FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 32, No 5 (2017), 715–727
https://doi.org/10.22190/FUMI1705715C

HYERS-ULAM STABILITY FOR A SPECIAL CLASS OF FUNCTIONAL EQUATIONS

Ahmed Charifi, Radosław Łukasik and Muaadh Almahalebi

Abstract. In this paper, we investigate the stability in the sense of Hyers-Ulam for a class of the following type functional equations:

$$\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) = Nf(x)+h(y),\ x,y\in S$$

where $\mathbb{K}$ is a complete valued field of characteristic zero, F is a complete normed space (Archimedean or ultrametric) over $\mathbb{K}$, $(S,+)$ is an abelian monoid, $f,h\colon S\to F$, Φ is a finite automorphism group of S, N is the cardinality of Φ and $a_\lambda\in S$.

Keywords: Normed space; functional equations; abelian monoid

1. Introduction

The stability of functional equations was initiated by the question of S. M. Ulam asked in 1940 [33]. The first important result of the stability theory was given by D. H. Hyers [18] who answered the question of Ulam for the additive functional equation in Banach spaces. This result was extended and generalized in several ways by many authors worldwide, see for example, [1]-[4],[10]-[16],[19]-[24],[29]-[32].

The concept of p-adic numbers was introduced by the German mathematician, K. Hensel [17] as a tool for solving problems in algebra and number theory. It seems that Hensel's main motivation was the analogy between the ring of integers $\mathbb{Z}$ together with its field of fractions $\mathbb{Q}$ and the ring $\mathbb{C}[X]$ of polynomials with complex coefficients together with its field of fractions $\mathbb{C}(X)$.

Hensel [17] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many important applications ([3]-[5], [27]-[28]).

Non-Archimedean functional analysis is a fast-growing discipline widely used not just within pure mathematics but also applied in other sciences including physics,

Received April 15, 2017; accepted August 06, 2017
2010 *Mathematics Subject Classification.* Primary 20B25; Secondary 39B82

biology and chemistry. In the following, we will recall briefly some fundamentals that will be needed later on.

Definition 1.1. Let $\mathbb{K}$ be a field equipped with a function (valuation) $|\cdot|\colon \mathbb{K} \to [0,\infty)$ such that for each $r, s \in \mathbb{K}$, the following conditions hold:

1. $|r| = 0$ if and only if $r = 0$,
2. $|rs| = |r| \cdot |s|$,
3. $|r+s| \leq |r| + |s|$.

Then, the pair $(\mathbb{K}, |\cdot|)$ is called a *valued field.* Moreover, we change the condition (3) instead of the following condition

4. $|r+s| \leq \max\{|r|, |s|\}, \quad r, s \in \mathbb{K}$,

then the pair $(\mathbb{K}, |\cdot|)$ will be called a *non-Archimedean (or ultrametric) field.*

It is known that any complete Archimedean field is isomorphic to either the real or the complex numbers and the valuation is equivalent to the usual one.

Definition 1.2. Let X be a vector space over a valued field $\mathbb{K}$. A function $\|\cdot\|\colon X \to [0,\infty)$ is a *norm* if it satisfies the following conditions:

1. $||x|| = 0$ if and only if $x = 0$,
2. $||rx|| = |r| \cdot ||x||, \ r \in \mathbb{K}, x \in X$,
3. $||x+y|| \leq ||x|| + ||y||, \ x, y \in X$.

Then $(X, ||\cdot||)$ is called *normed space.*
Moreover, if $\mathbb{K}$ is a non-Archimedean field, then we change the condition (3) instead of the following condition

4. $\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad x, y \in X$.

Hence, $(X, ||\cdot||)$ will be called a *non-Archimedean (or ultrametric) normed space.*

Definition 1.3. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

1. The sequence $\{x_n\}$ is said to be *convergent* if, for any $\varepsilon > 0$, there is a positive integer N and $x \in X$ such that $\|x_n - x\| \leq \varepsilon$ for all $n \geq N$. Then the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$ which is denoted by $lim_{n\to\infty} x_n = x$;
2. A sequence$\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is *a Cauchy sequence* if and only if the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero;

3. If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space* or an *ultrametric Banach space.*

Example 1.1. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$ where $0 \leq a_k \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\left|\sum_{k\geq n_x}^{\infty} a_k p^k\right|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field.

