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HYPERSURFACES OF A FINSLER SPACE WITH PROJECTIVE GENERALIZED KROPINA CONFORMAL CHANGE METRIC

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Abstract. In the present paper, we have studied a Finsler space whose metric is obtained from the metric of a Finsler space by generalized Kropina conformal change and obtained a necessary and sufficient condition for these Finsler spaces to be projectively related. Apart from other results, the relation between the hypersurfaces of the two Finsler spaces has been discussed.

Keywords: Finsler space, hypersurfaces, generalized Kropina conformal change, projective change, fundamental tensors

1. Introduction

In 1929, M. S. Knebelman[10] discussed the conformal geometry of generalized metric spaces. In 1980, Makoto Matsumoto[7] discussed projective changes of Finsler metrics. In 1984, C. Shibata[1] dealt with a change of Finsler metric known as β- change of metric. In 1985, Makoto Matsumoto[8] discussed the induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry. In 1986, T. Yamada[12] studied Finsler hypersurfaces satisfying certain conditions. In 2008, M. K. Gupta and P. N. Pandey[3][4] studied hypersurface of a Finsler space with Randers conformal metric and generalized Randers conformal metric. In 2009, these authors[5] studied hypersurfaces of conformally and h-conformally related Finsler spaces. In 2013, M. K. Gupta, Abhay Singh and P. N. Pandey[6] studied a hypersurface of a Finsler space with Randers change of Matsumoto metric.

The aim of the present paper is to study a Finsler space whose metric is obtained from the metric of the Finsler space by generalized Kropina conformal change and to obtain a necessary and sufficient condition for these Finsler spaces to be projectively related. Also planned is the study of the relation between the hypersurface of a Finsler space and the hypersurface of a Finsler space whose metric is obtained by the projective generalized Kropina conformal change.

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2. Preliminaries

Consider an n-dimensional smooth manifold M^n. Let F^n be an n-dimensional Finsler space equipped with a metric function $L(x^i, y^i)$ satisfying the requisite conditions (H. Rund [2] and P. L. Antonelli(ed.) [11]). Suppose l_i, g_{ij}, g^{ij}, h_{ij}, C_{ijk}, C^i_{jk} and G^i_{jk} denote the components of the corresponding normalized supporting element, metric tensor, inverse metric tensor, angular metric tensor, Cartan tensor, associate Cartan tensor and Berwald connection coefficients respectively. Then they satisfy the following relations

$$\text{(2.1)} \qquad \begin{cases} (a)\ l_i = \dot{\partial}_i L, & (b)\ g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad (c)\ C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \\ (d)\ C^i_{jk} = C_{jhk} g^{ih}, & (e)\ h_{ij} = L\dot{\partial}_i\dot{\partial}_j L, \quad (f)\ h_{ij} = g_{ij} - l_i l_j. \end{cases}$$

where g_{ij}, h_{ij}, C_{ijk} and C^i_{jk} are symmetric in their lower indices.Throughout the paper, we use the symbols $\dot{\partial}_i$ and ∂_i for the partial derivatives with respect to y^i and x^i.

The Cartan connection for a Finsler space F^n is given by $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$. The $h-$ covariant and $v-$ covariant derivatives of a covariant vector $X_i(x, y)$ with respect to Cartan connection are given by

$$\text{(2.2)} \qquad (a)\ X_{i|j} = \partial_j X_i - (\dot{\partial}_h X_i)G^h_j - F^r_{ij}X_r, \quad (b)\ X_i|_j = \dot{\partial}_j X_i - C^r_{ij}X_r,$$

and they satisfy the following

$$\text{(2.3)} \qquad \begin{cases} (a)\ l^i_{|j} = 0, & (b)\ g_{ij|k} = 0, & (c)\ h_{ij|k} = 0, \\ (d)\ L_{|i} = 0, & (e)\ l_i|_j = \frac{h_{ij}}{L}, & (f)\ L|_i = l_i, \\ (g)\ F^i_{jk} = F^i_{kj}, & (h)\ F^i_{jk}y^j = G^i_{jk}y^j = G^i_k, & (i)\ G^i_j y^j = 2G^i. \end{cases}$$

Let $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta(x, y) = b_i y^i$. Then the metric $L = \frac{\alpha^{n+1}}{\beta^n}$ $(n \neq 0, -1)$ is called a generaized Kropina metric[9].

Let us denote the symmetric and skew symmetric parts of the tensor $b_{i|j}$ by r_{ij} and s_{ij} respectively. Thus, we have

$$\text{(2.4)} \qquad (a)\ 2r_{ij} = b_{i|j} + b_{j|i}\,, \quad (b)\ 2s_{ij} = b_{i|j} - b_{j|i}.$$

Let $F^{*n} = (M^n, L^*)$ be another Finsler space over the same manifold M^n. If $L^*(x, y) = e^{\sigma(x)}L(x, y)$, then the change of metric is a conformal change and the function $\sigma(x)$ is conformal factor [10].

If the conformal change is given by

$$\text{(2.5)} \qquad L^*(x, y) = e^{\sigma(x)}\frac{L^{n+1}}{\beta^n}, \quad where \quad \beta = b_i(x)y^i,$$

then it is called a generalized Kropina conformal change[9].

A hypersurface M^{n-1} of the underlying manifold M^n may be represented parametrically by the equations $x^i = x^i(u^\alpha)$, where u^α are the Gaussian coordinates on M^{n-1} (Latin indices run from 1 to n, while Greek indices take values from 1 to $n-1$). The rank of the matrix of projection factors $B^i_\alpha = \partial x^i/\partial u^\alpha$ is supposed to be $n-1$. If the supporting element y^i at a point $u = (u^\alpha)$ of M^{n-1} is assumed to be tangential to M^{n-1} then y^i may be written as $y^i = B^i_\alpha(u)v^\alpha$ so that $v = (v^\alpha)$ is thought of as the supporting element at the point u^α of M^{n-1}. The function $\underline{L} = L\left(x(u), y(u,v)\right)$ gives rise to a Finsler metric on M^{n-1}. Thus, we get an $(n-1)$- dimensional Finsler space $F^{n-1} = \left(M^{n-1}, \underline{L}(u,v)\right)$.

