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# SOME CLASSES OF CONVEX FUNCTIONS ON TIME SCALES *

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**Abstract.** We have introduced diamond $\phi_{h-s,\mathbb{T}}$ derivative and diamond $\phi_{h-s,\mathbb{T}}$ integral on an arbitrary time scale. Moreover, various interconnections with the notion of classes of convex functions about these new concepts are also discussed.
**Keywords:** convex functions, time scale, dynamic derivatives.

## 1. Introduction

There has been a considerable amount of interest and publications in the theory and applications of dynamic derivatives on time scales. The study, which was initiated by Stephen Hilger (1988) in his PhD thesis, unifies traditional concepts of derivatives and differences, and it also reveals diversities in the corresponding results. The investigations are not only significant in the theoretical research of differential and difference equations, but also crucial in many computational and numerical applications (see for example [4], [14] and [25]).

The study helps to avoid proving a result twice, once for differential equations, and once for difference equations. Once we have proved a result for a general time scale, by choosing the set of real numbers $\mathbb{R}$, the derivative and integral are easily seen to be the 'usual' derivative and integral respectively. Furthermore, when we choose the time scale to be the set of integers $\mathbb{Z}$, the same general result yields a result for difference equations and integral respectively. Hence all results that are proved on a general time scale include results for both differential and difference equations.

The time scale theory has advanced fast since its introduction. Lately, the applications of its calculus in Engineering, Biology, Physics, Medical Sciences, Economics and Finance, Chemistry and Others, have come to light. For a good introduction to the theory of time scales and more details, see [1]-[6].

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## 2. Basic Concepts and Definitions

A *time scale*, denoted $\mathbb{T}$, is an arbitrary nonempty closed subset of the real numbers. We assume that a time scale is endowed with the topology inherited from $\mathbb{R}$ with the standard topology. Thus, the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, and the non negative integers $\mathbb{N}_0$ are examples of time scales, as are $[0,1]\cup[2,3]$, $[0,1]\cup\mathbb{N}$, and the Cantor Set, while the rational numbers $\mathbb{Q}$, irrational numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the open interval $(0,1)$, are not time scales.

Throughout this paper, we will denote a time scale by $\mathbb{T}$, and for any $I$, interval of $\mathbb{R}$ (open or closed, finite or infinite), $I_{\mathbb{T}} = I\cap\mathbb{T}$, a time scale interval.

**Definition 2.1.** Let $\mathbb{T}$ be a time scale. For all $t\in\mathbb{T}$, the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by

$$\sigma(t) = \inf\{\tau\in\mathbb{T} : \tau\geqslant t\}\,\forall\, t\in\mathbb{T},$$

while the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t) = \sup\{\tau\in\mathbb{T} : \tau\leqslant t\}\,\forall\, t\in\mathbb{T}.$$

We make the convention:
$\inf\emptyset = \sup\mathbb{T}$ (i.e, $\sigma(t)=t$ if $\mathbb{T}$ has a maximum $t$) and
$\sup\emptyset = \inf\mathbb{T}$ (i.e, $\rho(t)=t$ if $\mathbb{T}$ has a minimum $t$), where $\emptyset$ denotes the empty set.

The point $t$ is said to be *right-scattered* if $\sigma(t)>t$, respectively *left-scattered* if $\rho(t)<t$. Points that are right-scattered and left-scattered at the same time are called *isolated*. The point $t$ is called *right-dense* if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, respectively *left-dense* if $t>\inf\mathbb{T}$ and $\rho(t)=t$. Points that are simultaneously right-dense and left-dense are called *dense*.

The mappings $\mu,\nu:\mathbb{T}\to[0,+\infty]$ is defined by

$$\mu(t) = \sigma(t) - t$$

and

$$\nu(t) = t - \rho(t)\ \forall\ t\in\mathbb{T}$$

are called, the *forward and backward graininess functions* respectively.