The purpose of this paper is to keep continuity with our previous work in [6]-[9]. Indeed, we investigate the approximation of solutions of a class of Jensen type, quadratic type and Drygas type. These equations are extension forms of several equations, for examples,

$$f(x + y + a) = f(x) + h(y), \ x, y \in S, \tag{1.1}$$

$$f(x + y + a) + f(x - y + b) = 2f(x), \ x, y \in S, \tag{1.2}$$

$$f(x + y + a) + f(x + \sigma(y) + b) = 2f(x) + h(y), \ x, y \in S, \tag{1.3}$$

$$\sum_{\lambda\in\Phi} f(x + \lambda y) = Nf(x) + \sum_{\lambda\in\Phi} f(\lambda y), \ x, y \in S, \tag{1.4}$$

$$\sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) = Nf(x) + \sum_{\lambda\in\Phi} f(\lambda y), \ x, y \in S, \tag{1.5}$$

$$\sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) = Nf(x), \ x, y \in S \tag{1.6}$$

and

$$\sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) = Nf(x) + Nf(y), \ x, y \in S \tag{1.7}$$

where $(S, +)$ is an abelian monoid.

In a perspective of continuity of previous work, we study the stability for a class of the following type functional equations

$$\sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) = Nf(x) + h(y), \ x, y \in S. \tag{1.8}$$

2. BACKGROUND RESULTS

Throughout this paper, F is a complete normed space (Archimedean or ultrametric) over $\mathbb{K}$ which is a characteristic zero complete field and equipped with a non-trivial valuation $|\cdot|$, $(S,+)$ is an abelian monoid, Φ is a finite automorphism group of S and $N = |\Phi|$. In addition, when $\mathbb{K}$ is an ultrametric field, then there is a prime number p such that $|p| \neq 1$ and the mapping $x \to px,\ x \in S$ is assumed to be bijective on S. So, we can write $p^{-1}x \in S,\ x \in S$.

Let $n \in \mathbb{N}^*$ and $\mathcal{A}_n \colon S^n \to F$ be a function. Then we say that $\mathcal{A}_n$ is *n-additive* if and only if it is additive on each variable.
We say that $\mathcal{A}_n$ is *symmetric* if and only if

$$\mathcal{A}_n(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) = \mathcal{A}_n(x_1, x_2, ..., x_n)$$

for all $x_1, x_2, ..., x_n \in S$ and each permutation σ of $(1, 2, ..., n)$.

Let $k \in \mathbb{N}^*$ and $\mathcal{A}_k \colon S^k \to F$ be symmetric and k-additive, $\mathcal{A}_k^*(x) = \mathcal{A}_k(\underbrace{x, .., x}_{k})$ for all $x \in S$. Using additivity, we have $\mathcal{A}_k^*(rx) = r^k \mathcal{A}_k^*(x)$ whenever $x \in S$ and $r \in \mathbb{N}^*$. Also, we note that

$$\mathcal{A}_k(x+h, ..., x+h) = \sum_{i=0}^{k} \frac{k!}{(k-i)!i!} \mathcal{A}_k(\underbrace{x, ..., x}_{i}, \underbrace{h, ..., h}_{k-i}),\ x, h \in S.$$

The function $\mathcal{A}_k^*$ is called a *monomial function* of degree k associated to $\mathcal{A}_k$.

A function $p \colon S \to F$ is called a *generalized polynomial function* of degree n provided there exist $\mathcal{A}_0 \in F$ and monomial functions $\mathcal{A}_k^* \colon S \to F$ (for $1 \leq k \leq n$) such that

$$p(x) = \mathcal{A}_0 + \sum_{i=1}^{n} \mathcal{A}_k^*(x),$$

for all $x \in S$. We also need to recall the definition of the *linear difference operator* Δ_h on F^S by

$$\Delta_h f(x) = f(x+h) - f(x),\ x, h \in S.$$

Observe that these difference operators have important properties as the linearity property

$$\Delta_h(\alpha f + \beta g) = \alpha \Delta_h(f) + \beta \Delta_h(g),\ f, g \in F^S,\ \alpha, \beta \in \mathbb{K},$$

and the commutativity property

$$\Delta_{h_1} \Delta_{h_2} ... \Delta_{h_s} = \Delta_{h_1 h_2 ... h_s} = \Delta_{h_{\sigma(1)} h_{\sigma(2)} ... h_{\sigma(s)}},$$

where σ is a permutation of $(1, 2, ..., n)$. There is also other property as

$$\Delta_h^n f(x) = \sum_{i=1}^{n} (-1)^{n-i} \frac{n!}{(n-i)!i!} f(x+ih),\ x, h \in S.$$

At the end of this section, we come up with the following results.

