EXISTENCE OF SOLUTIONS FOR NONLINEAR IMPULSIVE DYNAMIC EQUATIONS ON A TIME SCALE

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Abstract. Let $\mathbb{T}$ be a time scale such that $0, t_i, T \in \mathbb{T}$, $i = 1, 2, \ldots, n$, and $0 < t_i < t_{i+1}$. Assume each t_i is dense. Using a fixed point theorem due to Krasnoselskii-Burton, we show that the nonlinear impulsive dynamic equation

$$\begin{cases} y^\Delta(t) = -a(t)h\left(y^\sigma(t)\right) + f(t, y(t)), \ t \in (0, T], \\ y(0) = 0, \\ y(t_i^+) = y(t_i^-) + I(t_i, y(t_i)), \ i = 1, 2, \ldots, n, \end{cases}$$

where $y(t_i^\pm) = \lim_{t \to t_i^\pm} y(t)$, and y^Δ is the Δ-derivative on $\mathbb{T}$, has a solution.

Keywords: Fixed point, large contraction, time scales, nonlinear impulsive dynamic equations.

1. Introduction

Let $\mathbb{T}$ be a time scale such that $0, t_i, T \in \mathbb{T}$, for $i = 1, 2, \ldots, n$, $0 < t_i < t_{i+1}$, and assume that t_i is dense in $\mathbb{T}$ for each $i = 1, 2, \ldots, n$. In this paper, we are interested in the analysis of qualitative theory of solutions to impulsive dynamic equations. Motivated by the papers [1]–[4], [8]–[15], [19] and the references therein, we consider the following totally nonlinear impulsive dynamic equation

$$(1.1) \qquad \begin{cases} y^\Delta(t) = -a(t)h\left(y^\sigma(t)\right) + f(t, y(t)), \ t \in (0, T], \\ y(0) = 0, \\ y(t_i^+) = y(t_i^-) + I(t_i, y(t_i)), \ i = 1, 2, \ldots, n, \end{cases}$$

where $y(t_i^\pm) = \lim_{t \to t_i^\pm} y(t)$, $y(t_i) = y(t_i^-)$, $a : \mathbb{T} \to \mathbb{R}$ is a positive rd-continuous real-valued function, $h : \mathbb{R} \to \mathbb{R}$ is continuous function, $f, I : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ satisfying the Lipschitz condition, $[0, T] = \{t \in \mathbb{T} : 0 \le t \le T\}$. Note that the intervals $[\alpha, \beta)$, $(\alpha, \beta]$ and (α, β) are defined similarly.

In 1988, Stephan Hilger [12] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger's initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems; see [6, 7] and references therein. The study of impulsive initial and boundary value problems is extensive. For the theory and classical results, we direct the reader to the monographs [5, 16, 17].

Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [8] Theorem 3) to show the existence of solutions on time scales for equation (1.1). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [8], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (1.1) into an integral equation written as a sum of two mapping; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii fixed point theorem, to show the existence of a solution for equation (1.1).

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the modification of Krasnoselskii's fixed point theorem established by Burton (see ([8] Theorem 3) and [9]). For details on Krasnoselskii's theorem we refer the reader to [18]. We present our main results in Section 3.

2. Preliminaries

The reader can find more details on the materials and basic properties used here in the first chapter of Bohner and Peterson book [6] and can find good examples on dynamic equations in Chapter 2 of [7].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{t \in \mathbb{T} : s < t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\triangle}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$. $f \in C_{rd}^1 = C_{rd}^1(\mathbb{T}, \mathbb{R})$ if $f^{\triangle} \in C_{rd}(\mathbb{T}, \mathbb{R})$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1. *([6, Theorem 1.20]) Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f+g)^{\triangle}(t) = g^{\triangle}(t) + f^{\triangle}(t)$.

(ii) $(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.

(iii) *The product rules*

$$\begin{aligned} (fg)^{\triangle}(t) &= f^{\triangle}(t)\,g(t) + f^{\sigma}(t)\,g^{\triangle}(t), \\ (fg)^{\triangle}(t) &= f(t)\,g^{\triangle}(t) + f^{\triangle}(t)\,g^{\sigma}(t). \end{aligned}$$

(iv) *If $g(t)\,g^{\sigma}(t) \neq 0$ then*

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t)\,g(t) - f(t)\,g^{\triangle}(t)}{g(t)\,g^{\sigma}(t)}.$$

The next theorem is the chain rule on time scales ([6, Theorem 1.93]).