The unit normal vector $N^i(u,v)$ at each point u^α of F^{n-1} is defined by

$$(2.6)\qquad (a)\ g_{ij}B^i_\alpha N^j = 0, \quad (b)\ g_{ij}N^iN^j = 1.$$

Let us define $B^\alpha_i = B^\alpha_i(u,v)$ by

$$(2.7)\qquad B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta\,.$$

This, in view of $g_{ij}B^j_\beta B^i_\gamma = g^{\beta\gamma}$, implies

$$(2.8)\qquad B^i_\alpha B^\beta_i = \delta^\beta_\alpha.$$

From (2.6), (2.7) and (2.8), we have

$$(2.9)\qquad \begin{cases} (a)\ B^i_\alpha N_i = 0, & (b)\ N^iB^\alpha_i = 0, \qquad (c)\ N^iN_i = 1, \\ (d)\ B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j, & (e)\ N_i = g_{ij}N^j. \end{cases}$$

The second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α for the induced Cartan connection $IC\Gamma = \left(F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma}\right)$ on F^{n-1} are given by

$$(2.10)\qquad H_{\alpha\beta} = N_i\left(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta\right) + M_\alpha H_\beta$$

and

$$(2.11)\qquad H_\alpha = N_i\left(B^i_{0\alpha} + G^i_jB^j_\alpha\right),$$

where $M_\alpha = C_{ijk}B^i_\alpha N^jN^k$, $B^i_{\alpha\beta} = \partial^2x^i/\partial u^\alpha\partial u^\beta$ and $B^i_{0\alpha} = B^i_{\beta\alpha}v^\beta$.

From (2.10) and (2.11), we have

$$(2.12)\qquad (a)\ H_{0\alpha} = H_{\beta\alpha}v^\beta = H_\alpha, \quad (b)\ H_{\alpha 0} = H_{\alpha\beta}v^\beta = H_\alpha + M_\alpha H_0.$$

3. Generalized Kropina Conformal Change

Let $F^{*n} = (M^n, L^*)$ be an n- dimensional Finsler space on the differentiable manifold M^n whose metric L^* is obtained from the metric of the Finsler space F^n by generalized Kropina conformal change (2.5).

Throughout the paper, the geometric objects associated with F^{*n} will be asterisked $*$.

Differentiating (2.5) partially with respect to y^i, we get

$$l_i^* = e^{\sigma(x)}\left\{(n+1)\frac{L^n}{\beta^n}l_i - n\frac{L^{n+1}}{\beta^{n+1}}b_i\right\}, \tag{3.1}$$

where $l_i^* = \dot{\partial}_i L^*$.

Differentiating (3.1) partially with respect to y^j and using (2.1)(e) and (2.1)(f), we have

$$h_{ij}^* = (n+1)e^{2\sigma(x)}\frac{L^{2n}}{\beta^{2n}}\left\{g_{ij} - n\frac{L}{\beta}(l_i b_j + l_j b_i) + n\frac{L^2}{\beta^2}b_i b_j + (n-1)l_i l_j\right\}, \tag{3.2}$$

where $h_{ij}^* = L^*\dot{\partial}_j l_i^*$.

Using (3.1), we find

$$l_i^* l_j^* = e^{2\sigma(x)}\left\{(n+1)^2\frac{L^{2n}}{\beta^{2n}}l_i l_j - n(n+1)\frac{L^{2n+1}}{\beta^{2n+1}}(l_i b_j + l_j b_i) + n^2\frac{L^{2(n+1)}}{\beta^{2(n+1)}}b_i b_j\right\}. \tag{3.3}$$

From (3.2), (3.3) and $g_{ij}^* = h_{ij}^* + l_i^* l_j^*$, we have

$$\begin{aligned} g_{ij}^* =& e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}}g_{ij} + e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}b_i b_j \\ & - e^{2\sigma(x)}2n(n+1)\frac{L^{2n+1}}{\beta^{2n+1}}(l_i b_j + l_j b_i) + e^{2\sigma(x)}2n(n+1)\frac{L^{2n}}{\beta^{2n}}l_i l_j. \end{aligned} \tag{3.4}$$

Since $g_{ij}^* g^{*jk} = \delta_i^k$, the inverse metric tensor g^{*ij} is given by

$$\begin{aligned} g^{*ij} =& \frac{1}{p}g^{ij} - \frac{L^2}{p\beta^2\left\{\frac{L^2b^2}{\beta^2} + \frac{1-n}{n}\right\}}b^i b^j + \frac{2L}{p\beta\left\{\frac{L^2b^2}{\beta^2} + \frac{1-n}{n}\right\}}(l^i b^j + l^j b^i) \\ & - \frac{2n}{p}\left\{\frac{\beta^2(n+1) - nL^2b^2}{\beta^2(1-n) + nL^2b^2}\right\}l^i l^j, \end{aligned} \tag{3.5}$$

where $p = e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}}$, $b^i = g^{ij}b_j$ and $b^2 = b^i b_i$.

Differentiating (3.4) partially with respect to y^k, we find

$$
\begin{aligned}
C^*_{ijk} = p\Big\{ & C_{ijk} - \frac{n}{\beta}(g_{jk}b_i + g_{ki}b_j + g_{ij}b_k) - \frac{n(2n+1)L^2}{\beta^3}b_ib_jb_k \\
& + \frac{n}{L}(g_{jk}l_i + g_{ki}l_j + g_{ij}l_k) + \frac{n(2n+1)L}{\beta^2}(b_ib_jl_k + b_jb_kl_i + b_kb_il_j) \\
& - \frac{2n^2}{\beta}(b_il_jl_k + b_jl_kl_i + b_kl_il_j) + \frac{2n(n-1)}{L}l_il_jk_k\Big\}.
\end{aligned} \tag{3.6}
$$

Transfecting (3.6) with g^{*jh}, we have

$$
\begin{aligned}
C^{*h}_{ik} = & C^h_{ik} + ALC_{ijk}b^j(2\beta l^h - Lb^h) - \frac{n}{\beta}(\delta^h_k b_i + \delta^h_i b_k) \\
& + \frac{n}{L}(\delta^h_k l_i + \delta^h_i l_k) - A\beta g_{ik}b^h + A(\beta^2 + n\beta^2 - nL^2b^2)g_{ik}l^h \\
& - AL^2\beta^2(4n^2 + 2n + 1)b^hb_ib_k - 2n^2A(\beta^2 + L^2b^2)(b_il_k + b_kl_i)l^h \\
& - AL[3n\beta^2 + 2n^2\beta^2 + n(2n+1)L^2b^2]l^hb_ib_k \\
& + AL\beta^2(4n - 4n^2 + 1)b^h(b_il_k + b_kl_i) \\
& + 2A(2n^2\beta^2 - n\beta^2 - nL^2b^2 - \beta^2)l^hl_il_k,
\end{aligned} \tag{3.7}
$$

where $A = \frac{n}{nL^2b^2 + \beta^2(1-n)}$.