Theorem 2.1. *Let $\delta \in \mathbb{R}_+$ and $f\colon S \to F$ satisfy the inequality*

$$\left\| \sum_{\lambda\in\Phi} f(x+\lambda y) - Nf(x) - \sum_{\lambda\in\Phi} f(\lambda y) \right\| \leq \delta,\ x,y\in S. \tag{2.1}$$

Then, there exists a unique generalized polynomial function $p\colon S \to F$ of degree at most N such that

1. *p is a solution of the equation*

$$\sum_{\lambda\in\Phi} p(x+\lambda y) = Np(x) + \sum_{\lambda\in\Phi} p(\lambda y),\ x,y\in S. \tag{2.2}$$

2. *In the ultrametric case, p satisfies the inequality*

$$\|f(x) - f(0) - p(x)\| \leq \frac{\delta}{|N|},\ x,y\in S$$

and in the Archimedean case, p satisfies the inequality

$$\|f(x) - f(0) - p(x)\| \leq \frac{(N+2)\delta}{N},\ x,y\in S.$$

For the Archimedean case, see [[26], Corollary 2.6] and [[25], Theorem 3]. Also, see [[5], Theorem 3.2] for the ultrametric case.

Lemma 2.1. *Let $\mathbb{K}$ be a valued field, X be a normed space over $\mathbb{K}$, $\delta \in \mathbb{R}_+$ and $P\colon S \to X$ be a generalized polynomial function. Suppose that*

$$\|P(x)\| \leq \delta,\ x\in S. \tag{2.3}$$

Then, P is a constant function.

Proof. Let

$$P(x) = \mathcal{A}_0 + \sum_{i=1}^{n} \mathcal{A}_i^*(x),\ x\in S,$$

where $\mathcal{A}_i^*\colon S \to X$ and $1 \leq i \leq n$, are monomial functions. Suppose that $\mathcal{A}_n$ is a nonzero function and p is a prime number such that $|p^\epsilon| < 1$ with $\epsilon \in \{-1,1\}$. Then we have

$$\begin{aligned}\|\mathcal{A}_n^*(x)\| &= \lim_{j\to\infty} |p^{\epsilon nj}| \left\|\mathcal{A}_n^*(p^{-\epsilon j}x)\right\| \\ &= \lim_{j\to\infty} |p^{\epsilon nj}| \left\|P(p^{-\epsilon j}x)\right\| \\ &\leq \lim_{j\to\infty} |p^{\epsilon nj}|\delta \\ &= 0,\ x\in S.\end{aligned}$$

Therefore, $P(x) = \mathcal{A}_0$, $x \in S$. □

3. MAIN RESULTS

In this section, we investigate the Hyers-Ulam stability of functional equations of the type (1.8) by using the operatorial approach. We present two cases for the space F. In the first part, we investigate the Hyers-Ulam stability of functional equations of the type (1.8) where F is an Archimedean normed space over a valued field $\mathbb{K}$ of characteristic zero and the second part gives us the Hyers-Ulam stability of functional equations of the type (1.8) on an ultrametric space F over an ultrametric valued field of characteristic zero. For this connection, see [14].