Suppose that $\mathbb{T}^k$, $\mathbb{T}_k$, and $\mathbb{T}^k_k$ are sets derived from the time scale $\mathbb{T}$ as follows:

$$\mathbb{T}^k := \begin{cases} \mathbb{T}-\{M\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximum point M} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

and

$$\mathbb{T}_k := \begin{cases} \mathbb{T}-\{m\}, & \text{if } \mathbb{T} \text{ has a right-scattered minimum point m} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

We set $\mathbb{T}_k^k = \mathbb{T}_k \cap \mathbb{T}^k$.

Given $f : \mathbb{T} \to \mathbb{R}$, the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e $f^\sigma = f \circ \sigma$. Also, the function $f^\rho : \mathbb{T} \to \mathbb{R}$ is defined by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, i.e, $f^\rho = f \circ \rho$.

**Definition 2.2.** (i). Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. The *delta derivative* of $f$ in $t$ is the number $f^\Delta(t)$ (when it exists) with the property that for any $\epsilon > 0$, there is a neighbourhood $U$ of $t$ such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)\,(\sigma(t) - s)\right| < \epsilon\,|\sigma(t) - s|$$

for all $s \in U$.

We call $f^\Delta(t)$ the delta (or Hilger) derivative of $f$ at $t$.

(ii). Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$. The *nabla derivative* of $f$ in $t$ is the number denoted by $f^\nabla(t)$ (when it exists), with the property that, for any $\epsilon > 0$, there is a neighbourhood $V$ of $t$ such that

$$\left|f(\rho(t)) - f(s) - f^\nabla(t)\,(\rho(t) - s)\right| < \epsilon\,|\rho(t) - s|\,,$$

for all $s \in V$.

(iii). The function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* on $\mathbb{T}^k$, provided $f^\Delta(t) = \lim_{s\to t} \frac{f(\sigma(t))-f(s)}{\sigma(t)-s}$ exists where $s \to t, s \in \mathbb{T} \setminus \{\sigma(t)\}$ for all $t \in \mathbb{T}^k$.

(iv). The function $f : \mathbb{T} \to \mathbb{R}$ is *nabla differentiable* on $\mathbb{T}_k$, provided $f^\nabla(t) = \lim_{s\to t} \frac{f(s)-f(\rho(t))}{s-\rho(t)}$ exists where $s \to t, s \in \mathbb{T} \setminus \{\rho(t)\}$ for all $t \in \mathbb{T}_k$.

(v). The function $f : \mathbb{T} \to \mathbb{R}$ is said to be differentiable on $\mathbb{T}_k^k$ provided $f^\Delta(t)$ and $f^\nabla(t)$ exists for all $t \in \mathbb{T}_k^k$.

**Remark 2.1.** 1. When $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f^\nabla(t) = f'(t)$ becomes the total differential operator (ordinary derivative).

2. When $\mathbb{T} = \mathbb{Z}$, then
   (i) $f^\Delta(t) = f(t+1) - f(t)$ and $f^{\Delta^r}(t) = f^{\Delta^{r-1}}(t+1) - f^{\Delta^r}(t)$ are the forward and r-th forward difference operators
   (ii) $f^\Delta(t) = f(t+\frac{1}{2}) - f(t-\frac{1}{2})$ is the central difference operator
   (iii) $f^\nabla(t) = f(t) - f(t-1)$ and $f^{\nabla^r}(t) = f^{\nabla^{r-1}}(t) - f^{\nabla^r}(t-1)$ are the backward and r-th backward difference operators
   (iv) The shift operator $E$, is defined as $Ef(t) = f(t+1)$ so that $E^{-1}f(t) = f(t-1)$ and $E^{-\frac{1}{2}}f(t+\frac{1}{2}) = f(t+1)$
   (v) Also, the mean operator, $\mu$, can be given as$f^\mu(t+\frac{1}{2}) = \frac{1}{2}\{f(t) + f(t+1)\}$.

   Several new important relationships may be established between these five operators $(i) - (v)$ above in terms of the shift operator, $E$ as follows:

(i) $f^{\Delta}(t) = (E-1)f(t)$
(ii) $f^{\nabla}(t) = (1-E^{-1})f(t)$
(iii) $f^{\delta}(t) = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})f(t)$
(iv) $f^{\mu}(t) = \frac{1}{2}\{E^{\frac{1}{2}} - E^{-\frac{1}{2}}\}f(t)$.

