FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 29, No 3 (2014), 281–293

WEIGHTED SHARP INEQUALITY FOR VECTOR-VALUED MULTILINEAR SINGULAR INTEGRAL OPERATOR

Qiaozhen Zhao and Junfeng Wang

Abstract. In this paper, we prove a sharp inequality for some vector-valued multilinear singular integral operators. By using this inequality, we obtain the weighted L^p-norm inequality and $LlogL$-type inequality for the vector-valued multilinear operators.

Keywords: Vector-valued multilinear operator; Singular integral operator; Sharp inequality; BMO; A_p-weight.

1. Introduction and Results

In this paper, we shall study some vector-valued multilinear singular integral operators which are defined as following.

Fixed $\varepsilon > 0$. Let $T : S \to S'$ be a linear operator and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y)dy$$

for every bounded and compactly supported function f, where K satisfies:

$$|K(x, y)| \leqslant C|x - y|^{-n}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leqslant C|y - z|^{\varepsilon}|x - z|^{-n-\varepsilon}$$

if $2|y - z| \leqslant |x - z|$. Let m_j be the positive integers ($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and A_j be the functions on R^n ($j = 1, \cdots , l$). For $1 < s < \infty$, the vector-valued multilinear operator associated to T is defined by

$$|T_A(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T_A(f_i)(x)|^s\right)^{1/s},$$

Received June 17, 2014.; Accepted July 07, 2014.
2010 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25

where

$$T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} K(x,y) f_i(y) dy$$

and

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leqslant m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

We also denote

$$|T(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^s \right)^{1/s} \text{ and } |f(x)|_s = \left(\sum_{i=1}^{\infty} |f_i(x)|^s \right)^{1/s}.$$

Suppose that $|T|_r$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ and weak (L^1, L^1)-bounded.

Note that when $m = 0$, T_A is just the vector-valued multilinear commutator of T and A (see [11], [12]). While when $m > 0$, T_A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1], [2],[3],[4], [5], [8],[14]). Hu and Yang (see [7]) proved a variant sharp estimate for the multilinear singular integral operators. In [11], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator when $A_j \in Osc_{expL^{r_j}}$. The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear singular integral operators when $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$. As the application, we obtain the weighted L^p-norm inequality and $LlogL$-type inequality for the vector-valued multilinear operators.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows,

$$f_Q = |Q|^{-1} \int_Q f(x) dx.$$

It is well-known that (see [6])

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and

$$\|f\|_{BMO} = \|f^\#\|_{L^\infty}.$$

For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x\in Q} |Q|^{-1} \int_Q |f(y)|dy,$$

For $k \in N$, we denote by M^k the operator M iterated k times, i.e.,

$$M^1(f)(x) = M(f)(x)$$

and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \text{ when } k \geqslant 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f,

$$||f||_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \leqslant 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in Q} ||f||_{\Phi,Q}.$$

The Young functions to be used in this paper are

$$\Phi(t) = t(1+logt)^r, \;\; \tilde{\Phi}(t) = exp(t^{1/r}),$$

the corresponding average and maximal functions denoted by $||\cdot||_{L(logL)^r,Q}$, $M_{L(logL)^r}$ and $||\cdot||_{expL^{1/r},Q}$, $M_{expL^{1/r}}$. Following [11], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \leqslant ||f||_{\Phi,Q}||g||_{\tilde{\Phi},Q}$$

and the following inequality, for $r, r_j \geqslant 1$, $j = 1, \ldots, l$ with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$||f||_{L(logL)^{1/r},Q} \leqslant M_{L(logL)^{1/r}}(f) \leqslant CM_{L(logL)^l}(f) \leqslant CM^{l+1}(f),$$

$$||b - b_Q||_{expL^r,Q} \leqslant C||b||_{BMO},$$

$$|b_{2^{k+1}Q} - b_{2Q}| \leqslant Ck||b||_{BMO}.$$

We denote the Muckenhoupt weights by A_p for $1 \leqslant p < \infty$ (see [6]).

We shall prove the following theorems.