3.1. Part 1: Archimedean case

Theorem 3.1. *Let $a_\lambda \in S$, $\lambda \in \Phi$, $\delta \in \mathbb{R}_+$ and $f, h\colon S \to F$ satisfy the inequality*

$$\left\| \sum_{\lambda\in\Phi} f(x+\lambda y + a_\lambda) - Nf(x) - h(y) \right\| \le \delta, \;\; x, y \in S. \tag{3.1}$$

Then there exists a unique generalized polynomial function $p\colon S \to F$ of degree at most N such that

1. *$p(0) = 0$ and p is a solution of the equation*
$$\sum_{\lambda\in\Phi} p(x+\lambda y + a_\lambda) = Np(x) + \sum_{\lambda\in\Phi} p(\lambda y + a_\lambda), \;\; x, y \in S. \tag{3.2}$$
2. *p satisfies the conditions*
$$\|f(x) - f(0) - p(x)\| \le \frac{4(N+2)\delta}{N}, \;\; x \in S \tag{3.3}$$
and
$$\left\| h(x) - \sum_{\lambda\in\Phi} p(\lambda x + a_\lambda) \right\| \le (4N+9)\delta, \;\; x \in S. \tag{3.4}$$

Proof. Putting $y = 0$ and $x = 0$ in (3.1) respectively, we obtain that

$$\left\| \textstyle\sum_{\lambda\in\Phi} f(x + a_\lambda) - Nf(x) - h(0) \right\| \le \delta, \;\; x \in S, \tag{3.5}$$

$$\left\| \textstyle\sum_{\lambda\in\Phi} f(\lambda y + a_\lambda) - Nf(0) - h(y) \right\| \le \delta, \;\; y \in S. \tag{3.6}$$

Furthermore, using the inequalities (3.5) and (3.6), we get that

$$\begin{aligned}
N \cdot &\left\| \textstyle\sum_{\lambda\in\Phi} f(x+\lambda y) - Nf(x) + Nf(0) - \sum_{\lambda\in\Phi} f(\lambda y) \right\| \\
&\le \textstyle\sum_{\lambda\in\Phi} \left\| Nf(x+\lambda y) + h(0) - \sum_{\mu\in\Phi} f(x+\lambda y + a_\mu) \right\| \\
&\quad + \textstyle\sum_{\lambda\in\Phi} \left\| \sum_{\mu\in\Phi} f(x + \mu\lambda y + a_\mu) - Nf(x) - h(\lambda y) \right\| \\
&\quad + \textstyle\sum_{\lambda\in\Phi} \left\| h(\lambda y) + Nf(0) - \sum_{\mu\in\Phi} f(\mu\lambda y + a_\mu) \right\| \\
&\quad + \textstyle\sum_{\lambda\in\Phi} \left\| \sum_{\mu\in\Phi} f(\lambda y + a_\mu) - Nf(\lambda y) - h(0) \right\| \\
&\le 4N\delta, \;\; x, y \in S.
\end{aligned}$$

Thus, by putting $f_0 = f - f(0)$, we get that f_0 satisfies the inequality (2.1) with 4δ instead of δ. In view of Theorem 2.1, there exists a generalized polynomial function p of degree at most N such that p satisfies the equation (2.2) and

$$\|f(x) - f(0) - p(x)\| \leq \frac{4(N+2)\delta}{N}, \ x \in S. \tag{3.7}$$

By the inequalities (3.6) and (3.7), we have

$$\begin{aligned}\left\|h(y) - \sum_{\lambda\in\Phi} p(\lambda y + a_\lambda)\right\| &\leq \left\|h(y) + Nf(0) - \textstyle\sum_{\lambda\in\Phi} f(\lambda y + a_\lambda)\right\| \\ &\quad + \textstyle\sum_{\lambda\in\Phi} \|f(\lambda y + a_\lambda) - f(0) - p(\lambda y + a_\lambda)\| \\ &\leq \delta + N\tfrac{4(N+2)\delta}{N} \\ &= (4N+9)\delta, \ \ y \in S.\end{aligned}$$

To prove that p is a solution of (3.2), we observe that

$$\begin{aligned}&\left\|\textstyle\sum_{\lambda\in\Phi} p(x + \lambda y + a_\lambda) - Np(x) - \sum_{\lambda\in\Phi} p(\lambda y + a_\lambda)\right\| \\ &\leq \textstyle\sum_{\lambda\in\Phi} \|p(x + \lambda y + a_\lambda) + f(0) - f(x + \lambda y + a_\lambda)\| \\ &+N\|f(x) - f(0) - p(x)\| + \left\|h(y) - \textstyle\sum_{\lambda\in\Phi} p(\lambda y + a_\lambda)\right\| \\ &+ \left\|\textstyle\sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) - Nf(x) - h(y)\right\| \\ &\leq (4N+2)\delta + (4N+2)\delta + (4N+9)\delta + \delta \\ &= (12N+14)\delta, \ \ x, y \in S.\end{aligned}$$