Theorem 2.2. *(Chain Rule) Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then $(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right)\nu^{\triangle}$.*

In the sequel we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(\nu(t))$ where, $\nu(t) := t - \tau(t)$. Our next theorem is the substitution rule ([6, Theorem 1.98]).

Theorem 2.3. *(Substitution) Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(T)$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in T$,*

$$\int_a^b f(t)\,\nu^{\triangle}(t)\,\triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\triangle}s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)\,p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)\,f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log\left(1 + \mu(z)p(z)\right) \triangle z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\triangle = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.1. *([6]) Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

(iii) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1 + \mu(t)p(t)}$;

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p^\triangle(.,s) = p e_p(.,s)$ *and* $\left(\dfrac{1}{e_p(.,s)}\right)^\triangle = -\dfrac{p(t)}{e_p^\sigma(.,s)}$.

Lemma 2.2. *([1]) If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\triangle u\right), \ \forall t \in \mathbb{T}.$$

Corollary 2.1. *([1]) If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\triangle u\right) < 1.$$

Krasnoselskii (see [8] or [18]) combined the contraction mapping theorem and Schauder's theorem and formulated the following hybrid and attractive result.

Theorem 2.4. *Let M be a closed convex nonempty subset of a Banach space $(S, \|.\|)$. Suppose that A and B map M into S such that*

(i) $\forall x, y \in M \Rightarrow Ax + By \in M$,

(ii) *A is continuous and AM is contained in a compact set,*

(iii) *B is a contraction with constant $\alpha < 1$.*

Then there is a $z \in M$ with $z = Az + Bz$.

This is a captivating result and has a number of interesting applications. In recent year much attention has been paid to this theorem. Burton [8] observed that Krasnoselskii result can be more interesting in applications with certain changes and formulated in Theorem 2.6 below (see [8] for the proof).

Definition 2.1. ([8]) Let (M, d) be a metric space and $B : M \to M$. B is said to be a large contraction if $\varphi, \psi \in M$, with $\varphi \neq \psi$ then $d(B\varphi, B\psi) < d(\varphi, \psi)$ and if for all $\varepsilon > 0$ there exists $\delta < 1$ such that

$$[\varphi, \psi \in M, \ d(\varphi, \psi) \geq \varepsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

Theorem 2.5. *([8]) Let (M, d) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and $L > 0$, such that $d(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.*

Theorem 2.6. *(Krasnoselskii-Burton) Let M be a closed bounded convex non-empty subset of a Banach space $(S, \|.\|)$. Suppose that A, B map M into M and that*

(i) $\forall x, y \in M \Rightarrow Ax + By \in M$,

(ii) A is continuous and AM is contained in a compact subset of M,

(iii) B is a large contraction.

Then there is a $z \in M$ with $z = Az + Bz$.

It is obvious that if we want to apply the above theorem we need to construct two mappings, one is large contraction and the other is compact.

3.   Existence of solutions

We will state and prove our main results in this section.

Define $t_{n+1} \equiv T$ and let $J_0 = [0, t_1]$ and for $k = 1, 2, \ldots, n$, let $J_k = (t_k, t_{k+1}]$. Define

$$PC = \{y : [0, T] \to \mathbb{R} \,|\, y \in C_{rd}(J_k), \ y(t_k^{\pm}) \text{ exist and } y(t_k^-) = y(t_k), \ k = 1, \ldots, n\},$$

and

$$PC^1 = \{y : [0, T] \to \mathbb{R} \,|\, y \in C_{rd}^1(J_k), \ k = 1, \ldots, n\},$$

where $C_{rd}(J_k)$ is the space of all real valued rd-continuous functions on J_k and $C_{rd}^1(J_k)$ is the space of all rd-continuously delta-differentiable functions on J_k. The set PC is a Banach space when it is endowed with the supremum norm

$$\|u\| = \max_{0 \leq k \leq n} \{\|u\|_k\},$$

where $\|u\|_k = \sup_{t \in J_k} |u(t)|$.