Theorem 1. Let $1 < s < \infty$, $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then there exists a constant $C > 0$ such that for any $f = \{f_i\} \in C_0^\infty(R^n)$, $0 < r < 1$ and $x \in R^n$,

$$(|T_A(f)|_s)_r^\#(x) \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M^{l+1}(|f|_s)(x).$$

Theorem 2. Let $1 < s < \infty$, $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then $|T_A|_s$ is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_p$, that is

$$\||T_A(f)|_s\|_{L^p(w)} \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \||f|_s\|_{L^p(w)},$$

Theorem 3. Let $1 < s < \infty$, $w \in A_1$, $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$w(\{x \in R^n : |T_A(f)(x)|_s > \lambda\}) \leqslant C \int_{R^n} \frac{|f(x)|_s}{\lambda} \left[1 + log^+ \left(\frac{|f(x)|_s}{\lambda} \right) \right]^l w(x)dx.$$

2. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1.([3]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leqslant C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2.([6]) Let $0 < p < q < \infty$ and for any function $f \geqslant 0$. We define that, for $1/r = 1/p - 1/q$

$$\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p} / \|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$\|f\|_{WL^q} \leqslant N_{p,q}(f) \leqslant (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

Lemma 3.([11]) Let $r_j \geqslant 1$ for $j = 1, \cdots, l$, we denote $1/r = 1/r_1 + \cdots + 1/r_l$. Then

$$\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_l(x) g(x)| dx \leqslant \|f\|_{expL^{r_1},Q} \cdots \|f\|_{expL^{r_l},Q} \|g\|_{L(logL)^{1/r},Q}.$$

Proof of Theorem 1. It suffices to prove for $f = \{f_i\} \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \|T_A(f)(x)|_s^r - C_0|dx\right)^{1/r} \leqslant C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{BMO}\right)M^{l+1}(|f|_s)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha\tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i\chi_{\tilde{Q}}$ and $h_i = f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned}
T_A(f_i)(x) &= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m}K(x, y)f_i(y)dy\\
&= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m}K(x, y)h_i(y)dy\\
&\quad + \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m}K(x, y)g_i(y)dy\\
&\quad - \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x, y)g_i(y)dy\\
&\quad - \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x, y)g_i(y)dy\\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x, y)g_i(y)dy,
\end{aligned}$$

then, by the Minkowski' inequality,

$$\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q\left||T_A(f)(x)|_s^r - |T_{\tilde{A}}(h)(x_0)|_s^r\right|dx\right]^{1/r}\\
&\leqslant \left[\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^\infty|T_A(f_i)(x) - T_{\tilde{A}}(h_i)(x_0)|^s dx\right)^{1/s}\right]^{1/r}\\
&\leqslant \left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^\infty\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m}K(x, y)g_i(y)dy\right|^s\right)^{1/s}dx\right]^{1/r}
\end{aligned}$$

$$+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy\right|^s\right)^{1/s}dx\right]^{1/r}$$

$$+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy\right|^s\right)^{1/s}dx\right]^{1/r}$$

$$+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)dy\right|^s\right)^{1/s}dx\right]^{1/r}$$

$$+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0)\right|^s\right)^{1/s}dx\right]^{1/r}$$

$$:= I_1+I_2+I_3+I_4+I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$R_m(\tilde{A}_j;x,y)\leqslant C|x-y|^m\sum_{|\alpha_j|=m}\|D^{\alpha_j}A_j\|_{BMO},$$

thus, by Lemma 2 and the weak type (1,1) of $|T|_r$, we obtain

$$\begin{aligned}
I_1 &\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{BMO}\right)\left(\frac{1}{|Q|}\int_Q|T(g)(x)|_s^r dx\right)^{1/r}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{BMO}\right)|Q|^{-1}\frac{\||T(g)|_s\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{BMO}\right)|Q|^{-1}\||T(g)|_s\|_{WL^1}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{BMO}\right)|\tilde{Q}|^{-1}\||g|_s\|_{L^1}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{BMO}\right)M(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_2, we get, by Lemma 2 and generalized Hölder's inequality,

$$\begin{aligned}
I_2 &\leqslant C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_s^r dx\right)^{1/r}\\
&\leqslant C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}\frac{\||T(D^{\alpha_1}\tilde{A}_1g)|_s\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\
&\leqslant C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}\||T(D^{\alpha_1}\tilde{A}_1g)|_s\|_{WL^1}\\
&\leqslant C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{BMO}\sum_{|\alpha_1|=m_1}|\tilde{Q}|^{-1}\|D^{\alpha_1}\tilde{A}_1|g|_s\|_{L^1}\\
&\leqslant C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{BMO}\sum_{|\alpha_1|=m_1}\|D^{\alpha_1}A_1-(D^{\alpha}A_1)_{\tilde{Q}}\|_{expL,\tilde{Q}}\||f|_s\|_{L(logL),\tilde{Q}}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{BMO}\right)M^2(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{BMO}\right)M^2(|f|_s)(\tilde{x}).$$