In the view of Lemma 2.1, for fixed $y \in S$, we have

$$\begin{aligned}&\textstyle\sum_{\lambda\in\Phi} p(x + \lambda y + a_\lambda) - Np(x) - \sum_{\lambda\in\Phi} p(\lambda y + a_\lambda) \\ &= \textstyle\sum_{\lambda\in\Phi} p(0 + \lambda y + a_\lambda) - Np(0) - \sum_{\lambda\in\Phi} p(\lambda y + a_\lambda) \\ &= 0, \ x \in S.\end{aligned}$$

To prove the uniqueness of p, let $\overline{p}$ be a generalized polynomial function such that $\overline{p}(0) = 0$ and satisfies the inequality (3.3). Then we get that

$$\begin{aligned}\|p(x) - \overline{p}(x)\| &= \|p(x) - f(0) - f(x) + f(x) + f(0) - \overline{p}(x)\| \\ &\leq \|p(x) - f(0) - f(x)\| + \|f(x) + f(0) - \overline{p}(x)\| \\ &\leq \tfrac{8(N+2)\delta}{N}, \ \ x \in S\end{aligned}$$

and according to Lemma 2.1, we get that $p - \overline{p}$ is constant which ends the proof. □

Corollary 3.1. *Let $a_\lambda \in S$, $\lambda \in \Phi$, $\delta \in \mathbb{R}_+$ and $f\colon S \to F$ satisfy the following inequality*

$$\left\|\sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) - Nf(x)\right\| \leq \delta, \ x, y \in S. \tag{3.8}$$

Then there exists, up to a constant, a unique generalized polynomial function $P\colon S \to F$ of degree at most $N-1$ such that P is a solution of the equation (1.6) and satisfies the following inequality

$$\|f(x) - P(x)\| \le \frac{4(N+2)\delta}{N}, \quad x \in S.$$

Proof. According to Theorem 3.1, there exists a unique generalized polynomial function P of degree at most N such that $P(0) = 0$ and satisfies the condition (3.3). And observe that the inequality (3.8) shows that $\| \sum_{\lambda\in\Phi}(f - f(0))(\lambda x)\|$ is bounded on S. Then, in view of [[6], Lemma 3.1] which is also valid in Archimedean case with $2^N \frac{\delta}{N}$ instead of $\frac{\delta}{|N|}$, the expression $\|\Delta_y^N(f - f(0))(x)\|$ is bounded on S. Consequently, according to Lemma 2.1, the generalized polynomial function P is of degree at most $N - 1$. □

Corollary 3.2. *Let $a_\lambda \in S$, $\lambda \in \Phi$, $\delta \in \mathbb{R}_+$ and $f\colon S \to F$ satisfy the following inequality*

$$\left\| \sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) - Nf(x) - \sum_{\lambda\in\Phi} f(\lambda y) \right\| \le \delta, \ x, y \in S.$$

Then there exists a unique generalized polynomial function $P\colon S \to F$ of degree at most N such that

1. *P is a solution of equation (1.5).*
2. *P satisfies the inequality*

$$\|f(x) - P(x)\| \le \frac{4(N+2)\delta}{N}, \ x \in S.$$

Corollary 3.3. *Let $a_\lambda \in S$, $\lambda \in \Phi$, $\delta \in \mathbb{R}_+$ and $f\colon S \to F$ satisfy the following inequality*

$$\left\| \sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) - Nf(x) - Nf(y) \right\| \le \delta, \ x, y \in S.$$

Then there exists a unique generalized polynomial function $P\colon S \to F$ of degree at most N such that P is a solution of equation (1.7) and satisfies the following inequality

$$\|f(x) - P(x)\| \le \frac{4(N+2)\delta}{N}, \ x \in S.$$

3.2. Non-Archimedean case

Theorem 3.2. *Let Φ be a finite automorphism group of S, $N = card\Phi$, $a_\lambda \in S$, $\lambda \in \Phi$, $\delta \in \mathbb{R}_+$ and $f, h\colon S \to F$ satisfy the inequality (3.1). Then there exists a unique generalized polynomial function $p\colon S \to F$ of degree at most N such that*

1. $p(0) = 0$ and p is a solution of equation (3.2).

