FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 32, No 5 (2017), 729–745
https://doi.org/10.22190/FUMI1705729G

ENTIRE FUNCTIONS SHARING POLYNOMIALS WITH THEIR DERIVATIVES

Goutam Kumar Ghosh

Abstract. In this paper we study the uniqueness of entire functions sharing two polynomials with their derivatives. The results of the paper improve the corresponding results of Chang and Fang (Kodai Math.J. 25(2002), 309–320) and Lahiri-Ghosh(Present author) (Analysis ,Munich. 31(2011), 47–59).

Keywords: entire function, polynomial, uniqueness

1. Introduction, definitions and results

Let f be a non-constant meromorphic function in the open complex plane $\mathbb{C}$. We denote by $n(r,\infty;f)$ the number of poles of f lying in $\mid z \mid< r$, the poles are counted according to their multiplicities. The quantity

$$N(r,\infty;f)=\int_0^r \frac{n(t,\infty;f)-n(0,\infty;f)}{t}dt+n(0,\infty;f)\log r$$

is called the integrated counting function or simply the counting function of poles of f.

Also $m(r,\infty;f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\mid f(re^{i\theta})\mid d\theta$ is called the proximity function of poles of f , where $\log^+ x=\log x$ if $x\geq 1$ and $\log^+ x=0$ if $0\leq x<1$.

The sum $T(r,f)=m(r,\infty;f)+N(r,\infty;f)$ is called the Nevanlinna characteristic function of f. We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ except possibly a set of finite linear measure.

For $a\in\mathbb{C}$, we put $N(r,a;f)=N\left(r,\infty;\frac{1}{f-a}\right)$ and $m(r,a;f)=m\left(r,\infty;\frac{1}{f-a}\right)$.

Received June 21, 2016; accepted August 06, 2016
2010 *Mathematics Subject Classification.* Primary 2010; Secondary 30D35

Let us denote by $\overline{n}(r, a; f)$ the number of distinct a-points of f lying in $\mid z \mid< r$, where $a \in \mathbb{C} \cup \{\infty\}$. The quantity

$$\overline{N}(r, a; f) = \int_0^r \frac{\overline{n}(t, a; f) - \overline{n}(0, a; f)}{t} dt + \overline{n}(0, a; f) \log r$$

denotes the reduced counting function of a-points of f.

Also by $\overline{N}_{(2}(r, a; f)$ we denote the reduced counting function of multiple a-points of f.

Let $A \subset \mathbb{C}$ and $n_A(r, a; f)$ be the number of a-points of f lying in $A \cap \{z : \mid z \mid< r\}$, where $a \in \mathbb{C} \cup \{\infty\}$ and the a-points are counted according to their multiplicities. We put

$$\overline{N}_A(r, a; f) = \int_0^r \frac{\overline{n}_A(t, a; f) - \overline{n}_A(0, a; f)}{t} dt + \overline{n}_A(0, a; f) \log r.$$

For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E(a; f)$ the set of a-points of f (counted with multiplicities) and by $\overline{E}(a; f)$ the set of distinct a-points of f.

For standard definitions and results of the value distribution theory the reader may consult [3] and [11].

In 1977 L.A.Rubel and C.C.Yang [9] first investigated the uniqueness of entire function sharing certain values with their derivatives. They proved the following result.

Theorem 1.1. *[9] Let f be a nonconstant entire function. If $E(a; f) = E(a; f^{(1)})$ and $E(b; f) = E(b; f^{(1)})$, for distinct finite complex numbers a and b, then $f \equiv f^{(1)}$.*

In 1979, E. Mues and N. Steinmetz [8] took up the case of IM shared values in the place of CM shared values and proved the following theorem.

Theorem 1.2. *[8] Let f be a nonconstant entire function. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(b; f) = \overline{E}(b; f^{(1)})$, for distinct finite complex numbers a and b, then $f \equiv f^{(1)}$.*

Afterwards in 1986 G. Jank, E. Mues and L. Volkman [4] considered the case of a single shared value by the first two derivatives of an entire function. They proved the following result:

Theorem 1.3. *[4] Let f be a nonconstant entire function and $a(\neq 0)$ be a finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

In 2002 J. Chang and M. Fang [2] extended Theorem 1.3 in the following way.

Theorem 1.4. *[2] Let f be a nonconstant entire function and a, b be two finite constants. If $\overline{E}(a;f) \subset \overline{E}(a;f^{(1)}) \subset \overline{E}(b;f^{(2)})$, then either $f = \lambda e^{\frac{bz}{a}} + \frac{ab-a^2}{b}$ or $f = \lambda e^{\frac{bz}{a}} + a$, where $\lambda(\neq 0)$ is a constant.*

In [12] it was observed by the following example that in Theorem 1.3 the second derivative can not be straightway replaced by a higher order derivative.

Example 1.1. Let $(k \geq 3)$ be a positive integer and $w(\neq 1)$ be a root of the algebraic equation $w^{k-1} = 1$. We put $f = e^{wz} + w - 1$, then $E(w;f) = E(w;f^{(1)}) = E(w;f^{(k)})$ but $f \not\equiv f^{(1)}$.

In this context Zhong [12] extended Theorem 1.3 to higher order derivatives and proved the following result.

Theorem 1.5. *[12] Let f be a nonconstant entire function and $a(\neq 0)$ be a finite complex number. If f and $f^{(1)}$ share the value a CM and $\overline{E}(a;f) \subset \overline{E}(a;f^{(n)}) \cap \overline{E}(a;f^{(n+1)})$ for $n(\geq 1)$, then $f \equiv f^{(n)}$.*

For $A \subset \mathbb{C} \cup \{\infty\}$, we denote by $N_A(r,a;f)(\overline{N}_A(r,a;f))$ the counting function (reduced counting function) of those $a-points$ of f which belong to A.

In 2011, I. Lahiri and G. K. Ghosh(Present author) [5] improved Theorem 1.5 in the following manner.

Theorem 1.6. *[5] Let f be a nonconstant entire function and a, b be two nonzero finite constants. Suppose further that $A = \overline{E}(a;f) \setminus \overline{E}(a;f^{(1)})$ and $B = \overline{E}(a;f^{(1)}) \setminus \{\overline{E}(a;f^{(n)}) \cap \overline{E}(b;f^{(n+1)})\}$ for $n(\geq 1)$. If each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity and $N_A(r,a;f)+N_B(r,a;f^{(1)}) = S(r,f)$, then $f = \lambda e^{\frac{bz}{a}} + \frac{ab-a^2}{b}$ or $f = \lambda e^{\frac{bz}{a}} + a$, where $\lambda(\neq 0)$ is a constant.*

Another similar type of results we may see [6].