3. Let $h > 0$. If $\mathbb{T} = h\mathbb{Z}$, then $f^{\Delta}(t) = \frac{f(t+h)-f(t)}{h}$ and $f^{\nabla}(t) = \frac{f(t)-f(t-h)}{h}$ are the $h$-derivatives.
4. Let $q > 1$. If $\mathbb{T} = q^{N_0}$, where $N_0 = 0, 1, 2, ...$, then $f^{\Delta}(t) = \frac{f(qt)-f(t)}{t(q-1)}$ and $f^{\nabla}(t) = \frac{q(f(t)-f(\frac{t}{q}))}{t(q-1)}$ are the $q$-derivatives.

The following lemma is useful in the sequel.

**Lemma 2.1.** *Let $f, g : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then the following holds.*

*(i) If $f^{\Delta}(t)$ exists, i.e, $f$ is delta differentiable at $t$, then $f$ is continuous at $t$.*

*(ii) If $f$ is left-continuous at $t$ and $t$ is right-scattered, then $f$ is delta differentiable at $t$ with $f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$.*

*(iii) If $t$ is right-dense, then $f^{\Delta}(t)$ exists, if and only if, the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\Delta}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.*

*(iv) If $f^{\Delta}$ exists on $\mathbb{T}^k$ and $f$ is invertible on $\mathbb{T}$, then $(f^{-1})^{\Delta} = -(f^{\sigma})^{-1} f^{\Delta} f^{-1}$ on $\mathbb{T}^k$.*

*(v) If $f^{\Delta}(t)$, $g^{\Delta}(t)$ exist and $(fg)(t)$ is defined, then $(fg)^{\Delta}(t) = f(\sigma(t))g^{\Delta}(t) + f^{\Delta}(t)g(t)$.*

*Also, given $f, g : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$. Then the following holds.*

*(i) If $f^{\nabla}(t)$ exists, i.e, $f$ is nabla differentiable at $t$, then $f$ is continuous at $t$.*

*(ii) If $f$ is right-continuous at $t$ and $t$ is left-scattered, then $f$ is nabla differentiable at $t$ with $f^{\nabla}(t) = \frac{f(t)-f(\rho(t))}{\nu(t)}$.*

*(iii) If $t$ is left-dense, then $f^{\nabla}(t)$ exists, if and only if , the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\nabla}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.*

*(iv) If $f^{\nabla}$ exists on $\mathbb{T}_k$ and $f$ is invertible on $\mathbb{T}$, then $(f^{-1})^{\nabla} = -(f^{\rho})^{-1} f^{\nabla} f^{-1}$ on $\mathbb{T}_k$.*

*(v) If $f^{\nabla}(t)$, $g^{\nabla}(t)$ exist and $(fg)(t)$ is defined, then $(fg)^{\nabla}(t) = f(\rho(t))g^{\nabla}(t) + f^{\nabla}(t)g(t)$.*

**Definition 2.3.** 1. A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it satisfies the following two conditions:

(i) $f$ is continuous at all right-dense point or maximal element of $\mathbb{T}$,

(ii) the left-sided limit $\lim_{s\to t^-} f(s) = f(t^-)$ exists (finite) at each left-dense point $t \in \mathbb{T}$.

We denote by $C_{rd}(\mathbb{T}, \mathbb{R})$ the set of rd-continuous functions.

2. A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be ld-continuous if it satisfies the following two conditions:

(i) $f$ is continuous at all left-dense point or minimal element of $\mathbb{T}$,

(ii) the right-sided limit $\lim_{s\to t^+} f(s) = f(t^+)$ exists (finite) at each right-dense point $t \in \mathbb{T}$.

We denote by $C_{ld}(\mathbb{T}, \mathbb{R})$ the set of ld-continuous function.
Obviously, the set of continuous functions on $\mathbb{T}$ contains both $C_{rd}$ and $C_{ld}$.