Thus, we have

Theorem 3.1. *The components of the metric tensor, inverse metric tensor, Cartan tensor and associate Cartan tensor of a Finsler space with generalized Kropina conformal changed metric are given by (3.4), (3.5), (3.6) and (3.7) respectively.*

Let us denote the difference of Cartan connection coefficients F^i_{jk} of the Finsler space F^n and Cartan connection coefficients F^{*i}_{jk} of the Finsler space F^{*n} by D^i_{jk}. Thus, we have

$$F^{*i}_{jk} = F^i_{jk} + D^i_{jk}. \tag{3.8}$$

Transvecting (3.8) by y^k and using (2.3)(h), we get

$$G^{*i}_j = G^i_j + D^i_{0j}, \tag{3.9}$$

where $D^i_{0j} = D^i_{kj}y^k$.

Transvecting (3.9) by y^j and using (2.3)(i), we get

$$2G^{*i} = 2G^i + D^i_{00}, \tag{3.10}$$

where $D^i_{00} = D^i_{0j} y^0$.

Differentiating (3.10) partially with respect to y^j and using (3.9), we have

$$\dot{\partial}_j D^i_{00} = 2D^i_{0j}, \tag{3.11}$$

The expressions for D^i_{00}, D^i_{0j} and D^i_{jk} are calculated as follows.

Differentiating (3.1) partially with respect to y^j, we find

$$L^*_{ij} = (n+1)e^{\sigma(x)}\frac{L^{n-1}}{\beta^n}\left\{LL_{ij} + nl_il_j - \frac{nL}{\beta}(l_ib_j + l_jb_i) + \frac{nL^2}{\beta^2}b_ib_j\right\}, \tag{3.12}$$

where $L^*_{ij} = \dot{\partial}_j l^*_i$ and $L_{ij} = \dot{\partial}_j l_i$.

Differentiating (3.12) partially with respect to y^k, we get

$$\begin{aligned} L^*_{ijk} =&(n+1)e^{\sigma(x)}\left\{\frac{L^n}{\beta^n}L_{ijk} + n(n-1)\frac{L^{n-2}}{\beta^n}l_il_jl_k\right.\\ &+ \frac{nL^{n-1}}{\beta^n}(L_{ij}l_k + L_{jk}l_i + L_{ki}l_j) - \frac{nL^n}{\beta^{n+1}}(L_{ij}b_k + L_{jk}b_i + L_{ki}b_j)\\ &- \frac{n^2L^{n-1}}{\beta^{n+1}}(l_il_jb_k + l_jl_kb_i + l_kl_ib_j)\\ &\left.+ \frac{n(n+1)L^n}{\beta^{n+2}}(l_ib_jb_k + l_jb_kb_i + l_kb_ib_j) - \frac{n(n+2)L^{n+1}}{\beta^{n+3}}b_ib_jb_k\right\}, \end{aligned} \tag{3.13}$$

where $L^*_{ijk} = \dot{\partial}_k L^*_{ij}$ and $L_{ijk} = \dot{\partial}_k L_{ij}$.

Differentiating (3.12) partially with respect to x^k, we get

$$\begin{aligned} \partial_k L^*_{ij} =&(n+1)e^{\sigma(x)}\left\{\begin{bmatrix} \frac{L^n}{\beta^n}L_{ij} + \frac{nL^{n-1}}{\beta^n}l_il_j - \frac{nL^n}{\beta^{n+1}}(l_ib_j + l_jb_i) \\ +\frac{nL^{n+1}}{\beta^{n+2}}b_ib_j \end{bmatrix}\sigma_k\right.\\ &+ \frac{L^n}{\beta^n}\partial_k L_{ij} + \begin{bmatrix} \frac{nL^{n-1}}{\beta^n}L_{ij} + \frac{n(n-1)L^{n-2}}{\beta^n}l_il_j \\ -\frac{n^2L^{n-1}}{\beta^{n+1}}(l_ib_j + l_jb_i) + \frac{n(n+1)L^n}{\beta^{n+2}}b_ib_j \end{bmatrix}\partial_k L\\ &+ \begin{bmatrix} \frac{n(n+1)L^n}{\beta^{n+2}}(l_ib_j + l_jb_i) - \frac{n(n+2)L^{n+1}}{\beta^{n+3}}b_ib_j \\ -\frac{nL^n}{\beta^{n+1}}L_{ij} - \frac{n^2L^{n-1}}{\beta^{n+1}}l_il_j \end{bmatrix}\partial_k\beta\\ &+ \left[\frac{nL^{n-1}}{\beta^n}l_j - \frac{nL^n}{\beta^{n+1}}b_j\right]\partial_k l_i + \left[\frac{nL^{n-1}}{\beta^n}l_i - \frac{nL^n}{\beta^{n+1}}b_i\right]\partial_k l_j\\ &\left.+ \left[\frac{nL^{n+1}}{\beta^{n+2}}b_j - \frac{nL^n}{\beta^{n+1}}l_j\right]\partial_k b_i + \left[\frac{nL^{n+1}}{\beta^{n+2}}b_i - \frac{nL^n}{\beta^{n+1}}l_i\right]\partial_k b_j\right\}, \end{aligned} \tag{3.14}$$

where $\sigma_k = \frac{\partial\sigma(x)}{\partial x^k}$.

From (2.4)(a) and (2.4)(b), we have

$$b_{i|j} = r_{ij} + s_{ij}, \tag{3.15}$$

which may be re-written as

$$\partial_j b_i = r_{ij} + s_{ij} + b_r F^r_{ij}. \tag{3.16}$$

Transvecting (3.16) with y^i, we have

$$(\partial_j b_i) y^i = r_{0j} + s_{0j} + b_r G^r_j, \tag{3.17}$$

where $'0'$ stands for the contraction with respect to y^i, i.e. $r_{0j} = r_{ij} y^i$ and $s_{0j} = s_{ij} y^i$.
Since the h-covariant derivative of L and l_i with respect to Cartan connection vanish identically, we have

$$\partial_k l_i = L_{ir} G^r_k + l_r F^r_{ik}. \tag{3.18}$$

and

$$\partial_k L = l_r G^r_k. \tag{3.19}$$

Differentiating $\beta = b_i y^i$ with respect to x^k and using (3.17), we have

$$\partial_k \beta = r_{0k} + s_{0k} + b_r G^r_k. \tag{3.20}$$

Since the h-covariant derivative of the tensor L^*_{ij} with respect to Cartan connection vanishes identically, we have

$$\partial_k L^*_{ij} - L^*_{ijr} G^{*r}_k - L^*_{ir} F^{*r}_{jk} - L^*_{jr} F^{*r}_{ik} = 0. \tag{3.21}$$