In the paper we extend Theorem 1.4 and Theorem 1.6 by considering shared polynomials instead of value sharing.

We now state the main result of the paper.

Theorem 1.7. *Let f be a nonconstant entire function. Also let $a(\not\equiv 0), b(\not\equiv 0)$ be two polynomials having degrees less than 2 and $deg(a) \neq deg(f)$. Suppose further that*

(i) $N_{A\cup B}(r,a;f) + N_A(r,a;f^{(1)}) = S(r,f)$, where $A = \overline{E}(a;f)\Delta\overline{E}(a;f^{(1)})$ and $B = \overline{E}(a;f) \setminus \overline{E}(b;f^{(2)})$,

(ii) $E_{1)}(a;f) \subset \overline{E}(a;f^{(1)})$, and

(iii) each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity.

Then $a \equiv b$ and $f = \lambda e^{z}$, where $\lambda(\neq 0)$ is a constant.

Theorem 1.8. *Let f be a nonconstant entire function. Also let $a(\not\equiv 0), b(\not\equiv 0)$ be two polynomials having degrees less than n and $deg(a) \neq deg(f)$. Suppose further that*

(i) $N_A(r,a;f) + N_{A\cup B}(r,a;f^{(1)}) = S(r,f)$, *where* $A = \overline{E}(a;f)\Delta\overline{E}(a;f^{(1)})$, $B = \overline{E}(a;f^{(1)}) \setminus \{\overline{E}(a;f^{(n)}) \cap \overline{E}(b;f^{(n+1)}) \cap \overline{E}(a;f^{(n+2)})\}$ *and $n \geq 1$ is an integer,*

(ii) $E_{1)}(a;f) \subset \overline{E}(a;f^{(1)})$, *and*

(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity.

Then $a \equiv b$ and either $f = \lambda e^z$ or $f = a + \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Putting $A = B = \emptyset$ in Theorem 1.7 and Theorem 1.8 we respectively obtain the following corollaries.

Corollary 1.1. *Let f be a nonconstant entire function. Also let $a(\not\equiv 0), b(\not\equiv 0)$ be two polynomials having degrees less than 2 and $deg(a) \neq deg(f)$. Further suppose that $E(a;f) = E(a;f^{(1)})$ and $\overline{E}(a;f) \subset \overline{E}(b;f^{(2)})$. Then $a \equiv b$ and $f = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.*

Corollary 1.2. *Let f be a nonconstant entire function. Also let $a(\not\equiv 0), b(\not\equiv 0)$ be two polynomials having degrees less than n and $deg(a) \neq deg(f)$. Further suppose that $E(a;f) = E(a;f^{(1)})$ and $\overline{E}(a;f) \subset \{\overline{E}(a;f^{(n)}) \cap \overline{E}(b;f^{(n+1)}) \cap \overline{E}(a;f^{(n+2)})\}$. Then $a \equiv b$ and either $f = \lambda e^z$ or $f = a + \lambda e^z$, where $\lambda(\neq 0)$ is a constant.*

We note that Corollary 1.1 is an improvement of Theorem 1.4 .

2. Lemmas

In this section we need the following lemmas.

Lemma 2.1. *[13] Let g ge a transcendental entire function and $\phi(\not\equiv 0)$ be a meromorphic function satisfying $T(r,\phi) = S(r,g)$. Then for any positive integer n*

$$T(r,g) \leq C_n\{N(r,0;g) + \overline{N}(r,0;g^{(n)} - \phi)\} + S(r,g),$$

where C_n is a constant depending only on n.

Lemma 2.2. *Let f ge a transcendental entire function and a, b be two meromorphic functions satisfying $T(r,a) + T(r,b) = S(r,f)$ and $b - a^{(n)} \not\equiv 0$. Then*

$$T(r,f) \leq C_n\{N(r,0;f - a) + \overline{N}(r,0;f^{(n)} - b)\} + S(r,f),$$

where C_n is a constant depending only on $n(\geq 1)$.

Proof. Putting $g = f - a$ and $\phi = b - a^{(n)}$ in Lemma 2.1 we get Lemma 2.2. □

Lemma 2.3. *[1] Let f be a meromorphic function and n be a positive integer. If there exist meromorphic functions $a_0(\not\equiv 0), a_1, a_2, \cdots, a_n$ such that*

$$a_0 f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n \equiv 0,$$

then

$$m(r,f) \le nT(r,a_0) + \sum_{j=1}^{n} m(r,a_j) + (n-1)\log 2.$$

Lemma 2.4. *{ [7]; see also p.28[11]} Let f be a nonconstant meromorphic function. If*

$$R(f) = \frac{a_0 f^p + a_1 f^{p-1} + \cdots + a_p}{b_0 f^q + b_1 f^{q-1} + \cdots + b_q}$$

is an irreducible rational function in f with the coefficients being small functions of f and $a_0 b_0 \not\equiv 0$, then

$$T(r, R(f)) = max\{p,q\} T(r,f) + S(r,f).$$

Lemma 2.5. *Let f, $a_0, a_1, a_2, \cdots, a_p, b_0, b_1, b_2, \cdots, b_q$ be meromorphic functions. If*

$$R(f) = \frac{a_0 f^p + a_1 f^{p-1} + \cdots + a_p}{b_0 f^q + b_1 f^{q-1} + \cdots + b_q} \qquad (a_0 b_0 \not\equiv 0),$$

then

$$T(r, R(f)) = O(T(r,f) + \sum_{i=0}^{p} T(r,a_i) + \sum_{j=0}^{q} T(r,b_j)).$$

Proof. The Lemma follows from the first fundamental theorem and the properties of the characteristic function. □

Lemma 2.6. *{p.68 [3]} Let f be a transcendental meromorphic function and $f^n P(z) = Q(z)$, where $P(z)$, $Q(z)$ are differential polynomials generated by f and the degree of Q is at most n. Then $m(r,P) = S(r,f)$.*

Lemma 2.7. *{p.69 [3]} Let f be a nonconstant meromorphic function and*

$$g(z) = f^n(z) + P_{n-1}(f),$$

where $P_{n-1}(f)$ is a differential polynomial generated by f and of degree at most $n-1$.

If $N(r,\infty;f) + N(r,0;g) = S(r,f)$, then $g(z) = h^n(z)$, where $h(z) = f(z) + \frac{a(z)}{n}$ and $h^{n-1}(z)a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h^{(1)}(z)$ for $f^{(1)}(z)$ etc. in the terms of degree $n-1$ in $P_{n-1}(f)$.