In this paper, we make the following conditions

(A) $a \in \mathcal{R}^+$ is rd-continuous and $a(t) > 0$ for all $t \in \mathbb{T}$.

(F) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ and there exists a positive constant k_1 such that

(3.1) $|f(t,x) - f(t,y)| \le k_1 \|x - y\|$, for $x, y \in \mathbb{R}$.

    (I) There exists a positive constant k_2 such that

(3.2) $|I(t,x) - I(t,y)| \le k_2 \|x - y\|$, for $x, y \in \mathbb{R}$.

Lemma 3.1. *The function $y \in PC^1$ is a solution of equation (1.1) if and only if $y \in PC$ is a solution of*

$$y(t) = \int_0^t e_{\ominus a}(t,s)a(s)H(y(s))\Delta s$$

(3.3)
$$+ \int_0^t e_{\ominus a}(t,s)f(s,y(s))\Delta s + \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i)I(t_i,y(t_i)),$$

where

(3.4) $H(y(s)) = y^\sigma(s) - h(y^\sigma(s))$.

Proof. For $t \in J_0$, the solution of (1.1) satisfying $y(0) = 0$ is

$$y(t) = \int_0^t e_{\ominus a}(t,s)a(s)H(y(s))\Delta s + \int_0^t e_{\ominus a}(t,0)f(s,y(s))\Delta s.$$

See [6] for details. To find the solution of (1.1) on J_1 we consider the initial value problem

$$y^\Delta(t) = -a(t)h(y^\sigma(t)) + f(t,y(t)), \ t \in J_1,$$
$$y(t_1^+) = \int_0^{t_1} e_{\ominus a}(t_1,s)a(s)H(y(s))\Delta s$$
$$+ \int_0^{t_1} e_{\ominus a}(t_1,s)f(s,y(s))\Delta s + I(t_1,y(t_1)).$$

The solution to this initial value problem is

$$y(t) = \int_0^t e_{\ominus a}(t,s)a(s)H(y(s))\Delta s$$
$$+ \int_0^t e_{\ominus a}(t,s)f(s,y(s))\Delta s + e_{\ominus a}(t,t_1)I(t_1,y(t_1)).$$

We proceed inductively to obtain that if $y \in PC^1$ is a solution of (1.1), then $y \in PC$ is a solution of

$$y(t) = \int_0^t e_{\ominus a}(t,s)a(s)H(y(s))\Delta s$$
$$+ \int_0^t e_{\ominus a}(t,s)f(s,y(s))\Delta s + \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i)I(t_i,y(t_i)).$$

The converse statement follows trivially and the proof is complete. □

To apply Theorem 2.6, we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset M_L of $\mathbb{B}$ and construct two mappings, one is a large contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (PC, \|.\|)$ and $M_L = \{\varphi \in \mathbb{B} : \|\varphi\| \leq L\}$, where L is positive constant. We express equation (3.3) as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) = (\mathbb{C}\varphi)(t),$$

where $A, B : M_L \to \mathbb{B}$ are defined by

$$(3.5) \qquad (A\varphi)(t) = \int_0^t e_{\ominus a}(t,s)f(s,\varphi(s))\Delta s + \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i)I\left(t_i, \varphi(t_i)\right),$$

and

$$(3.6) \qquad (B\varphi)(t) = \int_0^t e_{\ominus a}(t,s)a\left(s\right)H(\varphi(s))\Delta s.$$

We need the following assumptions

$$(3.7) \qquad k_1 L + |f\left(t,0\right)| \leq \alpha L a\left(t\right),$$

$$(3.8) \qquad nk_2 L + \sum_{i=1}^{n} |I\left(t_i, 0\right)| \leq \beta L,$$

$$(3.9) \qquad J\left(\alpha + \beta\right) \leq 1,$$

$$(3.10) \qquad \max\left(|H\left(-L\right)|, |H\left(L\right)|\right) \leq \frac{2L}{J},$$

where α, β and J are constants with $J \geq 3$.

We begin with the following proposition (see [1]) and for convenience we present, below, its proof. In the next proposition we prove that, for a well chosen function h, the mapping H in (3.4) is a large contraction on M_L. So, let us make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous on $U_L = [-L, L]$ and differentiable on $(-L, L)$.

(H2) h is strictly increasing on U_L.