2. p satisfies the conditions

$$\|f(x) - f(0) - p(x)\| \le \frac{\delta}{|N|^2}, \; x \in S \tag{3.9}$$

and

$$\left\| h(x) - \sum_{\lambda\in\Phi} p(\lambda x + a_\lambda) \right\| \le \frac{\delta}{|N|^2}, \; x \in S.$$

Proof. Putting $y = 0$ and $x = 0$ in (3.1) respectively, we obtain (3.5) and (3.6). Furthermore, by using these inequalities we get that

$$\begin{aligned}
&|N| \cdot \left\|\sum_{\lambda\in\Phi} f(x+\lambda y) - Nf(x) + Nf(0) - \sum_{\lambda\in\Phi} f(\lambda y)\right\| \\
&\le \max \left\{ \max_{\lambda\in\Phi} \left\| Nf(x+\lambda y) + h(0) - \sum_{\mu\in\Phi} f(x+\lambda y + a_\mu) \right\|, \right. \\
&\qquad \max_{\lambda\in\Phi} \left\| \sum_{\mu\in\Phi} f(x+\mu\lambda y + a_\mu) - Nf(x) - h(\lambda y) \right\|, \\
&\qquad \max_{\lambda\in\Phi} \left\| h(\lambda y) + Nf(0) - \sum_{\mu\in\Phi} f(\mu\lambda y + a_\mu) \right\|, \\
&\qquad \left. \max_{\lambda\in\Phi} \left\| \sum_{\mu\in\Phi} f(\lambda y + a_\mu) - Nf(\lambda y) - h(0) \right\| \right\} \\
&\le \delta, \quad x, y \in S.
\end{aligned}$$

Thus, by putting $f_0 = f - f(0)$, we get that f_0 satisfies the inequality (2.1) with $\frac{\delta}{|N|}$ instead of δ. In the view of Theorem 2.1, there exists a generalized polynomial function p of degree at most N such that p satisfies the equation (2.2) and

$$\|f(x) - f(0) - p(x)\| \le \frac{\delta}{|N|^2}, \; x \in S. \tag{3.10}$$

By the inequalities (3.6) and (3.10), we have

$$\begin{aligned}
&\left\| h(y) - \sum_{\lambda\in\Phi} p(\lambda y + a_\lambda) \right\| \le \max \left\{ \left\| h(y) + Nf(0) - \sum_{\lambda\in\Phi} f(\lambda y + a_\lambda) \right\|, \right. \\
&\left. \max_{\lambda\in\Phi} \| f(\lambda y + a_\lambda) - f(0) - p(\lambda y + a_\lambda) \| \right\} \le \max\left(\delta, \frac{\delta}{|N|^2}\right) = \frac{\delta}{|N|^2}, \; y \in S.
\end{aligned}$$

Next, we prove that p is solution of (3.2). Indeed, we observe that

$$\begin{aligned}
&\left\|\sum_{\lambda\in\Phi} p(x+\lambda y+a_\lambda) - Np(x) - \sum_{\lambda\in\Phi} p(\lambda y+a_\lambda)\right\| \\
&\le \max\Bigg\{ \max_{\lambda\in\Phi} \|p(x+\lambda y+a_\lambda)+f(0)-f(x+\lambda y+a_\lambda)\|, \\
&\quad |N|\cdot\|f(x)-f(0)-p(x)\|, \left\|h(y)-\sum_{\lambda\in\Phi} p(\lambda y+a_\lambda)\right\|, \\
&\quad \left\|\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda)-Nf(x)-h(y)\right\| \Bigg\} \\
&\le \max\left(\tfrac{\delta}{|N|^2}, \tfrac{\delta}{|N|}, \tfrac{\delta}{|N|^2}, \delta\right) = \tfrac{\delta}{|N|^2}, \ \ x,y\in S.
\end{aligned}$$