Let us note the special case, where $P_{n-1}(f) = a_0(z)f^{n-1}$ + terms of degree $n-2$ at most. Then $h^{n-1}(z)a(z) = a_0(z)h^{n-1}(z)$ and so $a(z) = a_0(z)$. Hence $g(z) = (f(z) + \frac{a_0(z)}{n})^n$.

Lemma 2.8. *{p.47 [3]} Let f be a nonconstant meromorphic function and a_1,a_2,a_3 be three distinct meromorphic functions satisfying $T(r, a_\mu) = S(r, f)$ for $\mu = 1, 2, 3$. Then*

$$T(r, f) \leq \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r, f).$$

Lemma 2.9. *{p.92, [11]} Suppose that $f_1, f_2, \cdots, f_n (n \geq 3)$ are meromorphic functions which are not constants except possibly for f_n. Further let $\sum\limits_{j=1}^{n} f_j(z) \equiv 1$. If $f_n \not\equiv 0$ and*

$$\sum_{j=1}^{n} N(r, 0; f_j) + (n-1)\sum_{j=1}^{n} \overline{N}(r, \infty; f_j) < \{\lambda + o(1)\}T(r, f_k)$$

for $0 < \lambda < 1$ and $k = 1, 2, \cdots, n-1$, then $f_n \equiv 1$.

3. Proof of the theorems

Proof. [Proof of Thorem 1.7] We verify that f cannot be a polynomial. If f is a polynomial, then $T(r, f) = O(\log r)$ and so $N_{A\cup B}(r, a; f) = S(r, f)$ implies that $A = B = \emptyset$. Therefore $\overline{E}(a; f)\Delta\overline{E}(a; f^{(1)}) = \{\overline{E}(a; f) - \overline{E}(a; f^{(1)})\} \cup \{\overline{E}(a; f^{(1)}) - \overline{E}(a; f)\} = \emptyset$ implies that $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$.

Let $deg(f) = m$ and $deg(a) = p$. If $m \geq p + 1$, then $deg(f - a) = m$ and $deg(f^{(1)} - a) \leq m - 1$. Since $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity, we arrive at a contradiction.

If $m \leq p-1$ and $deg(a) = p < 2$ then f will be a constant, which contradicts our assumption that f is nonconstant entire function. Therefore f is a transcendental entire function.

Let z_0 be a zero of $f-a$ and $f^{(1)}-a$ with multiplicity $q(\geq 2)$, since by hypotheses each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity. Then z_0 is a zero of $f^{(1)} - a^{(1)}$ with multiplicity $q - 1$. Hence z_0 is a zero of $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$ with multiplicity $q - 1$. Since $q \leq 2(q - 1)$, we have

$$N_{(2}(r, a; f) \leq 2N(r, 0; a - a^{(1)}) + N_A(r, a; f) = S(r, f). \tag{3.1}$$

Let $\lambda = \frac{f^{(1)}-a}{f-a}$ and $F = f - a$. Then by the hypotheses we get

$$\begin{aligned} N(r, 0; \lambda) + N(r, \infty; \lambda) &\leq N_A(r, 0; f - a) + N_A(r, 0; f^{(1)} - a) \\ &= S(r, f). \end{aligned} \tag{3.2}$$

Now

$$F^{(1)} = \lambda F + a - a^{(1)} = \lambda F + h, \tag{3.3}$$

where $h = a - a^{(1)}$. And also

$$\begin{aligned} F^{(2)} &= \lambda F^{(1)} + \lambda^{(1)} F + h^{(1)} \\ &= \lambda(\lambda F + h) + \lambda^{(1)} F + h^{(1)} \\ &= (\lambda^2 + d_1 \lambda) F + \lambda h + h^{(1)}, \end{aligned} \tag{3.4}$$

where $d_1 = \frac{\lambda^{(1)}}{\lambda}$ and $T(r, d_1) = N(r, 0; \lambda) + N(r, \infty; \lambda) + S(r, \lambda) = S(r, f)$.

Set

$$\tau = \frac{(a - a^{(1)})(f^{(2)} - a^{(2)}) - (b - a^{(2)})(f^{(1)} - a^{(1)})}{f - a}. \tag{3.5}$$

Then by the lemma of logarithmic derivative $m(r, \tau) = S(r, f)$. Now by (3.1) and by hypotheses we get $N(r, \tau) = S(r, f)$ and so $T(r, \tau) = S(r, f)$.

Since a is a polynomial and $deg(a) < 2$ so $a^{(2)} = 0$, then by (3.5) we get

$$\tau F = (a - a^{(1)}) F^{(2)} - (b - a^{(2)}) F^{(1)} = h F^{(2)} - b F^{(1)}.$$

Putting the value of $F^{(1)}$ and $F^{(2)}$ from (3.3) and (3.4) to the above equation we get

$$\{h\lambda^2 + (hd_1 - b)\lambda - \tau\} F = h^2 \left(\frac{b - a^{(1)}}{a - a^{(1)}} - \lambda\right). \tag{3.6}$$

If $h\lambda^2 + (hd_1 - b)\lambda - \tau \not\equiv 0$, then from (3.6) we get

$$F = -\frac{h^2 \lambda - (b - a^{(1)}) h}{h\lambda^2 + (hd_1 - b)\lambda - \tau}. \tag{3.7}$$

Then from (3.7) we get by Lemma 2.5, $T(r, F) = O(T(r, \lambda)) + S(r, f)$ and so $T(r, f) = T(r, F + a) \le T(r, F) + S(r, f)$ also $T(r, F) \le T(r, f) + S(r, f)$ i.e., $T(r, f) = T(r, F) + S(r, f) = O(T(r, \lambda)) + S(r, f)$ this implies that $S(r, f)$ is replaceable by $S(r, \lambda)$.

Also from (3.7) we see that F is a rational function in λ, which can be made irreducible. We put

$$F = \frac{P_l(\lambda)}{Q_{l+1}(\lambda)}, \tag{3.8}$$

where $P_l(\lambda)$ and $Q_{l+1}(\lambda)$ are relatively prime polynomials in λ of respective degrees l and $l + 1$. Also the coefficients of the both the polynomials are small functions of λ. Without loss of generality we assume that $Q_{l+1}(\lambda)$ is a monic polynomial. We further note that the counting function of the common zeros of $P_l(\lambda)$ and $Q_{l+1}(\lambda)$, if any, is $S(r,\lambda)$, because $P_l(\lambda)$ and $Q_{l+1}(\lambda)$ are relatively prime and the coefficients are small functions of λ.