**Definition 2.4.** (i). A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. In this case, the delta integral of $f$ is defined as

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s)$$

for all $s, t \in \mathbb{T}$.

(ii). A function $G : \mathbb{T} \to \mathbb{R}$ is called a *nabla antiderivative* of $g : \mathbb{T} \to \mathbb{R}$ if $G^{\nabla}(t) = g(t)$ for all $t \in \mathbb{T}_k$. In this case, the nabla integral of $g$ is defined as

$$\int_s^t g(\tau)\nabla\tau = G(t) - G(s)$$

for all $s, t \in \mathbb{T}$.

Every rd-continuous function has a delta antiderivative and every ld-continuous function has a nabla antiderivative (see [2], Theorem 1.74, [26] and [30]).

**Theorem 2.1.** *([2], Theorem 1.75).*

*(i) If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then $\int_t^{\sigma} tf(s)\Delta s = \mu(t)f(t)$.*

*(ii) If $g \in C_{ld}$ and $t \in \mathbb{T}_k$, then $\int_{\rho}^{t} tg(s)\nabla s = \nu(t)g(t)$.*

The following theorem shows some basic operations with the delta integral.

**Theorem 2.2.** *(see [7], Theorem 2.2). If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

*(i) $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$*

*(ii) $\int_a^b (\alpha f)t\Delta t = \alpha \int_a^b f(t)\Delta t$*

*(iii)* $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$

*(iv)* $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$

*(v)* $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$

*(vi)* $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$

*(vii)* $\int_a^a f(t)\Delta t = 0$

*(viii)* *If* $f(t) \geq 0$ *for all* $a \leq t < b$*, then* $\int_a^b f(t)\Delta t \geq 0$

*(ix)* *If* $|f(t)| \leq g(t)$ *on* $[a,b)$*, then* $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t.$

An analogous version of Theorem 2.2 holds for the nabla antiderivative of functions in $C_{ld}$.

In [7] and [9], Dinu established the diamond-$\alpha$ dynamic derivatives which are linear combinations of the delta and nabla dynamic derivatives on time scales. Motivated by this, we have introduced and discussed some basic properties of the diamond-$\phi_{h-s,\mathbb{T}}$ derivative.
We begin with the following new definition.

**Definition 2.5.** Let $\mathbb{T}$ be a time scale and $h : J_{\mathbb{T}} \to \mathbb{R}$ a given real-valued function. For $m, n \in \mathbb{T}_k^k$, set $\mu_{mn} = \sigma(m) - n$ and $\nu_{mn} = \rho(m) - n$. We define the diamond-$\phi_{h-s,\mathbb{T}}$ dynamic derivative of a function $f : \mathbb{T} \to \mathbb{R}$ in $t$ to be the number denoted by $f^{\diamond_{\phi_{h-s,T}}}(t)$(when it exists), with the property that for any $\epsilon > 0$, there is a neighbourhood $U$ of $m$ such that, for all $n \in U,\ 0 \leq s \leq 1$ and $0 < \omega < 1$,

$$\left| \left(\frac{h(\omega)}{\omega}\right)^{-s} [f(\sigma(m)) - f(n)]\nu_{mn} + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} [f(\rho(m)) - f(n)]\mu_{mn} - f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)\mu_{mn}\nu_{mn} \right| < \epsilon|\mu_{mn}\nu_{mn}|.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called diamond-$\phi_{h-s,\mathbb{T}}$ differentiable on $\mathbb{T}_k^k$ if $f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)$ exists for all $t \in \mathbb{T}_k^k$. If $f$ is differentiable on $\mathbb{T}$ in the sense of $\Delta$ and $\nabla$, then $f$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}_k^k$, and the diamond-$\phi_{h-s,\mathbb{T}}$ derivative $f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)$ is given by

$$f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(\phi(t)) = \left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\nabla}(\phi(t))$$

for all $s \in [0,1], \omega \in (0,1)$.

**Remark 2.2.** (i) $f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)$ reduces to the diamond-$\alpha$ derivative for $\phi(t) = t, s = 1$, $h(\omega) = 1$ and $\omega = \alpha$. Thus, every diamond-$\alpha$ differentiable on $\mathbb{T}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable function but the converse is not true.