(H3) $\sup\limits_{s \in (-L,L)} h'\left(s\right) \leq 1$.

Proposition 3.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying (H1)–(H3). Then the mapping H in (3.4) is a large contraction on the set M_L.*

Proof. Let $\phi, \varphi \in M_L$ with $\phi^\sigma \neq \varphi^\sigma$. Then $\phi^\sigma(t) \neq \varphi^\sigma(t)$ for some $t \in \mathbb{T}$. Define the set

$$D\left(\phi, \varphi\right) = \{t \in \mathbb{T} : \phi^\sigma\left(t\right) \neq \varphi^\sigma\left(t\right)\}.$$

Note that $\varphi^{\sigma}(t) \in U_L$ for all $t \in \mathbb{T}$ whenever $\varphi \in M_L$. Since h is strictly increasing

$$(3.11) \qquad \frac{h(\varphi^{\sigma}(t)) - h(\phi^{\sigma}(t))}{\varphi^{\sigma}(t) - \phi^{\sigma}(t)} = \frac{h(\phi^{\sigma}(t)) - h(\varphi^{\sigma}(t))}{\phi^{\sigma}(t) - \varphi^{\sigma}(t)} > 0,$$

holds for all $t \in D(\phi, \varphi)$. On the other hand, for all $t \in D(\phi, \varphi)$, we have

$$
\begin{aligned}
|(H\phi)(t) - (H\varphi)(t)| &= |\phi^{\sigma}(t) - h(\phi^{\sigma}(t)) - \varphi^{\sigma}(t) + h(\varphi^{\sigma}(t))| \\
(3.12) \qquad &= |\phi^{\sigma}(t) - \varphi^{\sigma}(t)| \left| 1 - \left(\frac{h(\phi^{\sigma}(t)) - h(\varphi^{\sigma}(t))}{\phi^{\sigma}(t) - \varphi^{\sigma}(t)} \right) \right|.
\end{aligned}
$$

For each fixed $t \in D(\phi, \varphi)$, define the set $U_t \subset U_L$ by

$$
U_t = \begin{cases} (\varphi^{\sigma}(t), \phi^{\sigma}(t)), & \text{if } \phi^{\sigma}(t) > \varphi^{\sigma}(t), \\ (\phi^{\sigma}(t), \varphi^{\sigma}(t)), & \text{if } \varphi^{\sigma}(t) > \phi^{\sigma}(t), \end{cases} \quad \text{for } t \in D(\phi, \varphi).
$$

The mean value theorem implies that for each fixed $t \in D(\phi, \varphi)$ there exists a real number $c_t \in U_t$ such that

$$\frac{h(\phi^{\sigma}(t)) - h(\varphi^{\sigma}(t))}{\phi^{\sigma}(t) - \varphi^{\sigma}(t)} = h'(c_t).$$

By (H2) and (H3), we have

$$(3.13) \quad 1 \geq \sup_{t \in (-L,L)} h'(t) \geq \sup_{t \in U_t} h'(t) \geq h'(c_t) \geq \inf_{s \in U_t} h'(s) \geq \inf_{t \in (-L,L)} h'(t) \geq 0.$$

Consequently, by (3.11)–(3.13), we obtain

$$(3.14) \qquad |(H\phi)(t) - (H\varphi)(t)| \leq \left| 1 - \inf_{u \in (-L,L)} h'(u) \right| |\phi^{\sigma}(t) - \varphi^{\sigma}(t)|,$$

for all $t \in D(\phi, \varphi)$. Hence, the mapping H is a large contraction in the supremum norm. Indeed, fix $\epsilon \in (0,1)$ and assume that ϕ and φ are two functions in M_L satisfying

$$\|\phi - \varphi\| = \sup_{t \in D(\phi, \varphi)} |\phi(t) - \varphi(t)| \geq \epsilon.$$