Since $N(r,\infty;F) = S(r,f) = S(r,\lambda)$, we see from (3.8) that $N(r,0;Q_{l+1}(\lambda)) = S(r,\lambda)$. Also by (3.2) we know that $N(r,\infty;\lambda) = S(r,f) = S(r,\lambda)$. So by Lemma 2.7 we get

$$Q_{l+1}(\lambda) = (\lambda + \frac{c}{l+1})^{l+1}, \tag{3.9}$$

where c is the coefficient of λ^l in $Q_{l+1}(\lambda)$.

If $c \not\equiv 0$, then by Lemma 2.8 we obtain

$$\begin{aligned} T(r,\lambda) &\leq \overline{N}(r,0;\lambda) + \overline{N}(r,\infty;\lambda) + \overline{N}(r,-\frac{c}{l+1};\lambda) + S(r,\lambda) \\ &= \overline{N}(r,0;Q_{l+1}(\lambda)) + S(r,\lambda) \\ &= S(r,\lambda), \end{aligned}$$

a contradiction. Therefore $c \equiv 0$ and we get from (3.8) and (3.9)

$$F = \frac{P_l(\lambda)}{\lambda^{l+1}}. \tag{3.10}$$

Differentiating (3.10) we obtain $F^{(1)} = d_1 \frac{\lambda P_l^{(1)}(\lambda) - (l+1)P_l(\lambda)}{\lambda^{l+1}}$, where $d_1 = \frac{\lambda^{(1)}}{\lambda}$ and $T(r,d_1) = O(\overline{N}(r,0;\lambda) + \overline{N}(r,\infty;\lambda)) + m(r,d_1) = S(r,f) + S(r,\lambda) = S(r,\lambda)$. So by Lemma 2.4 we have

$$T(r,F^{(1)}) = (l+1-p)T(r,\lambda) + S(r,\lambda) \tag{3.11}$$

for some integer p, $0 \leq p \leq l$.

Again since $F^{(1)} = \lambda F + h$, where $h = a - a^{(1)} \not\equiv 0$, we get by (3.10) $F^{(1)} = \frac{P_l(\lambda)}{\lambda^l} + h$ and so by Lemma 2.4 we have

$$T(r,F^{(1)}) = (l-p)T(r,\lambda) + S(r,\lambda), \tag{3.12}$$

where p is same as in (3.11). Now from (3.11) and (3.12) we get $T(r,\lambda) = S(r,\lambda)$, a contradiction.

If $h\lambda^2 + (hd_1 - b)\lambda - \tau \equiv 0$, then by (3.6) and $h^2 \not\equiv 0$ we deduce that $\lambda \equiv \frac{b-a^{(1)}}{a-a^{(1)}}$. Since $\lambda = \frac{f^{(1)}-a}{f-a}$ so

$$\frac{f^{(1)}-a}{f-a} = \frac{b-a^{(1)}}{a-a^{(1)}}. \tag{3.13}$$

Then (3.13) can be written as

$$\frac{f^{(1)}-a^{(1)}}{f-a} + \frac{a^{(1)}-a}{f-a} = \frac{b-a^{(1)}}{a-a^{(1)}}$$

and so $m(r,a;f)=O(\log r)+S(r,f)=S(r,f)$, because f is transcendental. Hence

$$T(r,f)=N(r,a;f)+S(r,f). \tag{3.14}$$

Also (3.13) can be written in the form

$$\frac{f^{(1)}-f}{f-a}=\frac{b-a}{a-a^{(1)}}. \tag{3.15}$$

If $a\not\equiv b$ then the number of common zeros of $f^{(1)}-a$ and $f-a$ are at most finite. So

$$N(r,a;f)\leq N_A(r,a;f)+N(r,a;f\mid f^{(1)}=a)=S(r,f)+O(\log r)=S(r,f),$$

which contradicts (3.14).

Hence $a\equiv b$ and so by (3.15) we conclude that $f\equiv f^{(1)}$. Therefore $f=ce^z$, where $c(\neq 0)$ is a constant. This proves the theorem. $\square$

Proof. [Proof of Theorem 1.8] We verify that f cannot be a polynomial. If f is a polynomial, then $T(r,f)=O(\log r)$ and so $N_{A\cup B}(r,a;f)=S(r,f)$ implies that $A=B=\emptyset$. Therefore $\overline{E}(a;f)\Delta\overline{E}(a;f^{(1)})=\{\overline{E}(a;f)-\overline{E}(a;f^{(1)})\}\cup\{\overline{E}(a;f^{(1)})-\overline{E}(a;f)\}=\emptyset$ implies that $\overline{E}(a;f)=\overline{E}(a;f^{(1)})$.

Let $deg(f)=m$ and $deg(a)=p$. If $m\geq p+1$, then $deg(f-a)=m$ and $deg(f^{(1)}-a)\leq m-1$. Since $\overline{E}(a;f)=\overline{E}(a;f^{(1)})$ and each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity, we arrive at a contradiction.

If $m\leq p-1$. Then $deg(f-a)=deg(f^{(1)}-a)=p$. Since $\overline{E}(a;f)=\overline{E}(a;f^{(1)})$ and each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity, we have $f^{(1)}-a=k(f-a)$, where $(k\neq 0)$ is a constant. If $k\neq 1$, then $kf-f^{(1)}\equiv(k-1)a$, which is impossible as $deg(k-1)a=p>m=deg(kf-f^{(1)})$. If $k=1$, then $f^{(1)}\equiv f$ but f is a polynomial, which is a contradiction. Therefore f is a transcendental entire function.

Case 1. Let $f^{(n)}\not\equiv f^{(n+1)}$. Then we have two possibilities either $af^{(n+1)}\equiv bf^{(n)}$ or $af^{(n+1)}\not\equiv bf^{(n)}$.

Sub-Case 1.1. Let $af^{(n+1)}\equiv bf^{(n)}$. If $\overline{E}(a:f^{(n)})\cap\overline{E}(b;f^{(n+1)})=\emptyset$, then $N(r,a;f^{(1)})=N_B(r,a;f^{(1)})=S(r,f)$ and so by (3.1) and hypothesis we get

$$\begin{aligned}
N(r,a;f) &\leq N_A(r,a;f)+N(r,a;f\mid f^{(1)}=a)\\
&\leq N_{1)}(r,a;f\mid f^{(1)}=a)+N_{(2}(r,a;f\mid f^{(1)}=a)+S(r,f)\\
&\leq \overline{N}(r,a;f\mid f^{(1)}=a)+N_{(2}(r,a;f)+S(r,f)\\
&\leq N(r,a;f^{(1)})+S(r,f)\\
&= S(r,f).
\end{aligned}$$

Where $N_{1)}(r,a;f\mid f^{(1)}=a)$ denotes the simple a-points of f which are also a-points of $f^{(1)}$.