Using (3.1), (3.8), (3.9), (3.10), (3.12), (3.13), (3.14), (3.16), (3.18), (3.19) and (3.20) in (3.21) then transvecting the resulting equation with y^k, we have

$$\begin{aligned} &L^*_{ij}\sigma_0 + (n+1)e^{\sigma(x)}\Bigg\{ \begin{bmatrix} \frac{-nL^n}{\beta^{n+1}} L_{ij} + \frac{n(n+1)L^n}{\beta^{n+2}}(l_i b_j + l_j b_i) \\ -\frac{n^2 L^{n-1}}{\beta^{n+1}} l_i l_j - \frac{n(n+2)L^{n+1}}{\beta^{n+3}} b_i b_j \end{bmatrix} r_{00} \\ &+ \left[\tfrac{nL^{n+1}}{\beta^{n+2}} b_j - \tfrac{nL^n}{\beta^{n+1}} l_j\right](r_{i0} + s_{i0}) + \left[\tfrac{nL^{n+1}}{\beta^{n+2}} b_i - \tfrac{nL^n}{\beta^{n+1}} l_i\right](r_{j0} + s_{j0}) \\ &- L^*_{ijr} D^r_{00} - L^*_{ir} D^r_{0j} - L^*_{jr} D^r_{0i} \Bigg\} = 0, \end{aligned} \tag{3.22}$$

where $\sigma_0 = \sigma_k y^k$ and $r_{00} = r_{0i} y^i$.

Differentiating (3.1) partially with respect to x^j, we have

$$\begin{aligned} \partial_j l^*_i =& l^*_i \sigma_j + e^{\sigma(x)}\left\{ \frac{n(n+1)L^{n-1}}{\beta^n} l_i - \frac{n(n+1)L^n}{\beta^{n+1}} b_i \right\}\partial_j L \\ &+ e^{\sigma(x)}\left\{ \frac{n(n+1)L^{n+1}}{\beta^{n+2}} b_i - \frac{n(n+1)L^n}{\beta^{n+1}} l_i \right\}\partial_j \beta \\ &+ e^{\sigma(x)}\left\{ \frac{(n+1)L^n}{\beta^n}\partial_j l_i - \frac{nL^{n+1}}{\beta^{n+1}}\partial_j b_i \right\}. \end{aligned} \tag{3.23}$$

Since the h-covariant derivative of the vector l^*_i with respect to Cartan connection vanishes identically, we have

$$\partial_j l^*_i - L^*_{ir} G^{*r}_j - l^*_r F^{*r}_{ij} = 0. \tag{3.24}$$

Using (3.1), (3.8), (3.10), (3.12) and (3.23) in (3.24), we have

$$\begin{aligned} l^*_i \sigma_j + e^{\sigma(x)} \Big\{ & \Big[-\frac{n(n+1)L^n}{\beta^{n+1}} l_i + \frac{n(n+1)L^{n+1}}{\beta^{n+2}} b_i \Big] (r_{0j} + s_{0j}) \\ & - \frac{nL^{n+1}}{\beta^{n+1}} b_{i|j} \Big\} - L^*_{ir} D^r_{0j} - l^*_r D^r_{ij} = 0, \end{aligned} \tag{3.25}$$

which implies

$$\begin{aligned} e^{\sigma(x)} \frac{nL^{n+1}}{\beta^{n+1}} b_{i|j} = {} & l^*_i \sigma_j - L^*_{ir} D^r_{0j} - l^*_r D^r_{ij} \\ & + e^{\sigma(x)} \Big\{ -\frac{n(n+1)L^n}{\beta^{n+1}} l_i + \frac{n(n+1)L^{n+1}}{\beta^{n+2}} b_i \Big\} (r_{0j} + s_{0j}). \end{aligned} \tag{3.26}$$

From (2.4)(a) and (3.26), we have

$$\begin{aligned} 2e^{\sigma(x)} \frac{nL^{n+1}}{\beta^{n+1}} r_{ij} = e^{\sigma(x)} \Big\{ & \Big[(n+1)\tfrac{L^n}{\beta^n} l_i - n\tfrac{L^{n+1}}{\beta^{n+1}} b_i \Big] \sigma_j \\ & + \Big[(n+1)\tfrac{L^n}{\beta^n} l_j - n\tfrac{L^{n+1}}{\beta^{n+1}} b_j \Big] \sigma_i \\ & + n(n+1) \Big[\tfrac{L^{n+1}}{\beta^{n+2}} b_i - \tfrac{L^n}{\beta^{n+1}} l_i \Big] (r_{0j} + s_{0j}) \\ & + n(n+1) \Big[\tfrac{L^{n+1}}{\beta^{n+2}} b_j - \tfrac{L^n}{\beta^{n+1}} l_j \Big] (r_{0i} + s_{0i}) \Big\} \\ & - L^*_{ir} D^r_{0j} - L^*_{jr} D^r_{0i} - 2l^*_r D^r_{ij}. \end{aligned} \tag{3.27}$$

Subtracting (3.27) from (3.22) and contracting with $y^i y^j$, we have

$$(n+1)\beta l_r D^r_{00} - nLb_r D^r_{00} = -nLr_{00} + L\beta\sigma_0. \tag{3.28}$$

Let us denote $l_r D^r_{00}$ by R and $b_r D^r_{00}$ by S. Thus, we have

$$(n+1)\beta R - nLS = -nLr_{00} + L\beta\sigma_0, \tag{3.29}$$

From (2.4)(b) and (3.26), we have

$$
\begin{aligned}
2e^{\sigma(x)}\frac{nL^{n+1}}{\beta^{n+1}}s_{ij} = e^{\sigma(x)}\Big\{ & \left[(n+1)\tfrac{L^n}{\beta^n}l_i - n\tfrac{L^{n+1}}{\beta^{n+1}}b_i\right]\sigma_j \\
& - \left[(n+1)\tfrac{L^n}{\beta^n}l_j - n\tfrac{L^{n+1}}{\beta^{n+1}}b_j\right]\sigma_i \\
& + n(n+1)\left[\tfrac{L^{n+1}}{\beta^{n+2}}b_i - \tfrac{L^n}{\beta^{n+1}}l_i\right](r_{0j}+s_{0j}) \\
& - n(n+1)\left[\tfrac{L^{n+1}}{\beta^{n+2}}b_j - \tfrac{L^n}{\beta^{n+1}}l_j\right](r_{0i}+s_{0i})\Big\} \\
& - L^*_{ir}D^r_{0j} + L^*_{jr}D^r_{0i}.
\end{aligned} \tag{3.30}
$$