In the view of Lemma 2.1, for fixed $y \in S$, we have

$$\begin{aligned}
&\sum_{\lambda\in\Phi} p(x+\lambda y+a_\lambda) - Np(x) - \sum_{\lambda\in\Phi} p(\lambda y+a_\lambda) \\
&= \sum_{\lambda\in\Phi} p(0+\lambda y+a_\lambda) - Np(0) - \sum_{\lambda\in\Phi} p(\lambda y+a_\lambda) = 0, \ \ x\in S.
\end{aligned}$$

To prove the uniqueness of p, let $\overline{p}$ be a generalized polynomial function such that $\overline{p}(0) = 0$ and satisfies the inequality (3.9). Then we get that

$$\begin{aligned}
&\|p(x)-\overline{p}(x)\| = \|p(x)+f(0)-f(x)+f(x)-f(0)-\overline{p}(x)\| \\
&\le \max\{\|p(x)+f(0)-f(x)\|, \|f(x)-f(0)-\overline{p}(x)\|\} \le \tfrac{\delta}{|N|^2}, \ \ x\in S
\end{aligned}$$

and according to Lemma 2.1, we get that $p-\overline{p}$ is constant which ends the proof. □

Corollary 3.4. *Let Φ be a finite subgroup of the group of automorphisms of S , $N = card(\Phi)$, $a_\lambda \in S$, $(\lambda \in \Phi)$ and $\delta \in \mathbb{R}^+$. Assume that $f : S \to F$ satisfies the following inequality*

$$\left\|\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) - Nf(x)\right\| \le \delta,$$

for all $x, y \in S$. Then there exists, up to a constant, a unique generalized polynomial function $P : S \to F$ of degree at most $N-1$ such that P is a solution of the equation (1.6) and

$$\|f(x)-P(x)\| \le \frac{\delta}{|N|^2} \ \ \text{for all } x\in S.$$

Proof. The proof is analogous to the proof of Corollary 3.2. □

Corollary 3.5. *Let Φ be a finite automorphism group of S, $N = card\Phi$, $a_\lambda \in S$ $(\lambda \in \Phi)$, $\delta \in \mathbb{R}^+$ and $f \in F^S$ such that*

$$\left\|\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) - Nf(x) - \sum_{\lambda\in\Phi} f(\lambda y)\right\| \le \delta, \ \ x,y\in S.$$

Then there exists a unique generalized polynomial function $P \in F^S$ of degree at most N such that

1. P is a solution of the equation

$$\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) = Nf(x) + \sum_{\lambda\in\Phi} f(\lambda y), \quad x, y \in S.$$

2. P satisfies the inequality

$$\|f(x) - P(x)\| \leq \frac{\delta}{|N|^2}, \quad x \in S.$$

Corollary 3.6. *Let Φ be a finite subgroup of the group of automorphisms of S, $N = card(\Phi)$, $a_\lambda \in S$ and $(\lambda \in \Phi)$. Assume that $f : S \to F$ satisfies the following inequality*

$$\left\| \sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) - Nf(x) - Nf(y) \right\| \leq \delta,$$

for all $x, y \in S$. Then there exists a unique generalized polynomial function $P : S \to F$ of degree at most N such that P is a solution of the equation (1.7) and

$$\|f(x) - P(x)\| \leq \frac{\delta}{|N|^2} \quad \textit{for all } x \in S.$$

REFERENCES

1. J. Aczél, J. K. Chung and C. T. Ng, *Symmetric second differences in product form on groups*, in: Th. M. Rassias (Ed.), Topics in mathematical analysis, 1-22, Ser. Pure Math. **11**, World Scientific, Singapore-Teaneck, N.Y., 1989.
2. T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**, 1950, 64-66.
3. L. M. Arriola and W. A. Beyer, *Stability of the Cauchy functional equation over p-adic fields*, Real Analysis Exchange, **31** (1), 2005, 125-132.
4. J. Baker, *A general functional equation and its stability*, Proc. Amer. Math. Soc., **133**(6), 2005, 1657-1664.
5. A. Chahbi, A. Charifi, B. Bouikhalene and S. Kabbaj, *Operatorial approach to the non-Archimedean stability of a Pexider K-quadratic functional equation*, Arab J. Math. Sci., **21** (1), 2015, 67-83.
6. A. Charifi, M. Almahalebi, J. M. Rassias and S. Kabbaj, *Ultrametric approximations of a Drygas functional equation*, Adv. Pure Appl. Math., **7** (3), 2016, 155-164.
7. A. Charifi, B. Bouikhalene and E. Elqorachi, *Hyers-Ulam-Rassias stability of a generalized Pexider functional equation*, Banach J. Math. Anal., **1** (2), 2007, 176-185.
8. A. Charifi, B. Bouikhalene, E. Elqorachi and A. Redouani, *Hyers-Ulam-Rassias stability of a generalized Jensen functional equation*, Austral. J. Math. Anal. Appl. , **6** (1) , 2009, 1-16.
9. A. Charifi, S. Kabbaj and D. Zeglami, *Non-Archimedian stability of generalized Jensen's and quadratic equations,* Acta Universitatis Apulensis, **45**, 2016, 11-29.