Hence by Lemma 2.2 we get $T(r,f) = S(r,f)$, a contradiction. Therefore $\overline{E}(a;f^{(n)}) \cap \overline{E}(b;f^{(n+1)}) \neq \emptyset$.
Differentiating both sides of $af^{(n+1)} \equiv bf^{(n)}$ we get $af^{(n+2)}+a^{(1)}f^{(n+1)} \equiv bf^{(n+1)}+b^{(1)}f^{(n)}$, which implies

$$\begin{aligned} af^{(n+2)} &\equiv b(\frac{b}{a}f^{(n)}) + b^{(1)}f^{(n)} - a^{(1)}(\frac{b}{a}f^{(n)}) \\ &\equiv (\frac{b^2}{a} + b^{(1)} - \frac{a^{(1)}b}{a})f^{(n)}. \end{aligned} \tag{3.16}$$

If z_0 is a zero of $f^{(1)} - a$ which is also a zero of $f^{(n)} - a$, $f^{(n+1)} - b$ and $f^{(n+2)} - a$, then (3.16) shows that z_0 is a zero of $a^2 - b^2 - ab^{(1)} + a^{(1)}b$, provided that $a^2 - b^2 - ab^{(1)} + a^{(1)}b \not\equiv 0$.

If $a^2 - b^2 - ab^{(1)} + a^{(1)}b \not\equiv 0$, then

$$\begin{aligned} N(r,a;f) &\leq N_A(r,a;f) + N(r,a;f \mid f^{(1)} = a) \\ &\leq n\overline{N}(r,a;f \mid f^{(1)} = a) + S(r,f) \\ &\leq nN_B(r,a;f^{(1)}) + nN(r,a;f^{(1)} \mid f^{(n)} = a, f^{(n+1)} = b, f^{(n+2)} = a) + S(r,f) \\ &= O(\log r) + S(r,f) \\ &= S(r,f). \end{aligned}$$

Also

$$\begin{aligned} \overline{N}(r,a;f^{(1)}) &\leq N_{A\cup B}(r,a;f^{(1)}) + \overline{N}(r,a;f^{(1)} \mid f^{(n)} = a, f^{(n+1)} = b, f^{(n+2)} = a) \\ &= O(\log r) + S(r,f) \\ &= S(r,f). \end{aligned}$$

So by Lemma 2.2 we get $T(r,f) = S(r,f)$, a contradiction.

Therefore $a^2 - b^2 - ab^{(1)} + a^{(1)}b \equiv 0$ and so $(\frac{a}{b})^2 + (\frac{a}{b})^{(1)} \equiv 1$, which implies $\frac{a}{b} \equiv \frac{e^{2z}-c}{e^{2z}+c}$, where c is a constant. Since a, b are polynomials so $\frac{a}{b}$ is a rational function, we have $c = 0$ and so $a \equiv b$. Therefore $af^{(n+1)} \equiv bf^{(n)}$ implies $f^{(n+1)} \equiv f^{(n)}$, a contradiction. Hence $af^{(n+1)} \not\equiv bf^{(n)}$.

Sub-Case 1.2. If $af^{(n+1)} \not\equiv bf^{(n)}$. Now by the hypothesis we get

$$\begin{aligned} \overline{N}(r,a;f^{(1)}) &\leq N(r,\frac{b}{a};\frac{f^{(n+1)}}{f^{(n)}}) + N_{A\cup B}(r,a;f^{(1)}) \\ &\leq T(r,\frac{f^{(n+1)}}{f^{(n)}}) + S(r,f) \\ &= N(r,0;f^{(n)}) + S(r,f). \end{aligned} \tag{3.17}$$

Again,

$$\begin{aligned} m(r,a;f) &\leq m(r,0;f^{(n)}) + S(r,f) \\ &= T(r,f^{(n)}) - N(r,0;f^{(n)}) + S(r,f) \\ &\leq T(r,f) - N(r,0;f^{(n)}) + S(r,f). \end{aligned}$$

implies

$$N(r,0;f^{(n)}) \le N(r,a;f) + S(r,f). \tag{3.18}$$

From (3.17) and (3.18) and Lemma 2.2 we get

$$T(r,f) \le 2C_n N(r,a;f) + S(r,f). \tag{3.19}$$

In the proof of Theorem 1.7 we have $\lambda = \frac{f^{(1)}-a}{f-a}$. Then $f^{(1)} - a = \lambda f - \lambda a$, so

$$F^{(1)} = \lambda_1 F + \mu_1, \tag{3.20}$$

where $F = f - a$, $\lambda_1 = \lambda$ and $\mu_1 = a - a^{(1)} = h$, say. Taking the derivatives of (3.20) and using (3.20) repeatedly we get

$$F^{(k)} = \lambda_k F + \mu_k, \tag{3.21}$$

where $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k$ and $\mu_{k+1} = \mu_k^{(1)} + \mu_1\lambda_k$ for $k = 1, 2, \ldots$.

Now we shall prove that $T(r,\lambda) = S(r,f)$. If λ is constant, then obviously $T(r,\lambda) = S(r,f)$. So we suppose that λ is nonconstant. From the hypothesis we get

$$\begin{aligned} N(r,0;\lambda) + N(r,\infty;\lambda) &\le N_A(r,0;f-a) + N_A(r,0;f^{(1)}-a) \\ &= S(r,f). \end{aligned} \tag{3.22}$$

Put $k = 1$ in $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k$ we get $\lambda_2 = \lambda^2 + d_1\lambda$ where $d_1 = \frac{\lambda^{(1)}}{\lambda}$. Again putting $k = 2$ in $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k$ we get $\lambda_3 = \lambda_2^{(1)} + \lambda_1\lambda_2$, so $\lambda_3 = \lambda^3 + 3d_1\lambda^2 + d_2\lambda$, where $d_2 = d_1^2 + d_1^{(1)}$. Similarly $\lambda_4 = \lambda_3^{(1)} + \lambda_1\lambda_3 = \lambda^4 + 6d_1\lambda^3 + (6d_1^2 + 3d_1^{(1)} + d_2)\lambda^2 + (d_2^{(1)} + d_1d_2)\lambda$. Therefore, in general, we get for $k \ge 2$

$$\lambda_k = \lambda^k + \sum_{j=1}^{k-1} \alpha_j \lambda^j, \tag{3.23}$$

where $T(r,\alpha_j) = O(\overline{N}(r,0;\lambda) + \overline{N}(r,\infty;\lambda)) + S(r,\lambda) = S(r,f)$ for $j = 1, 2, \cdots, k-1$.