Adding (3.22) and (3.30), using $LL_{ir} = g_{ir} - l_i l_r$ and transvecting with $b^i y^j$, we have

$$
\begin{aligned}
& n(n+1)L(L^2b^2-\beta^2)r_{00} + \Big\{(n+1)\beta^2 - nL^2b^2\Big\}L\beta\sigma_0 - 2nL^3\beta s_{i0}b^i \\
& + L^3\beta^2\sigma_i b^i = (n+1)\Big\{(1-n)\beta^2 + nL^2b^2\Big\}\Big\{LS - \beta R\Big\}.
\end{aligned} \tag{3.31}
$$

(3.29) and (3.31) constitute the system of algebraic equations in R and S. Solving these equations, we have

$$
S = \frac{n\left[L^2b^2 - 2\beta^2\right]r_{00} - 2nL^2\beta s_{i0}b^i + L^2\beta^2\sigma_i b^i + 2\beta^3\sigma_0}{[(1-n)\beta^2 + nL^2b^2]} \tag{3.32}
$$

and

$$
R = \frac{-n(n+1)L\beta r_{00} - 2n^2L^3 s_{i0}b^i + nL^3\beta\sigma_i b^i + L\left[(n+1)\beta^2 + nL^2b^2\right]\sigma_0}{(n+1)\left[(1-n)\beta^2 + nL^2b^2\right]}. \tag{3.33}
$$

Transvecting (3.30) with y^j and using $LL_{ir} = g_{ir} - l_i l_r$, we have

$$
\begin{aligned}
2n\frac{L^{n+1}}{\beta^{n+1}}s_{i0} = & \left[(n+1)\tfrac{L^n}{\beta^n}l_i - n\tfrac{L^{n+1}}{\beta^{n+1}}b_i\right]\sigma_0 - \frac{L^{n+1}}{\beta^n}\sigma_i \\
& + n(n+1)\left[\tfrac{L^{n+1}}{\beta^{n+2}}b_i - \tfrac{L^n}{\beta^{n+1}}l_i\right]r_{00} - \Big\{(n+1)\frac{L^{n-1}}{\beta^n}g_{ir} \\
& + (n^2-1)\frac{L^{n-1}}{\beta^n}l_i l_r - n(n+1)\frac{L^n}{\beta^{n+1}}l_i b_r \\
& - n(n+1)\frac{L^n}{\beta^{n+1}}l_r b_i + n(n+1)\frac{L^{n+1}}{\beta^{n+2}}b_i b_r\Big\}D^r_{00}.
\end{aligned} \tag{3.34}
$$

Transvecting (3.34) with g^{ij}, we have

$$\begin{aligned} 2n\frac{L^{n+1}}{\beta^{n+1}}s^j_0 =& \left[(n+1)\tfrac{L^n}{\beta^n}l^j - n\tfrac{L^{n+1}}{\beta^{n+1}}b^j\right]\sigma_0 - \frac{L^{n+1}}{\beta^n}\sigma^j \\ &+ n(n+1)\left[\tfrac{L^{n+1}}{\beta^{n+2}}b^j - \tfrac{L^n}{\beta^{n+1}}l^j\right]r_{00} - \Bigg\{(n+1)\frac{L^{n-1}}{\beta^n}\delta^j_r \\ &+ (n^2-1)\frac{L^{n-1}}{\beta^n}l^j l_r - n(n+1)\frac{L^n}{\beta^{n+1}}l^j b_r \\ &- n(n+1)\frac{L^n}{\beta^{n+1}}l_r b^j + n(n+1)\frac{L^{n+1}}{\beta^{n+2}}b^j b_r\Bigg\}D^r_{00}, \end{aligned} \tag{3.35}$$

where $s^j_0 = s_{i0}g^{ij}$ and $\sigma^j = \sigma_i g^{ij}$.

From (3.35), we have

$$\begin{aligned} D^j_{00} =& \frac{y^j}{L\beta}\Big\{ - nLr_{00} - (n-1)\beta R + nLS + L\beta\sigma_0\Big\} \\ &+ \frac{nLb^j}{\beta^2}\Big\{Lr_{00} + \beta R - LS - \frac{L\beta}{(n+1)}\sigma_0\Big\} \\ &- \frac{L^2}{(n+1)}\sigma^j - \frac{2nL^2}{(n+1)\beta}s^j_0. \end{aligned} \tag{3.36}$$

Differentiating (3.36) partially with respect to y^k and using (3.11), we have

$$\begin{aligned} D^j_{0k} =& \delta^j_k\Big\{\frac{-nr_{00}}{2\beta} - \frac{(n-1)R}{2L} + \frac{nS}{2\beta} + \frac{\sigma_0}{2}\Big\} + y^j\Big\{\frac{-nr_{0k}}{\beta} \\ &+ \frac{nr_{00}}{2\beta^2}b_k + \frac{(n-1)R}{2L^2}l_k - \frac{(n-1)}{2L}R_k - \frac{nS}{2\beta^2}b_k + \frac{n}{2\beta}S_k + \frac{\sigma_k}{2}\Big\} \\ &+ nb^j\Big\{\frac{Lr_{00}}{\beta^2}l_k + \frac{L^2r_{0k}}{\beta^2} - \frac{L^2r_{00}}{\beta^3}b_k + \frac{R}{2\beta}l_k + \frac{L}{2\beta}R_k - \frac{LR}{2\beta^2}b_k \\ &- \frac{LS}{\beta^2}l_k + \frac{L^2S}{\beta^3}b_k - \frac{L^2}{2\beta^2}S_k - \frac{L}{\beta(n+1)}\sigma_0 l_k + \frac{L^2}{2\beta^2(n+1)}\sigma_0 b_k \\ &- \frac{L^2}{2\beta(n+1)}\sigma_k\Big\} - \frac{Ll_k}{(n+1)}\sigma^j + \frac{L^2}{(n+1)}\sigma^t C^j_{tk} - \frac{2nL}{(n+1)\beta}l_k s^j_0 \\ &+ \frac{nL^2}{(n+1)\beta^2}b_k s^j_0 - \frac{nL^2}{(n+1)\beta}\left(s^j_k - 2s^t_0 C^j_{tk}\right), \end{aligned} \tag{3.37}$$

where $s^j_k = s_{ik}g^{ij}$, $R_k = \dot\partial_k R$ and $S_k = \dot\partial_k S$.