10. P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math., **27**, 1984, 76-86.
11. J. K. Chung, B. R. Ebanks, C. T. Ng and P. K. Sahoo, *On a quadratic trigonometric functional equation and some applications*, Trans. Amer. Math. Soc., **347** (4), 1995, 1131-1161.
12. S. Czerwik, *On the stability of the quadratic mapping in normed spaces.* Abh. Math. Sem. Univ. Hamburg, **62**, 1992, 59-64.
13. B. R. Ebanks, P. L. Kannappan and P. K. Sahoo, *A common generalization of functional equations characterizing normed and quasi-inner-product spaces*, Canad. Math. Bull., **35**, 1992, 321-327.
14. G. Z. Eskandani, H. Vaezi and Y. N. Dehghan, *Stability of a mixed additive and quadratic functional equation in non-Archimedean Banach modules*, Taiwanese J. Math. **14** (4), 2010, 1309-1324 .
15. Z. Gajda, On the stability of additive mappings, Internat. J. Math. Sci., **14**, 1991, 431-434.
16. P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (3), 1994, 431-436.
17. K. Hensel, *Über eine neue Begrundung der Theorie der algebraischen Zahlen*, Jahresber. Deutsch. Math. Verein, **6**, 1897, 83-88.
18. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27**, 1941, 222-224.
19. D. H. Hyers, *Transformations with bounded n-th differences*, Pacific J. Math., **11**, 1961, 591-602.
20. D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math., **44**, 1992, 125-153.
21. D. H. Hyers, G. I. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
22. K. W. Jun and Y. H. Lee, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl., **238**, 1999, 305-315.
23. C. F. K. Jung, *On generalized complete metric spaces*, Bull. Amer. Math. Soc. , **75**, 1969, 113-116.
24. P. L. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math., **27** (3-4), 1995, 368-372.
25. R. Łukasik, *Some generalization of Cauchy's and the quadratic functional equations*, Aequationes Math., **83**, 2012, 75-86.
26. R. Łukasik, *The stability of some generalization of the quadratic functional equation*, Georgian Math. J., **21** (4), 2014, 463-474
27. S. Mazur and W. Orlicz, *Grundlegende Eigenschaften der Polynomischen Operationen*, Erst Mitteilung, Studia Math., **5**, 1934, 50-68.
28. M. S. Moslehian and Th. M. Rassias, *Stability of functional equations in non-Archimedea spaces*, Appl. Anal. Discrete Math., **1**, 2007, 325-334.
29. J. M. Rassias, *Solution of a problem of Ulam*, J. Approx. Theory, **57**, 1989, 268-273.
30. J. M. Rassias, *On the Ulam stability of mixed type mappings on restricted domains*, J. Math. Anal. Appl., **276** (2), 2002, 747-762.

31. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**, 1978, 297-300.

32. Th. M. Rassias, *Handbook of functional equations: stability theory*, Springer, 2014.

33. S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publ. New York, 1961.

Ahmed Charifi
Faculty of Science
Department of Mathematics
University of Ibn Tofail
BP 133 Kenitra, Morocco.
charifi2000@yahoo.fr

Radosław Łukasik
Institute of Mathematics
University of Silesia ul. Bankowa
14 40-007 Katowice, Poland.
rlukasik@math.us.edu.pl

Muaadh Almahalebi
Faculty of Science
Department of Mathematics
University of Ibn Tofail
BP 133 Kenitra, Morocco.
muaadh1979@hotmail.fr