Again put $k = 1$ in $\mu_{k+1} = \mu_k^{(1)} + \mu_1\lambda_k$ we get $\mu_2 = \mu_1^{(1)} + \mu_1\lambda_1 = h\lambda + h^{(1)}$. Also putting $k = 2$ in $\mu_{k+1} = \mu_k^{(1)} + \mu_1\lambda_k$ we obtain by (3.23), $\mu_3 = \mu_2^{(1)} + \mu_1\lambda_2 = h\lambda^{(1)} + h^{(1)}\lambda + h^{(2)} + h(\lambda^2 + d_1\lambda) = h\lambda^2 + (h^{(1)} + 2hd_1)\lambda + h^{(2)}$. Similarly $\mu_4 = h\lambda^3 + (5hd_1 + h^{(1)})\lambda^2 + (h^{(2)} + 2hd_1 + 2hd_1^2 + 2hd_1^{(1)} + h^{(1)}d_1 + h^{(1)})\lambda + h^{(3)}$. Therefore, in general, for $k \ge 2$

$$\mu_k = \sum_{j=1}^{k-1} \beta_j \lambda^j + h^{(k-1)}, \tag{3.24}$$

where $T(r,\beta_j) = O(\overline{N}(r,0;\lambda)+\overline{N}(r,\infty;\lambda))+S(r,\lambda) = S(r,f)$ for $j = 1,2,\cdots,k-1$ and $\beta_{k-1} = h$.

Now we suppose that $n \geq 2$. Then we set

$$\Psi = \frac{af^{(n+1)} - bf^{(n)}}{f-a}. \tag{3.25}$$

Then clearly $m(r,\Psi) = S(r,f)$. Now by (3.1) and by hypotheses we get $N(r,\Psi) \leq N_{(2}(r,a;f) + N_A(r,a;f) + S(r,f) = S(r,f)$ and so $T(r,\Psi) = S(r,f)$. Then from (3.25) we get

$$\Psi(f-a) - af^{(n+1)} + bf^{(n)} \equiv 0. \tag{3.26}$$

Since $deg(a)$ is less than n and a is a polynomial so (3.26) can be rewritten as

$$\Psi F - aF^{(n+1)} + bF^{(n)} \equiv 0. \tag{3.27}$$

Using (3.21),(3.23),(3.24) and (3.27) we get $\Psi F - a(\lambda_{n+1}F+\mu_{n+1})+b(\lambda_n F+\mu_n) \equiv 0$ implies $\Psi F - a\{(\lambda^{n+1} + \sum_{j=1}^{n} \alpha_j\lambda^j)F + \sum_{j=1}^{n} \beta_j\lambda^j + h^{(n)}\} + b\{(\lambda^n + \sum_{j=1}^{n-1} \alpha_j\lambda^j)F + \sum_{j=1}^{n-1} \beta_j\lambda^j + h^{(n-1)}\} \equiv 0$ implies

$$\begin{aligned} &\{\Psi - a\lambda^{n+1} - (a\alpha_n - b)\lambda^n - (a-b)\sum_{j=1}^{n-1}\alpha_j\lambda^j\}F \\ &-\{a\beta_n\lambda^n + (a-b)\sum_{j=1}^{n-1}\beta_j\lambda^j - bh^{(n-1)}\} \equiv 0. \end{aligned} \tag{3.28}$$

If $\Psi - a\lambda^{n+1} - (a\alpha_n - b)\lambda^n - (a-b)\sum_{j=1}^{n-1}\alpha_j\lambda^j \equiv 0$, then by Lemma 2.3 we get $m(r,\lambda) \leq O(\log r) = S(r,f)$, because f is a transcendental entire function. Therefore by (3.22) we have $T(r,\lambda) = S(r,f)$.

Next suppose that

$$\Psi - a\lambda^{n+1} - (a\alpha_n - b)\lambda^n - (a-b)\sum_{j=1}^{n-1}\alpha_j\lambda^j \not\equiv 0.$$

Then from (3.28) we get

$$F = \frac{a\beta_n\lambda^n + (a-b)\sum_{j=1}^{n-1}\beta_j\lambda^j - bh^{(n-1)}}{\Psi - a\lambda^{n+1} - (a\alpha_n - b)\lambda^n - (a-b)\sum_{j=1}^{n-1}\alpha_j\lambda^j}. \tag{3.29}$$

Following the similar argument of the Theorem 1.7 and using (3.29) we can show that $T(r,\lambda) = S(r,\lambda)$, a contradiction. Therefore we establish that $T(r,\lambda) = S(r,f)$ for $n \geq 2$.

Next suppose that $n = 1$. Let $\phi = \frac{af^{(2)} - bf^{(1)}}{f-a}$, then

$$\phi F - aF^{(2)} + bF^{(1)} \equiv 0. \tag{3.30}$$

Since $deg(a)$ and $deg(b) < n$ so a and b must be constant and so by hypothesis we have $T(r,\phi) = S(r,f)$. Using (3.21), (3.23), (3.24) and (3.30) we get

$$\{\phi - a\lambda^2 + (b - a\alpha_1)\lambda\}F + \{bh - a\beta_1\lambda - ah^{(1)}\} \equiv 0. \tag{3.31}$$

Following the similar argument of the preceding case and using (3.31) we can show that $m(r,\lambda) = S(r,f)$. So by (3.22) we have $T(r,\lambda) = S(r,f)$.

Since $T(r,\lambda) = S(r,f)$, we see that $T(r,\lambda_k) + T(r,\mu_k) = S(r,f)$ for $k = 1, 2, \ldots$, where λ_k and μ_k are defined in (3.21). Let z_0 be a zero of $F = f - a$ such that $z_0 \notin A \cup B$. For $k = n$ we have from (3.21) $F^{(n)} = \lambda_n F + \mu_n$ and since $a^{(n)} \equiv 0$ then we have

$$f^{(n)} = \lambda_n(f - a) + \mu_n. \tag{3.32}$$

Since $z_0 \notin A \cup B$ then z_0 must be a zero of $f - a$, $f^{(1)} - a$, $f^{(n)} - a$, $f^{(n+1)} - b$ and $f^{(n+2)} - a$. That is $f(z_0) = a(z_0)$ and $f^{(n)}(z_0) = a(z_0)$. Then from (3.32) we get $f^{(n)}(z_0) = \lambda_n(z_0)(f - a)(z_0) + \mu_n(z_0)$ then we get, $a(z_0) = \mu_n(z_0)$. If $a(z) \not\equiv \mu_n(z)$, we get

$$\begin{aligned} \overline{N}(r,a;f) &\leq N_A(r,0;f-a) + N(r,0;a-\mu_n) + S(r,f) \\ &= S(r,f) \end{aligned}$$