Differentiating (3.37) partially with respect to y^h, we have

(3.38)

$$\begin{aligned}
\dot\partial_h D^j_{0k} &= \delta^j_k\left\{\frac{-nr_{0h}}{\beta}+\frac{nr_{00}}{2\beta^2}b_h+\frac{(n-1)R}{2L^2}l_h-\frac{(n-1)}{2L}R_h-\frac{nS}{2\beta^2}b_h+\frac{n}{2\beta}S_h+\frac{\sigma_h}{2}\right\}\\
&+\delta^j_h\left\{\frac{-nr_{0k}}{\beta}+\frac{nr_{00}}{2\beta^2}b_k+\frac{(n-1)R}{2L^2}l_k-\frac{(n-1)}{2L}R_k-\frac{nS}{2\beta^2}b_k+\frac{n}{2\beta}S_k+\frac{\sigma_k}{2}\right\}\\
&+y^j\left\{\frac{-nr_{hk}}{\beta}+\frac{nb_r y^s(\dot\partial_h F^r_{ks})}{\beta}+\frac{n}{\beta^2}\left(r_{0k}b_h+r_{0h}b_k\right)-\frac{(n-1)R}{2L^3}\left(2l_kl_h-LL_{kh}\right)\right.\\
&\left.+\frac{(n-1)}{2L^2}\left(R_hl_k+R_kl_h\right)-\frac{n}{\beta^3}b_kb_h\left(r_{00}-S\right)+\frac{n}{2\beta}S_{kh}-\frac{n}{2\beta^2}\left(S_hb_k+S_kb_h\right)\right\}\\
&+nb^j\left\{\frac{2L}{\beta^2}\left(r_{0h}l_k+r_{0k}l_h\right)+\frac{L^2}{\beta^2}r_{kh}-\frac{L^2b_ry^s(\dot\partial_hF^r_{ks})}{\beta^2}+\frac{L}{\beta}R_{kh}-\frac{L^2}{2\beta^2}S_{kh}\right.\\
&-\frac{2L^2}{\beta^3}\left(r_{0h}b_k+r_{0k}b_h\right)+\frac{L}{\beta^4}b_kb_h\left[3Lr_{00}+\beta R-3LS-\frac{L\beta\sigma_0}{(n+1)}\right]+\frac{1}{2\beta}\left(R_kl_h+R_hl_k\right)\\
&+\frac{l_kl_h}{\beta^2}\left[r_{00}-S-\frac{\beta\sigma_0}{(n+1)}\right]+\frac{L_{kh}}{\beta^2}\left[Lr_{00}-LS+\frac{\beta R}{2}-\frac{L\beta\sigma_0}{(n+1)}\right]-\frac{L}{2\beta^2}\left(R_kb_h+R_hb_k\right)\\
&+\frac{(l_kb_h+l_hb_k)}{\beta^3}\left[-2Lr_{00}+2LS-\frac{\beta R}{2}+\frac{L\beta\sigma_0}{(n+1)}\right]-\frac{L}{\beta^2}\left(S_kl_h+S_hl_k\right)+\frac{L^2}{\beta^3}\left(S_kb_h+S_hb_k\right)\\
&\left.-\frac{L}{\beta(n+1)}\left(\sigma_hl_k+\sigma_kl_h\right)+\frac{L^2}{2\beta^2(n+1)}\left(\sigma_hb_k+\sigma_kb_h\right)\right\}-\frac{g_{kh}}{\beta(n+1)}\left(\beta\sigma^j+2ns^j_0\right)\\
&+\frac{L}{\beta(n+1)}\left(\beta\sigma^r+2ns^r_0\right)\left[L\left(\dot\partial_hC^j_{rk}-2C^t_{rh}C^j_{tk}\right)+2\left(l_kC^j_{rh}+l_hC^j_{rk}\right)\right]-\frac{2nL^2}{\beta^3(n+1)}b_kb_hs^j_0\\
&+\frac{2nL}{\beta^2(n+1)}s^j_0\left(l_kb_h+l_hb_k\right)-\frac{2nL}{\beta(n+1)}\left(l_ks^j_h+l_hs^j_k\right)+\frac{nL^2}{\beta^2(n+1)}\left(b_ks^j_h+b_hs^j_k\right)\\
&-\frac{2nL^2}{\beta^2(n+1)}s^r_0\left(b_kC^j_{rh}+b_hC^j_{rk}\right)+\frac{2nL^2}{\beta(n+1)}\left(s^r_kC^j_{rh}+s^r_hC^j_{rk}\right),
\end{aligned}$$

where $R_{kh}=\dot\partial_hR_k$ and $S_{kh}=\dot\partial_hS_k$.

Differentiating (3.9) with respect to y^k and using (3.8) and $G^i_{jk}=\left(\dot\partial_kF^i_{jr}\right)y^r+F^i_{jk}$, we have

$$\dot\partial_jD^i_{0k}=\left(\dot\partial_kD^i_{jr}\right)y^r+D^i_{jk} \tag{3.39}$$

Thus, we have

Theorem 3.2. *The difference tensor D^i_{jk} of the Cartan connection coefficients F^{*i}_{jk} of the Finsler space F^{*n} with the generalized Kropina conformal changed metric L^* and the Cartan connection coefficients F^i_{jk} of the Finsler space F^n with the metric L is given by (3.39) together with (3.32), (3.33) and (3.38).*

4. Relation between Projective change and Generalized Kropina Conformal Change

Definition 4.1. Let us consider two Finsler spaces $F^n = (M^n, L)$ and $F^{*n} = (M^n, L^*)$ on the same manifold M^n. Then the transformation from F^n to F^{*n} which maps every geodesic of F^n to some geodesic of F^{*n} is known as projective change and the Finsler spaces F^n and F^{*n} are called projectively related Finsler spaces

It is well known that the change $L \longrightarrow L^*$ is projective if

$$G^{*i} = G^i + P(x,y)y^i, \tag{4.1}$$

where $P(x,y)$ is a homogeneous scalar function of degree one in y^i, called as projective factor.

Partial differentiation of (4.1) with respect to y^j gives

$$G^{*i}_j - G^i_j = P_j y^i + P\delta^i_j, \tag{4.2}$$

A geodesic of F^n is given by the system of differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i(x,y) = \tau y^i, \tag{4.3}$$

where $\tau = \frac{1}{L}\frac{dL}{dt}$, $y^i = \frac{dx^i}{dt}$ and t is the parameter.