(ii) If $f$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable for $0 \leq s \leq 1, 0 < \omega < 1$, then $f$ is both $\Delta$ and $\nabla$ differentiable.

(iii) Let $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. The diamond-$\phi_{h-s,\mathbb{T}}$ integral of $f$ from $a$ to $b$ is defined by

$$\int_a^b f(\phi(t)) \diamond_{\phi_{h-s,\mathbb{T}}} t = \left(\frac{h(\omega)}{\omega}\right)^{-s} \int_a^b f(\phi(t))\Delta t + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} \int_a^b f(\phi(t))\nabla t$$

for all $s \in [0,1], \omega \in (0,1)$, provided that $f$ has a delta and nabla integral on $[a,b]_{\mathbb{T}}$ or $I_{\mathbb{T}}$.

Obviously, each continuous function has a diamond-$\phi_{h-s,\mathbb{T}}$ integral. The combined derivative $\diamond_{\phi_{h-s,\mathbb{T}}}$ is not a dynamic derivative, since we do not have a $\diamond_{\phi_{h-s,\mathbb{T}}}$ antiderivative. Generally,

$$\left(\int_a^b f(\phi(t)) \diamond_{\phi_{h-s,\mathbb{T}}} t\right)^{\diamond_{\phi_{h-s,\mathbb{T}}}} \neq f(\phi(t)), t \in \mathbb{T}.$$

**Example 2.1.** Let $\mathbb{T} = [0,1] \cup \{2,4\}$. Define a diamond-$\phi_{h-1,\mathbb{T}}$ function $f : \mathbb{T} \to \mathbb{R}$ by $f(\phi(t)) = 1$ and $h : \mathbb{T} \to \mathbb{R}$ by $h(\omega) = 1$. For $\phi(t), \omega \in \mathbb{T}$, then

$$\left(\int_0^b f(\phi(t)) \diamond_{\phi_{h-s,\mathbb{T}}} t\right)^{\diamond_{\phi_{h-s,\mathbb{T}}}} \Bigg|_{t=1} \neq f(\phi(t)).$$

We give some basic properties of the diamond-$\phi_{h-s,\mathbb{T}}$ integral which are similar to Theorem 2.2 of [7] and its analogue for the nabla integral.

**Proposition 2.1.** *Let $a, b, c \in \mathbb{T}, \beta \in \mathbb{R}$, and $f, g$ be continuous functions on $I_{\mathbb{T}}$, then*

*(i)* $\int_a^b (f(t) + g(t)) \diamond_{\phi_{h-s,\mathbb{T}}} t = \int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t + \int_a^b g(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$

*(ii)* $\int_a^b (\beta f)t \diamond_{\phi_{h-s,\mathbb{T}}} t = \beta \int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$

*(iii)* $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t = -\int_b^a f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$

*(iv)* $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t = \int_a^c f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t + \int_c^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$

*(v)* $\int_a^a f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t = 0$.

**Lemma 2.2.** *(i) If $f(t) \geq 0$ for all $t$, then $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t \geq 0$.*

*(ii) If $f(t) \leq g(t)$ for all $t$, then $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t \leq \int_a^b g(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$.*

*(iii) If $f(t) \geq 0$ for all $t \in I_{\mathbb{T}}$, then $f = 0$ if and only if $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t = 0$.*

*(iv) If $|f(t)| \leq g(t)$ on $[a,b)$, then $|\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t| \leq \int_a^b g(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$.*

(v) *If in (iv), we choose* $g(t) = |f(t)|$ *on* $[a, b]$, *we have*
$|\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t| \leq \int_a^b |f(t)| \diamond_{\phi_{h-s,\mathbb{T}}} t$.

*Proof.* 1. Assume that $f$ and $g$ are continuous functions on $I_{\mathbb{T}}$, then, $\int_a^b f(t)\Delta t \geq 0$ and $\int_a^b f(t)\nabla t \geq 0$ since $f(t) \geq 0$ for all $t \in I_{\mathbb{T}}$. Since $s \in [0, 1], \omega \in (0, 1)$, the result follows.