If $|\phi^{\sigma}(t) - \varphi^{\sigma}(t)| \leq \epsilon/2$ for some $t \in D(\phi, \varphi)$, then from (3.13) and (3.14), we get

$$(3.15) \qquad |(H\phi)(t) - (H\varphi)(t)| \leq |\phi^{\sigma}(t) - \varphi^{\sigma}(t)| \leq \frac{1}{2}\|\phi - \varphi\|.$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\epsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\epsilon}{2} < |\phi^{\sigma}(t) - \varphi^{\sigma}(t)|$ for some $t \in D(\phi, \varphi)$, then from (H2) and (H3) we conclude that

$$1 \geq \frac{h(\phi^{\sigma}(t)) - h(\varphi^{\sigma}(t))}{\phi^{\sigma}(t) - \varphi^{\sigma}(t)} > \lambda,$$

where,

$$\lambda = \frac{1}{2L} \min \left\{ h \left(u + \frac{\epsilon}{2} \right) - h \left(u \right), \ u \in [-L, L] \right\} > 0.$$

Therefore, from (3.12), we have

$$(3.16) \qquad |(H\phi)(t) - (H\varphi)(t)| \leq (1 - \lambda) \|\phi - \varphi\|.$$

Consequently, it follows from (3.15) and (3.16) that

$$|(H\phi)(t) - (H\varphi)(t)| \leq \eta \|\phi - \varphi\|,$$

where

$$\eta = \max \left\{ \frac{1}{2}, 1 - \lambda \right\} < 1.$$

The proof is complete. $\square$

We shall prove that the mapping $\mathbb{C}$ has a fixed point which solves (1.1), whenever its derivative exists.

Lemma 3.2. *For A defined in (3.5), suppose that (A)–(I) and (3.7)–(3.9) hold. Then $A : M_L \to M_L$ is continuous in the supremum norm and maps M_L into a compact subset of M_L.*

Proof. We first show that $A : M_L \to M_L$. It is clear that $A : M_L \to PC$. In view of (3.1) and (3.2) we arrive at

$$
\begin{aligned}
|f(t, x)| &\leq |f(t, x) - f(t, 0)| + |f(t, 0)| \leq k_1 \|x\| + |f(t, 0)|, \\
|I(t, x)| &\leq |I(t, x) - I(t, 0)| + |I(t, 0)| \leq k_2 \|x\| + |I(t, 0)|.
\end{aligned}
$$

So, for any $\varphi \in M_L$, we have

$$
\begin{aligned}
|(A\varphi)(t)| &\leq \int_0^t e_{\ominus a}(t, s) |f(s, \varphi(s))| \Delta s + \sum_{\{i : t_i < t\}} e_{\ominus a}(t, t_i) |I(t_i, \varphi(t_i))|, \\
&\leq \int_0^t e_{\ominus a}(t, s) (k_1 L + |f(s, 0)|) \Delta s + \sum_{i=1}^n e_{\ominus a}(t, t_i) (k_2 L + |I(t_i, 0)|), \\
&\leq \alpha L \int_0^t e_{\ominus a}(t, s) a(s) \Delta s + n k_2 L + \sum_{i=1}^n |I(t_i, 0)| \\
&\leq \alpha L + \beta L = (\alpha + \beta) L \leq \frac{L}{J} < L.
\end{aligned}
$$

Thus, $A\varphi \in M_L$. Consequently, we have $A : M_L \to M_L$.

We show that A is continuous in the supremum norm. Toward this, let $\varphi, \psi \in M_L$. Note that from $a(t) > 0$ we have $\max_{s \in [0,t]} \{e_{\ominus a}(t,s)\} \leq 1$. So,

$$
\begin{aligned}
|(A\varphi)(t) - (A\psi)(t)| &\leq \int_0^t e_{\ominus a}(t,s) |f(s, \varphi(s)) - f(s, \psi(s))| \, \Delta s \\
&\quad + \sum_{\{i : t_i < t\}} e_{\ominus a}(t, t_i) |I(t_i, \varphi(t_i)) - I(t_i, \psi(t_i))|, \\
&\leq k_1 T \|\varphi - \psi\| + n k_2 \|\varphi - \psi\| \\
&\leq (k_1 T + n k_2) \|\varphi - \psi\|.
\end{aligned}
$$

Let $\epsilon > 0$ be arbitrary. Define $\theta = \epsilon / K$ with $K = k_1 T + n k_2$, where k_1 and k_2 are given by (3.1) and (3.2). Then, for $\|\varphi - \psi\| < \theta$ we obtain

$$
\|A\varphi - A\psi\| \leq K \|\varphi - \psi\| < \epsilon.
$$

This proves that A is continuous.