The Euler-Lagrange equations for the Finsler space F^{*n} is given

$$\frac{\partial L^*}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L^*}{\partial y^i}\right) = 0. \tag{4.4}$$

Using (2.5) in (4.4), we find

$$\frac{\partial}{\partial x^i}\left(e^{\sigma(x)}\frac{L^{n+1}}{\beta^n}\right) - \frac{d}{dt}\left[\frac{\partial}{\partial y^i}\left(e^{\sigma(x)}\frac{L^{n+1}}{\beta^n}\right)\right] = 0, \tag{4.5}$$

which implies

$$\begin{aligned}
&e^{\sigma(x)}(n+1)\frac{L^n}{\beta^n}\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i}\right] + e^{\sigma(x)}\frac{L^{n+1}}{\beta^n}\frac{\partial\sigma(x)}{\partial x^i}\\
&- n(n+1)e^{\sigma(x)}\frac{L^{n-1}}{\beta^{n-2}}\left[\frac{\partial}{\partial y^i}\left(\frac{L}{\beta}\right)\right]\left[\frac{d}{dt}\left(\frac{L}{\beta}\right)\right]\\
&- ne^{\sigma(x)}\frac{L^{n+1}}{\beta^{n+1}}\left[\frac{\partial\beta}{\partial x^i} - \frac{d}{dt}\frac{\partial\beta}{\partial y^i}\right] = 0,
\end{aligned} \tag{4.6}$$

which reduces to

$$\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i}\right] + A_i = 0. \tag{4.7}$$

where A_i is the covariant vector defined as

$$\begin{aligned} A_i =& \frac{L}{(n+1)}\frac{\partial \sigma(x)}{\partial x^i} - n\frac{\beta^2}{L}\left[\frac{\partial}{\partial y^i}\left(\frac{L}{\beta}\right)\right]\left[\frac{d}{dt}\left(\frac{L}{\beta}\right)\right] \\ &- \frac{n}{(n+1)}\frac{L}{\beta}\left[\frac{\partial \beta}{\partial x^i} - \frac{d}{dt}\frac{\partial \beta}{\partial y^i}\right]. \end{aligned} \tag{4.8}$$

Thus, we conclude

Theorem 4.1. *A Finsler space $F^n = (M^n, L)$ and the Finsler space $F^{*n} = (M^n, L^*)$ whose metric L^* is obtained from the generalized Kropina conformal change of the metric L are projectively related if and only if the covariant vector A_i given by (4.8) vanishes identically.*

5. Hypersurfaces given by projective Generalized Kropina Conformal Change

Consider Finslerian hypersurfaces $F^{n-1} = \left(M^{n-1}, \underline{L}(u,v)\right)$ of F^n and $F^{*(n-1)} = \left(M^{n-1}, \underline{L^*}(u,v)\right)$ of F^{*n}. The functions $B^i_\alpha(u)$ may be considered as the components of $n-1$ linearly independent vectors tangent to F^{n-1}. Since N^i is the unit normal vector at a point u^α of F^{n-1}, the unit normal vector $N^{*i}(u,v)$ of $F^{*(n-1)}$ and the inverse projection factor $B^{*\alpha}_i$ along $F^{*(n-1)}$ are uniquely determined by

$$(a)\ g^*_{ij}B^i_\alpha N^{*j} = 0, \quad (b)\ g^*_{ij}N^{*i}N^{*j} = 1. \tag{5.1}$$

and

$$B^{*\alpha}_i = g^{*\alpha\beta}g^*_{ij}B^j_\beta\,. \tag{5.2}$$

where $g^{*\alpha\beta}$ is the inverse of metric tensor $g^*_{\alpha\beta}$ of $F^{*(n-1)}$.

From (5.1)(a), (5.1)(b) and (5.2), we have

$$(a)\ B^i_\alpha B^{*\beta}_i = \delta^\beta_\alpha, \quad (b)\ B^i_\alpha N^*_i = 0, \quad (c)\ N^{*i}B^{*\alpha}_i = 0, \quad (d)\ N^{*i}N^*_i = 1. \tag{5.3}$$

From (5.3), we have

$$B^i_\alpha B^{*\alpha}_j + N^{*i}N^*_j = \delta^i_j. \tag{5.4}$$

Transvection of (2.6)(a) with v^α gives

$$y_j N^j = 0. \tag{5.5}$$

Transvecting (3.4) with N^iN^j and using (2.6)(b) and (5.5), we have

$$g^*_{ij}N^iN^j = e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}} + e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_iN^i)^2. \tag{5.6}$$

which implies that $\frac{N^i}{\sqrt{e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}} + e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_iN^i)^2}}$ is a unit vector.

Transvecting (3.4) with $B^i_\alpha N^j$ and using (2.6)(a) and (5.5), we have

$$\begin{aligned} g^*_{ij}B^i_\alpha N^j = (b_jN^j)\Big\{ & e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_iB^i_\alpha) \\ & - e^{2\sigma(x)}2n(n+1)\frac{L^{2n+1}}{\beta^{2n+1}}l_iB^i_\alpha\Big\}, \end{aligned} \tag{5.7}$$

which shows that N^j is normal to $F^{*(n-1)}$ iff

$$(b_jN^j)\Big\{e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_iB^i_\alpha) - e^{2\sigma(x)}2n(n+1)\frac{L^{2n+1}}{\beta^{2n+1}}l_iB^i_\alpha\Big\} = 0. \tag{5.8}$$

This implies that either $e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_iB^i_\alpha) - e^{2\sigma(x)}2n(n+1)\frac{L^{2n+1}}{\beta^{2n+1}}l_iB^i_\alpha = 0$ or $b_jN^j = 0$.