which contradicts (3.19).
Therefore

$$a(z) \equiv \mu_n(z). \tag{3.33}$$

Again differentiate (3.32) we get $f^{(n+1)} = \lambda_n^{(1)}(f-a) + \lambda_n(f^{(1)} - a^{(1)}) + \mu_n^{(1)}$. Now at the point z_0 we get $f^{(n+1)}(z_0) = \lambda_n^{(1)}(z_0)(f-a)(z_0) + \lambda_n(z_0)(f^{(1)} - a^{(1)})(z_0) + \mu_n^{(1)}(z_0)$ then by hypothesis $b(z_0) = \lambda_n(z_0)(a(z_0) - a^{(1)}(z_0)) + \mu_n^{(1)}(z_0)$. If $b(z) \not\equiv \lambda_n(z)(a(z) - a^{(1)}(z)) + \mu_n^{(1)}(z)$, we get

$$\begin{aligned} \overline{N}(r,a;f) &\leq N_A(r,0;f-a) + N(r,0;b - \lambda_n(a - a^{(1)}) - \mu_n^{(1)}) + S(r,f) \\ &= S(r,f) \end{aligned}$$

which contradicts (3.19).
Therefore

$$b(z) = \lambda_n(z)(a(z) - a^{(1)}(z)) + \mu_n^{(1)}(z). \tag{3.34}$$

Differentiate (3.33) we get

$$a^{(1)}(z) \equiv \mu_n^{(1)}(z). \tag{3.35}$$

From (3.34) and (3.35) we get $b(z) = \lambda_n(z)(a(z) - a^{(1)}(z)) + a^{(1)}(z)$ implies

$$\lambda_n = \frac{b - a^{(1)}}{a - a^{(1)}}. \tag{3.36}$$

Hence from (3.32), (3.33) and (3.36) we get

$$f^{(n)} = \frac{b - a^{(1)}}{a - a^{(1)}}(f - a) + a. \tag{3.37}$$

Now (3.37) can be rewritten in the form

$$\frac{f^{(n)} - f}{f - a} = \frac{b - a}{a - a^{(1)}}. \tag{3.38}$$

Also from (3.37) we get

$$\frac{1}{f - a} = \frac{1}{a}\left(\frac{f^{(n)}}{f - a} - \frac{b - a^{(1)}}{a - a^{(1)}}\right)$$

and so

$$m(r, a; f) \leq O(\log r) + S(r, f) = S(r, f).$$

Hence

$$T(r, f) = N(r, a; f) + S(r, f). \tag{3.39}$$

If $a \not\equiv b$ then by (3.38) we conclude that the number of common zeros of $f - a$ and $f^{(n)} - a$ at most finite.

Therefore by hypothesis we get

$$\begin{aligned} N(r, a; f) &\leq N_A(r, a; f) + N(r, a; f \mid f^{(n)} = a) \\ &= O(\log r) + S(r, f) \\ &= S(r, f), \end{aligned}$$

which contradicts (3.39). Hence $a \equiv b$. Then from (3.38) we get $f \equiv f^{(n)}$.

Since $f \equiv f^{(n)}$, then we have

$$f = c_0 e^z + c_1 e^{\alpha z} + c_2 e^{\alpha^2 z} + \cdots + c_{n-1} e^{\alpha^{n-1} z}, \tag{3.40}$$

where $c_i (0 \leq i \leq n - 1)$ are constants and $\alpha^i (0 \leq i \leq n - 1)$ are distinct roots of $z^n = 1$.

Now we put $\rho = \frac{f-f^{(1)}}{f-a} = \frac{f^{(n)}-f^{(n+1)}}{f-a}$. The by the lemma of logarithmic derivative we get $m(r,\rho) = S(r,f)$ and by (3.1) and hypothesis $N(r,\rho) \leq N_A(r,a;f) + N_{(2}(r,a;f \mid f^{(1)} = a) = S(r,f)$. Therefore $T(r,\rho) = S(r,f)$. We suppose that $\rho \not\equiv 0$. Then using (3.40) we get

$$d_0 e^z + d_1 e^{\alpha z} + d_2 e^{\alpha^2 z} + \cdots + d_{n-1} e^{\alpha^{n-1} z} \equiv 1, \tag{3.41}$$

where $d_0 = \frac{c_0}{a}$ and $d_j = \frac{c_j(\rho-1+\alpha^j)}{a\rho}$ for $j = 1, 2, \cdots, n-1$.

We denote by $d_{k_j}(j = 0, 1, 2, \cdots, u; u < n)$ the nonzero term of $d_k(k = 0, 1, 2, \cdots, n-1)$. Let $f_j = d_{k_j} e^{\alpha^{k_j} z}$ for $j = 0, 1, 2, \cdots, u$. Then from (3.41) we get

$$f_0 + f_1 + f_2 + \cdots + f_u \equiv 1. \tag{3.42}$$

If the left hand side of (3.42) contains only one term, say f_0, then we have $d_{k_0} e^{\alpha^{k_0} z} \equiv 1$ and so

$$T(r, e^{\alpha^{k_0} z}) = T(r, \frac{1}{d_{k_0}}) = S(r,f) = S(r, e^{\alpha^{k_0} z}),$$

a contradiction.

Next we suppose that the left hand side of (3.42) contains exactly two terms, then we have $f_0 + f_1 \equiv 1$, say. So by Lemma 2.8 we have

$$\begin{aligned} T(r, e^{\alpha^{k_1} z}) &\leq \overline{N}(r, 0; e^{\alpha^{k_1} z}) + \overline{N}(r, \infty; e^{\alpha^{k_1} z}) + \overline{N}(r, \frac{1}{d_{k_1}}; e^{\alpha^{k_1} z}) + S(r, e^{\alpha^{k_1} z}) \\ &= \overline{N}(r, 0; e^{\alpha^{k_0} z}) + S(r, e^{\alpha^{k_1} z}) \\ &= S(r, e^{\alpha^{k_1} z}), \end{aligned}$$

a contradiction.

If the left hand side of (3.42) contains more than two terms, then by Lemma 2.9 we get $d_{k_j} e^{\alpha^{k_j} z} \equiv 1$ for some $j \in \{0, 1, \cdots, u\}$.

Therefore

$$T(r, e^{\alpha^{k_j} z}) = T(r, \frac{1}{d_{k_j}}) = S(r,f) = S(r, e^{\alpha^{k_j} z}),$$

a contradiction.

Hence $\rho \equiv 0$ and so $f \equiv ce^z$ for some constant $c(\neq 0)$.

