned} \tag{4.3}
$$

Now assume that ϕ is a strictly increasing concave function. So ϕ' is a bounded function on $\mathbb{R}^+$. Denote λ the maximum of ϕ'. By the Cauchy Schwarz's, Young's and Poincaré's inequalities and the fact that ϕ' is bounded and since E is an increasing function, we have

$$
\Big|E^p\phi'\int_\Omega uu'(x,t)dx\Big| \leq \lambda c_1 E^{p+1}(t), \tag{4.4}
$$

where $c_1 = \max\left(1, \frac{C_s^2}{l}\right)$. From (4.4) we deduce the following estimates

$$
\begin{aligned}
\Big|\int_S^T (E^p\phi')'\int_\Omega uu'dxdt\Big| = & \Big|\int_S^T pE'E^{p-1}\phi'\int_\Omega uu'(x,t)dxdt \\
& + \int_S^T E^p\phi''\int_\Omega uu'(x,t)dxdt\Big| \\
& \leq \lambda c_1 p\int_S^T E^p(-E')dt + c_1 E^{p+1}(S)\int_S^T \phi''(t)dt \\
& \leq \lambda c_2 E(S)^{p+1},
\end{aligned} \tag{4.5}
$$

where $c_2 = c_1\max(p, 1)$ and

$$
\Big|\frac{\alpha}{2}\int_S^T E^p\phi'\int_0^t g(s)\|\nabla u\|^2 dsdt\Big| \leq \lambda c_3 E(S)^{p+1}, \tag{4.6}
$$

where $c_3 = \frac{(k_0 - l)}{l}$. By the hypothesis (H2), Young's and Poincaré's inequalities and

(4.4), we have

$$
\begin{aligned}
&\Big|\int_S^T E^p\phi'\mu_1(t)\int_\Omega uu'(x,t)dxdt\Big| \\
&\qquad\qquad\le \lambda M\beta E^{p+1}+\lambda\int_S^T E^p(-E')dt \\
&\qquad\qquad\le \lambda c_4 E^{p+1}(S),
\end{aligned}
\tag{4.7}
$$

where $c_4 = Mc_1$ and

$$
\Big|\int_S^T E^p\phi'\mu_2(t)\int_\Omega uz(x,1,t)dxdt| \le \lambda c_5 E^{p+1}(S),
\tag{4.8}
$$

where $c_5 = \max\left(\beta M\frac{c_s^2}{l}, 1\right)$ and

$$
\frac{\tau}{2}\int_S^T\int_0^1 E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho dt \le \tau\lambda E(^{p+1}S),
\tag{4.9}
$$

therefore

$$
\begin{aligned}
&\frac{\tau}{2}\int_S^T\int_0^1\int_\Omega e^{-2\tau\rho}(E^p\phi'\xi(t))'z^2dxd\rho dt \\
&= \frac{\tau}{2}\int_S^T\int_0^1 pE'E^{p-1}\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho dt \\
&+ \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega e^{-2\tau\rho}E^p\phi''\xi(t)z^2dxd\rho dt \\
&+ \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega e^{-2\tau\rho}E^p\phi'\xi'(t)z^2dxd\rho dt \\
&\le \lambda\tau p\int_S^T E^p(-E')dt + \tau E^{p+1}(s)\int_S^T \phi''(t)dt + \tau\lambda E^{p+1}(s) \\
&\le \lambda c_6 E(S)^{p+1},
\end{aligned}
\tag{4.10}
$$

where $c_6 = \tau\max(1,p)$ and

$$
\begin{aligned}
\frac{1}{2}\int_S^T E^p\xi(t)\int_\Omega e^{-2\tau}z^2(x,1,t)dxdt &\le \lambda\int_S^T E^p\xi(t)\int_\Omega z^2(x,1,t)dxdt \\
&\le \lambda\int_S^T E^p(-E')dt \\
&\le \lambda E(S)^{p+1}
\end{aligned}
\tag{4.11}
$$

and we have

$$\begin{aligned}&\frac{1}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega z^2(x,0,t)dxdt\\&=\frac{1}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega u'^2(x,t)dxdt\\&\leq \tau\lambda(2-\beta)E(S)^{p+1}\end{aligned} \tag{4.12}$$

$$\frac{3}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega u'^2dxdt \leq 3\lambda E^{p+1}(S). \tag{4.13}$$

From (4.3) and the estimates (4.5), (4.6), (4.8),(4.2), (4.9), (4.12) we obtain

$$\int_S^T E(t)^{p+1}\phi' dt \leq CE(S)^{p+1}, \tag{4.14}$$

where $C = \lambda \max(c_i, 3, \tau(2-\beta)),\ i = 1, ..., 6$. Applying the lemma 2.3 we get the decay property. This ends the proof of Theorem 4.1. □

5. Acknowledgements

The authors express their sincere thanks to the reviewers for their valuable suggestions. The authors are also thankful to the editor-in-chief for his important comments which helped to improve the presentation of the paper.