It remains to show that A is compact. Let $\varphi_n \in M_L$, where n is a positive integer and let

$$(3.17) \qquad \gamma = \max_{t \in [0,T]} \{a(t)\}, \ \mu = \max_{t \in [0,T]} |f(t, 0)|.$$

Then, as above, we see that

$$\|A\varphi_n\| \leq L.$$

Moreover, a direct calculation shows that

$$(A\varphi_n)^{\triangle}(t) = -a(t)(A\varphi_n)(t) + f(t, \varphi_n(t)).$$

By invoking (3.1) and (3.17) we obtain

$$\left| (A\varphi_n)^{\triangle}(t) \right| \leq (\gamma + k_1) L + \mu \leq D,$$

for some positive constant D. Hence the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly. Thus A is continuous and $A M_L$ is contained in a compact subset of M_L. $\square$

Lemma 3.3. *Let B be defined by (3.6) and that (H1)–(H3) and (3.10) hold. Then $B : M_L \to M_L$ is a large contraction.*

Proof. We first show that $B : M_L \to M_L$. Clearly, $B : M_L \to PC$. So, for any $\varphi \in M_L$, we get by (3.6) that

$$
\begin{aligned}
|(B\varphi)(t)| &\leq \int_0^t e_{\ominus a}(t,s) a(s) |H(\varphi(s))| \, \triangle s \\
&\leq \max(|H(-L)|, |H(L)|) \int_0^t e_{\ominus a}(t,s) a(s) \, \triangle s \\
&\leq \frac{2L}{J} < L.
\end{aligned}
$$

Thus $B\varphi \in M_L$. Consequently, we have $B : M_L \to M_L$.

It remains to show that B is large contraction with a unique fixed point in M_L. Form the proof of Proposition 3.1 we have for $\phi, \varphi \in M_L$, with $\phi \neq \varphi$

$$
\begin{aligned}
\left|(B\phi)(t) - (B\varphi)(t)\right| &\leq \int_0^t e_{\ominus a}(t,s)\, a(s)\left|H(\phi(s)) - H(\varphi(s))\right| \triangle s \\
&\leq \|\phi - \varphi\| \int_0^t e_{\ominus a}(t,s)\, a(s)\, \triangle s \\
&\leq \|\phi - \varphi\|.
\end{aligned}
$$

Then $\|B\phi - B\varphi\| \leq \|\phi - \varphi\|$. Now, let $\epsilon \in (0,1)$ be given and let $\phi, \varphi \in M_L$ with $\|\phi - \varphi\| \geq \epsilon$. From the proof of Proposition 3.1, we have found a $\eta < 1$, such that

$$
\begin{aligned}
\left|(B\varphi)(t) - (B\psi)(t)\right| &\leq \int_0^t e_{\ominus a}(t,s)\, a(s)\left|H(\phi(s)) - H(\varphi(s))\right| \triangle s \\
&\leq \eta \|\phi - \varphi\| \int_0^t e_{\ominus a}(t,s)\, a(s)\, \triangle s \\
&\leq \eta \|\varphi - \psi\|.
\end{aligned}
$$

Then $\|B\phi - B\varphi\| \leq \eta \|\varphi - \psi\|$. Consequently, B is a large contraction on M_L. $\square$

Theorem 3.1. *Let $M_L = \{\varphi \in PC : \|\varphi\| \leq L\}$, where L is a positive constant. Suppose (A)–(I), (H1)–(H3) and (3.7)–(3.10) hold. Then equation (1.1) has a solution φ in the subset M_L.*

Proof. By Lemma 3.2, $A : M_L \to M_L$ is continuous and AM_L is contained in a compact set. Also, from Lemma 3.3, the mapping $B : M_L \to M_L$ is a large contraction. Next, note that if $\phi, \varphi \in M_L$, we have

$$
\|A\phi + B\varphi\| \leq \|A\phi\| + \|B\varphi\| \leq \frac{L}{J} + \frac{2L}{J} \leq L.
$$

Thus $A\phi + B\varphi \in M_L$.

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem (Theorem 2.6) are satisfied. Thus there exists a fixed point $\varphi \in M_L$ such that $\varphi = A\varphi + B\varphi$. Hence the equation (1.1) has a solution in M_L. $\square$

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