Case 2. Let $f^{(n)} \equiv f^{(n+1)}$. Since f is transcendental, we have $f^{(n)} \not\equiv 0$. After integration we obtain $f^{(n)} = ce^z$ and so $f = ce^z + P_{n-1}(z)$, where $c(\neq 0)$ is a constant and $P_{n-1}(z)$ is a polynomial of degree at most $n-1$.

Sub-Case 2.1. If $P_{n-1} \not\equiv a$, then by Lemma 2.8 we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,a;f) + \overline{N}(r,P_{n-1};f) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}(r,a;f) + S(r,f). \end{aligned} \tag{3.43}$$

Since $N_A(r,a;f)=S(r,f)$ we see from (3.43) that $\overline{E}(a;f)\cap\overline{E}(a;f^{(1)})$ contains infinitely many points. Also since $N_{A\cup B}(r,a;f^{(1)})=S(r,f)$, but $\overline{E}(a;f)\cap\overline{E}(a;f^{(1)})$ contains infinitely many points so $\overline{E}(a;f^{(1)})\cap\overline{E}(b;f^{(n+1)})$ contains infinitely many points. Since $f=ce^z+P_{n-1}(z)=f^{(n+1)}+P_{n-1}(z)$, we get $\overline{E}(a;f)\cap\overline{E}(b;f^{(n+1)})=\overline{E}(a-P_{n-1};f^{(n+1)})\cap\overline{E}(b;f^{(n+1)})$.
This set contains infinitely many points only if $a-P_{n-1}\equiv b$. Hence $P_{n-1}\equiv a-b$. We suppose that $P_{n-1}\equiv a-b\not\equiv 0$. Then

$$\begin{aligned}
\overline{N}(r,a;f) &\le N_A(r,a;f)+N(r,a;f\mid f^{(1)}=a)\\
&\le N_{A\cup B}(r,a;f^{(1)})+N(r,a;f\mid f^{(n)}=a,f^{(n+1)}=b)+S(r,f)\\
&= N(r,a;f\mid f^{(n)}=a,f^{(n+1)}=b)+S(r,f)\\
&\le N(r,a;f\mid f^{(n)}=a)+S(r,f)\\
&\le N(r,0;f-f^{(n)})+S(r,f)\\
&= N(r,0;f-f^{(n+1)})+S(r,f)\\
&= N(r,0;P_{n-1})+S(r,f)\\
&= S(r,f),
\end{aligned}$$

which contradicts (3.43). Therefore $P_{n-1}\equiv 0$ and so $a\equiv b$ and $f=ce^z$ where $c(\ne 0)$ is a constant.

Sub-Case 2.2. Next we suppose that $P_{n-1}\equiv a$. Then $f\equiv ce^z+a$ and $f^{(n)}=f^{(n+1)}=ce^z$. Let z_0 be a zero of $f^{(n)}-a$, which is also a zero of $f^{(n+1)}-b$. Then z_0 is a zero of $a-b$. If $a-b\not\equiv 0$, then by Lemma 2.8 we get

$$\begin{aligned}
T(r,f^{(n)}) &\le \overline{N}(r,0;f^{(n)})+\overline{N}(r,\infty;f^{(n)}+\overline{N}(r,0;f^{(n)}-a)+S(r,f)\\
&= \overline{N}(r,0;a-b)+S(r,f)\\
&= S(r,f). \qquad (3.44)
\end{aligned}$$

Also $f=f^{(n)}+a$ implies $T(r,f^{(n)})=T(r,f)+S(r,f)$. Therefore by (3.44) we get
$T(r,f)=S(r,f)$, a contradiction. Hence $a\equiv b$. This proves the theorem. □

4. An open question

For further study, we propose the following question.

Is it possible to replace the set $B=\overline{E}(a;f^{(1)})\setminus\{\overline{E}(a;f^{(n)})\cap\overline{E}(b;f^{(n+1)})\cap\overline{E}(a;f^{(n+2)})\}$ in Theorem 1.8 by the set $B=\overline{E}(a;f^{(1)})\setminus\{\overline{E}(a;f^{(n)})\cap\overline{E}(b;f^{(n+1)})\}$ as in Theorem 1.6 ?

5. Acknowledgments

The author is thankful to the reviewers for thorough checking the manuscript and for valuable suggestions towards the improvement of the paper.

REFERENCES

1. J. Chang and M. Fang: *Entire functions that share a small function with their derivatives.* Complex Var. Theory Appl. **49** (2004), 871–895.
2. J. Chang and M. Fang: *Uniqueness of entire functions and fixed points.* Kodai Math J. **25** (2002),309–320.
3. W. K. Hayman: *Meromorphic Functions.* The Clarendon Press, Oxford , 1964.
4. G. Jank and E. Mues and L. Volkman: *Meromorphe Functionen, die mit ihrer ersten und zweiten Ableitung einen endlichen wert teilen.* Complex Var. Theory Appl. **6** (1986), 51–71.
5. I. Lahiri and G. K. Ghosh: *Entire functions sharing values with their derivatives.* Analysis ,Munich. **31** (2011), 47–59.
6. I. Lahiri and I. Kaish: *An entire function sharing a polynomial with its derivatives.* Boll. Unione Mat. Ital. DOI 10.1007/s40574-016-0077-x.
7. A. Mohonko: *The Nevanlinna characteristics of certain meromorphic functions.* Teor. Funktsii Funktsional. Anal. i Prilozhen. **14** (1971), 83–87.
8. E. Mues and N. Steinmetz: *Meromorphe functionen, die mit ihrer ableitung werte teilen.* Manuscr. Math. **29** (1979), 195–206.
9. J. A. Rubel and C. C. Yang: *Values shared by entire function and its their derivative.* In: "Complex Analysis, Kentucky, 1976". Lecture Notes in Mathematics. Springer, Berlin 599(1977), pp. 101–103.
10. G. Valiron: *Lectures on General Theory of Integral Functions.* Edouard Privat, Toulouse ,1923.
11. C. C. Yang and H. X. Yi: *Uniqueness Theory of Meromorphic Functions.* Science Press and Kluwer Academic Publishers, 2003.
12. H. Zhong: *Entire functions that share one value with their derivatives.* Kodai Math.J. **18** (1995), 250–259.
13. J. Zhu: *The general form of Hayman's inequality and fixed points of meromorphic functions.* Kexue Tongbao. **33(4)** (1988), 265–269.

Goutam Kumar Ghosh
Department of Mathematics, Dr. Bhupendra Nath Dutta Smriti Mahavidyalaya
P. O.Hatgobindapur PIN- 713407
Burdwan, West Bengal, India
g80g@rediffmail.com