REFERENCES

1. A. Benaissa, A.Benguessoum and S. A. Messaoudi: *Energy decay of solutions for a wave equation with a constant weak delay and a weak internal feedback,* Electronic Journal of Qualitative Theory of Differential Equations, No. 11, (2014), 1-13.
2. R. Dacto, J. Lagnese and M. P. Polis: *An example on the effect of time delays in boundary feedback stabilization of wave equations,* SIAM J. Control Optim, 24(1986), 152–156.
3. M. Ferhat and A. Hakem: *On convexity for energy decay rates of a viscoelastic wave equation with a dynamic boundary and nonlinear delay term,* Facta. Univ. Ser. Math. Inform, Vol. 30, No 1 (2015), 67–87.
4. M. Kirane and S. H. Belkacem: *Existence and asymptotic stability of a viscoelastic wave equation with a delay,* Z. Angew. Math. Phys, DOI 10.1007/s00033-011-0145-0.
5. J. L. Lions: Quelques mthodes de rsolution des problmes aux limites non linaires, Dunod, Paris 1969.
6. Wenjun Liu: *General Decay Rate Estimate For The Energy Of A Weak Viscoelastic Equation With An Internal Time-Varying Delay Term,* Taiwanese Journal of Mathematics, Vol. 17, (**6**), (2013), 2101–2115.

7. P. Martinez: *A new method to obtain decay rate estimates for dissipative systems,* ESAIM Control Optim. Calc. Var, 4(1999), 419–444.

8. S. Nicaise and C. Pignotti: *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks,* SIAM J. Control Optim, 45(2006), 1561– 1585.

9. S. Nicaise and J. Valein: *Stabilization of second order evolution equations with unbounded feedback with delay,* ESAIM Control Optim. Calc. Var, 16(2010), 420–456.

10. S. Nicaise and C. Pignotti: *Stabilization of the wave equation with boundary or internal distributed delay,* Differ. Int. Equ, 21(2008), 935958. MR2483342.

11. S. Nicaise and C. Pignotti: *Interior Feedback Stabilization Of Wave Equation With Time Dependent Delay,* Electronic Journal of Differential Equations, Vol 2011, (**41**), (2011), 1-20.

12. F. Tahamtani and A. Peyravi: *Asymptotic Behavior And Blow-Up Of Solutions For A Nonlinear Viscoelastic Wave Equation With Boundary Dissipation,* Taiwanese Journal of Mathematics, Vol. 17, No. 6, (2013), 1921–1943.

13. C. Q. Xu, S.P. Yung and L. K. Li: *Stabilization of the wave system with input delay in the boundary control,* ESAIM Control Optim. Calc. Var, 12(2006), 770–785.

14. Z. Y. Zhang and X. J. Miao: *Global existence and uniform decay for wave equation with dissipative term and boundary damping,* Comput. Math. Appl.59(2),(2010), 1003–1018.

15. Z. Y. Zhang, X. J. Miao and D. M. Yu: *On solvability and stabilization of a class of hyperbolic hemivariational inequalities in elasticity,* Funkcialaj Ekvacioj 54, (2011), 297–314.

16. Zai-Yun Zhang, Zhen-Hai Liu and Xiang-Yang Gan: *Global existence and general decay for a nonlinear viscoelastic equation with nonlinear localized damping and velocity-dependent material density,* Applicable Analysis, 2012, 2021–2048.

17. Zai-yun ZHANG and Xiu-jin Miao: *Existence and asymptotic behavior of solutions to generalized Kirchhoff equation,* Nonlinear Studies, Vol. 19, n. 1, (2012), 57–70.

18. Zai-yun Zhang, Zhen-hai Liu, Xiu-jin Miao and Yue-zhong Chen: *Stability analysis of heat flow with boundary time-varying delay effect,* Nonliear Anal. TMA 73,(2010), 1878–1889.

19. Zai-yun Zhang, Zhen-hai Liu, Xiu-jin Miao and Yue-zhong Chen:*A note on decay properties for the solutions of A class of partial differential equation with memory,* Journal of Applied Mathematics and Computing, 37, (2010), 85–102.

20. Zai-Yun Zhang and Jian-Hua Huang: *On solvability of the dissipative Kirchhoff equation with nonlinear boundary damping,* Bulletin of the Korean Mathematical Society, 51(1), (2014), 189–206.

21. Z. Y. Zhang, Z. H. Liu, X. J. Miao and Y. Z. Chen: *Global existence and uniform stabi- lization of a generalized dissipative Klein-Gordon equation type with boundary damping,* Journal of Mathematics and Physics, 52 (2011), no. 2, 023502.

Remil Melouka
Laboratory ACEDP
Djillali Liabes university
22000 Sidi Bel Abbes, ALGERIA
benhocinemomo@yahoo.fr

Hakem Ali
Laboratory ACEDP
Djillali Liabes university
22000 Sidi Bel Abbes, ALGERIA
hakemali@yahoo.com