Similarly, for I_4, taking $r, r_1, r_2 \geqslant 1$ such that $1/r = 1/r_1 + 1/r_2$, we obtain, by Lemma 3,

$$\begin{aligned}
I_4 &\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)(x)|_s^r dx\right)^{1/r}\\
&\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1}\frac{\||T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)|_s\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\
&\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1}\||T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)|_s\|_{WL^1}\\
&\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1}\|D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2|g|_s\|_{L^1}\\
&\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\prod_{j=1}^{2}\left\|D^{\alpha_j}A_j-(D^{\alpha_j}A_j)_{\tilde{Q}}\right\|_{expL^{r_j},\tilde{Q}}\cdot\||f|_s\|_{L(logL)^{1/r},\tilde{Q}}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{BMO}\right)M^3(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$
\begin{aligned}
T_{\tilde{A}}(h_i)(x) - T_{\tilde{A}}(h_i)(x_0) &= \int_{R^n}\left(\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y)h_i(y)dy \\
&+\int_{R^n}\left(R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x_0,y)\right)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy \\
&+\int_{R^n}\left(R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y)\right)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy \\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y) - \frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}K(x_0,y)\right] \\
&\quad\times D^{\alpha_1}\tilde{A}_1(y)h_i(y)dy \\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y) - \frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}K(x_0,y)\right] \\
&\times D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy \\
&+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0,y)\right] \\
&\quad\times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy \\
&= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.
\end{aligned}
$$

By Lemma 1 and the following inequality (see [13])

$$
|b_{Q_1} - b_{Q_2}| \leqslant C\log(|Q_2|/|Q_1|)\|b\|_{BMO} \text{ for } Q_1 \subset Q_2,
$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_m(\tilde{A};x,y)| &\leqslant C|x-y|^m \sum_{|\alpha|=m}(\|D^\alpha A\|_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|) \\
&\leqslant Ck|x-y|^m \sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}.
\end{aligned}
$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on K,

$$|I_5^{(1)}| \leqslant C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{m+n+\varepsilon}}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)||h_i(y)|dy$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}}\right)|f_i(y)|dy$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f_i(y)|dy,$$

thus, by the Minkowski' inequality,

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(1)}\right|^s\right)^{1/s}$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_s dy$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M(|f|_s)(\tilde{x}).$$

For $I_5^{(2)}$, by the formula (see [3]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\leqslant C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}\|D^\alpha A\|_{BMO},$$

thus

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(2)}\right|^s\right)^{1/s}$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|_s dy$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M(|f|_s)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(3)}\right|^r\right)^{1/r} \leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M(|f|_s)(\tilde{x}).$$

For $I_5^{(4)}$, recall that

$$|b_{2^{k+1}Q}-b_{2Q}| \leqslant Ck\|b\|_{BMO},$$

we get

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}\left|I_5^{(4)}\right|^s\right)^{1/s}\\
\leqslant& C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|\\
&\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||h(y)|_s dy\\
&+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}|\\
&\times D^{\alpha_1}\tilde{A}_1(y)||h(y)|_s dy\\
\leqslant& C\sum_{|\alpha|=m_2}\|D^\alpha A_2\|_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_s|D^{\alpha_1}\tilde{A}_1(y)|dy\\
\leqslant& C\sum_{|\alpha|=m_2}\|D^\alpha A_2\|_{BMO}\\
&\times\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\|D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}\|_{expL,2^k\tilde{Q}}\cdot\||f|_s\|_{L(logL),2^k\tilde{Q}}\\
\leqslant& C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})\||f|_s\|_{L(logL),2^k\tilde{Q}}\\
\leqslant& C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M^2(|f|_s)(\tilde{x}).
\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(5)}\right|^s\right)^{1/s} \leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M^2(|f|_s)(\tilde{x}).$$

For $I_5^{(6)}$, similarly to the proof of I_4, we get

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(6)}\right|^s\right)^{1/s}$$

$$\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m}\right|$$

$$\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||h(y)|_s dy$$

$$\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_s|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)|dy$$

$$\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\prod_{j=1}^{2}\left\|D^{\alpha_j}A_j-(D^{\alpha_j}A_j)_{\tilde{Q}}\right\|_{expL^{r_j},2^k\tilde{Q}}\cdot\||f|_s\|_{L(logL)^{1/r},2^k\tilde{Q}}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{BMO}\right)M^3(|f|_s)(\tilde{x}).$$

Thus

$$|I_5|\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{BMO}\right)M^3(|f|_s)(\tilde{x}).$$

This completes the proof of Theorem 1.