Transvecting $e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_iB^i_\alpha) - e^{2\sigma(x)}2n(n+1)\frac{L^{2n+1}}{\beta^{2n+1}}l_iB^i_\alpha = 0$ with v^α and using $y^i = B^i_\alpha v^\alpha$, we have

$$e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}b_iy^i - e^{2\sigma(x)}2n(n+1)\frac{L^{2n+1}}{\beta^{2n+1}}l_iy^i = 0, \tag{5.9}$$

which gives

$$-ne^{2\sigma(x)}\frac{L^{2(n+1)}}{\beta^{2n+1}} = 0, \tag{5.10}$$

which is not possible. Hence we have

$$b_jN^j = 0. \tag{5.11}$$

Thus, the vector N^j is normal to $F^{*(n-1)}$ if and only if b_j is tangent to F^{n-1}. From (5.5), (5.7) and (5.10), we can say that $\frac{N^i}{\sqrt{e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}}}}$ is a unit normal vector of $F^{*(n-1)}$. Therefore, in view of (5.1)(a) and (5.1)(b), we have

$$N^{*i} = \frac{N^i}{\sqrt{e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}}}} \tag{5.12}$$

Transvecting (3.4) with N^{*j} and using (5.5), (5.11) and (5.12), we have

$$N_i^* = g_{ij}^* N^{*j} = \sqrt{e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}}}\, N_i\,. \tag{5.13}$$

Hence, we conclude

Theorem 5.1. *Let F^{*n} be the Finsler space obtained from F^n by a generalized Kropina conformal change. If $F^{*(n-1)}$ and F^{n-1} are the hypersurfaces of these spaces then the vector b_i is tangential to the hypersurface F^{n-1} if and only if every vector normal to F^{n-1} is also normal to $F^{*(n-1)}$.*

Suppose the generalized Kropina conformal change of metric is projective. We shall call such change of metric as projective generalized Kropina conformal change of metric.

From (3.9) and (4.2), we have

$$D_{0j}^i = P_j y^i + P\delta_j^i. \tag{5.14}$$

Transvecting (5.14) with $N_i B_\alpha^j$ and using (2.9)(b), (2.9)(e) and (5.5), we have

$$N_i D_{0j}^i B_\alpha^j = 0. \tag{5.15}$$

If each geodesic of the hypersurface F^{n-1} with respect to the induced metric is also a geodesic of a Finsler space F^n then F^{n-1} is known as totally geodesic hypersurface [2]. A totally geodesic hypersurface is characterised by $H_\alpha = 0$.

The normal curvature vector H_α^* on $F^{*(n-1)}$ is given by

$$H_\alpha^* = N_i^*\left(B_{0\alpha}^i + G_j^{*i} B_\alpha^j\right), \tag{5.16}$$

Using (3.9), (5.13) in (5.16), we have

$$H_\alpha^* = H_\alpha\sqrt{e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}}} + N_i D_{0j}^i B_\alpha^j\sqrt{e^{2\sigma(x)}(n+1)\frac{L^{2n}}{\beta^{2n}}}. \tag{5.17}$$

From (5.15) and (5.17), we have

$$H_\alpha^* = e^{\sigma(x)}\frac{L^n}{\beta^n}\sqrt{(n+1)}H_\alpha, \tag{5.18}$$

which in view of (2.5) gives

$$H_\alpha^* = \sqrt{(n+1)}\frac{L^*}{L}H_\alpha. \tag{5.19}$$

Since $\sqrt{(n+1)}\frac{L^*}{L} \neq 0$, the vanishing of H_α implies and implied by the vanishing of H^*_α.

This leads to:

Theorem 5.2. *Let F^{*n} be the Finsler space obtained from the Finsler space $F^n(n > 3)$ by a projective generalized Kropina conformal change then the hypersurface $F^{*(n-1)}$ of F^{*n} is totally geodesic if and only if the hypersurface F^{n-1} of F^n is totally geodesic.*

6. Hypersurfaces of projectively flat Finsler spaces

Consider a projective generalized Kropina conformal change. If there exists a projective change $L \longrightarrow L^*$ of a Finsler space $F^n = (M^n, L)$ such that the Finsler space $F^{*n} = (M^n, L^*)$ is a locally Minkowskian space then F^n is called projectively flat space.

In 1986, Yamada[12] proved that if F^n is projectively flat then the totally geodesic hypersurface F^{n-1} of F^n is also projectively flat.

In 1980, Matsumoto[7] showed that a Finsler space $F^n(n > 2)$ is projectively flat iff Weyl torsion tensor W^i_{jk} and Douglas tensor D^i_{jkh} vanish,i.e.

$$(a)\ W^i_{jk} = 0, \quad (b)\ D^i_{jkh} = 0. \tag{6.1}$$

Under the projective change, Weyl torsion tensor W^i_{jk} and Douglas tensor D^i_{jkh} are invariant, i.e.

$$(a)\ W^{*i}_{jk} = W^i_{jk}, \quad (b)\ D^{*i}_{jkh} = D^i_{jkh}. \tag{6.2}$$

From theorem 5.2 and equations (6.1) and (6.2), we conclude

Theorem 6.1. *Let F^{*n} be the Finsler space obtained from the Finsler space $F^n(n > 3)$ by a projective generalized Kropina conformal change and F^n be projectively flat. If $F^{*(n-1)}$ and F^{n-1} are the hypersurfaces of these spaces and F^{n-1} is totally geodesic then $F^{*(n-1)}$ is projectively flat.*

REFERENCES

[1] C. Shibata, On invariant tensors of β- changes of Finsler metrics, *J. Math. Kyoto Univ.*, **24(1)** (1984) 163-188.

[2] H. Rund, *The differential geometry of Finsler Spaces*, Springer-Verlag, (1959).

[3] M. K. Gupta and P. N. Pandey, On hypersurface of a Finsler space with Randers conformal metric, *Tensor N.S.*, **70(3)** (2008) 229-240.

[4] M. K. Gupta and P. N. Pandey, On Hypersurface of a Finsler space with a special metric, *Acta. Math. Hungar.* **120** (2008) 165-177.

[5] M. K. Gupta and P. N. Pandey, Hypersurfaces of conformally and h-conformally related Finsler space, *Acta Math. Hungar.*, **123(3)** (2009) 257-264.

[6] M. K. Gupta, Abhay Singh and P. N. Pandey, On a Hypersurface of a Finsler Space with Randers Change of Matsumoto Metric, *Geometry* (2013).

[7] Makoto Matsumoto, Projective changes of Finsler metrics and projectively flat Finsler spaces, *Tensor N. S.*, **34** (1980) 303-315.

[8] Makoto Matsumoto, The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry, *J. Math. Kyoto Univ.*, **25(1)** (1985) 107-144.

[9] Makoto Matsumoto, Theory of Finsler spaces with $(\alpha\beta)$-metric, *Reports on Mathematical Physics*, **31(1)** (1992) 43-83.

[10] M. S. Knebelman, Conformal geometry of generalized metric spaces, *Proc. Nat. Acad. Sci.*, **15** (1929) 376-379.

[11] P. L. Antonelli (ed.), *Handbook of Finsler Geometry*, Kluwer Academic Publishers, Dordrecht, (2003).

[12] T. Yamada, On Finsler hypersufaces satisfying certain conditions, *Tensor N. S.*, **43** (1986) 56-65.

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