By Theorem 1 and the $L^p(w)$-boundedness of M^3, we may obtain the conclusions of Theorem 2. By Theorem 1 and [8][10], we may obtain the conclusions of Theorem 3.

3. Example

In this section we shall apply Theorems 1, 2 and 3 of the paper to the Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [4],[6],[13]), the vector-valued multilinear operator related to T is defined by

$$|T_A(f)(x)|_r=\left(\sum_{i=1}^{\infty}|T_A(f_i)(x)|^r\right)^{1/r},$$

where

$$T_A(f_i)(x)=\int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(A_j;x,y)}{|x-y|^m}K(x,y)f_i(y)dy.$$

Then

$$(1)\quad (|T^A(f)|_s)_r^{\#}(x)\leqslant C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{BMO}\right)M^{l+1}(|f|_s)(x)$$

for any $f \in C_0^\infty(R^n)$, $1 < s < \infty$ and $0 < r < 1$;

$$(2)\quad \||T^A(f)|_s\|_{L^p(w)} \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \|f\|_{L^p(w)}$$

for any $1 < s < \infty$, $w \in A_p$ and $1 < p < \infty$;

$$(3)\quad w(\{x \in R^n : |T^A(f)(x)|_s > \lambda\}) \leqslant C \int_{R^n} \frac{|f(x)|_s}{\lambda} \left[1 + log^+ \left(\frac{|f(x)|_s}{\lambda} \right) \right]^l w(x)dx$$

for any $1 < s < \infty$, $w \in A_1$ and all $\lambda > 0$.

REFERENCES

1. J. Cohen: *A sharp estimate for a multilinear singular integral on R^n*. Indiana Univ. Math. J. **30** (1981), 693–702.
2. J. Cohen and J. Gosselin: *On multilinear singular integral operators on R^n*. Studia Math. **72** (1982), 199–223.
3. J. Cohen and J. Gosselin: *A BMO estimate for multilinear singular integral operators*. Illinois J. Math. **30** (1986), 445–465.
4. R. Coifman and Y. Meyer: *Wavelets, Calderón-Zygmund and multilinear operators*. Cambridge Studies in Advanced Math. 48, Cambridge University Press, Cambridge, 1997.
5. Y. Ding and S. Z. Lu: *Weighted boundedness for a class rough multilinear operators*. Acta Math. Sinica **17** (2001), 517–526.
6. J. Garcia-Cuerva and J. L. Rubio de Francia: *Weighted norm inequalities and related topics*. North-Holland Math. 116, Amsterdam, 1985.
7. G. Hu and D. C. Yang: *A variant sharp estimate for multilinear singular integral operators*. Studia Math. **141** (2000), 25–42.
8. D. Lu: *Sharp estimates and boundedness for multilinear integral operators*. Facta Universitatis, Series Math. and Informatics. **27**(2012), 255–270.
9. C. Pérez: *Endpoint estimate for commutators of singular integral operators*. J. Func. Anal. **128** (1995), 163–185.
10. C. Pérez and G. Pradolini: *Sharp weighted endpoint estimates for commutators of singular integral operators*. Michigan Math. J. **49** (2001), 23–37.
11. C. Pérez and R. Trujillo-Gonzalez: *Sharp weighted estimates for multilinear commutators*. J. London Math. Soc. **65** (2002), 672–692.
12. C. Pérez and R. Trujillo-Gonzalez: *Sharp weighted estimates for vector-valued singular integral operators and commutators*. Tohoku Math. J. **55** (2003), 109–129.
13. E. M. Stein: *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*. Princeton Univ. Press, Princeton NJ, 1993.
14. X. S. Zhou: *Sharp weighted boundedness for vector-valued multilinear singular integral operator*. Facta Universitatis, Series Math. and Informatics. **27**(2012), 345–356.

Qiaozhen Zhao
Mathematics teaching and research group
Jiaxing Nanyang Polytechnic Institute
Zhejiang, Jia xing, 314003, P. R. of China
zzhaoo@126.com

Junfeng Wang
Information Technology Branch Institute
Jiaxing Vocational Technical College
Zhejiang, Jia xing, 314036, P. R. of China
yidaidiguo@qq.com