2. Let $h(t) = g(t) - f(t)$, then $\int_a^b h(t) \diamond_{\phi_{h-s,\mathbb{T}}} t \geq 0$ and the result follows from properties (i) and (ii) above.

3. If $f(t) = 0$ for all $t \in I_{\mathbb{T}}$, the result is immediate. Now suppose that there exists $t_0 \in I_{\mathbb{T}}$ such that $f(t_0) > 0$. It is easy to see that at least one of the integrals $\int_a^b f(t)\Delta t$ or $\int_a^b f(t)\nabla t$ is strictly positive. Then we have the contradiction $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t > 0$.

$\square$

Dinu [8], defined the concept of a convex function on time scales. He equally included some results connecting this concept with the idea of convex functions on a classic interval and convex sequences.

In this paper, we have given the notion of classes of convex functions on time scales, consequently established the various interconnections that exist among them, and then related their properties with the concept of classes of convex functions on classic intervals.

## 3. Some classes of convex functions on time scales

Previous research works have shown that the history of convex functions is tied to the concept of Jensen convex or mid-point convex functions, which deals with the arithmetic mean (see for instance [10], [15], [17] and [19]). We shall state the analogue of Jensen convexity for time scales.

**Theorem 3.1.** *Let* $I_{\mathbb{T}} \subset \mathbb{T}$ *be a time scale interval. A function* $f : \mathbb{T} \to \mathbb{R}$ *is called convex in the Jensen sense or* $J$*-convex or mid-point convex on* $I_{\mathbb{T}}$ *if for all* $t_1, t_2 \in I_{\mathbb{T}}$, *the inequality,*

$$f\left(\frac{t_1 + t_2}{2}\right) \leq \frac{f(t_1) + f(t_2)}{2} \tag{3.1}$$

*holds.*

**Remark 3.1.** (i) If $\mathbb{T} = \mathbb{R}$, then our version is the same as the classical Jensen inequality. However, if $\mathbb{T} = \mathbb{Z}$, then it reduces to the well-known arithmetic-geometric mean inequality.

(ii) The extensions of the inequality (3.1) to the convex combination of finitely many points and next to random variables associated to arbitrary probability spaces are known as the discrete Jensen and integral inequalities on time scales respectively (see [3], [15] and [16]).

**Definition 3.1.** We say that $f : \mathbb{T} \to \mathbb{R}$ is Godunova-Levin convex on time scales or that $f$ belongs to the class $Q(I_{\mathbb{T}})$ if $f$ is nonnegative, and that for all $t_1, t_2 \in I_{\mathbb{T}}$, and $\omega \in (0, 1)$,

$$f(\omega t_1 + (1-\omega)t_2) \leq \frac{1}{\omega} f(t_1) + \frac{1}{1-\omega} f(t_2). \tag{3.2}$$

**Remark 3.2.** (i) For $\mathbb{T} = \mathbb{R}$, the definition 3.1 above is exactly the definition of a Godunova-Levin function on a classic interval (see [12], [13], [15], and [21]).

(ii) All nonnegative monotonic and nonnegative convex functions on time scales belong to this class $Q(I_{\mathbb{T}})$.

(iii) If $f \in Q(I_{\mathbb{T}}) = SX(h_{-1}, I_{\mathbb{T}})$ and $t_1, t_2, t_3 \in I_{\mathbb{T}}$, then

$$f(t_1)(t_1 - t_2)(t_1 - t_3) + f(t_2)(t_2 - t_1)(t_2 - t_3) + f(t_3)(t_3 - t_1)(t_3 - t_2) \geq 0. \tag{3.3}$$

In fact, (3.3) is equivalent to (3.2) so it can alternatively be used in the definition of the class $Q(I_{\mathbb{T}})$.

**Definition 3.2.** A function $f : \mathbb{T} \to \mathbb{R}$ is a $P$-function on $I_{\mathbb{T}}$ or $f \in P(I_{\mathbb{T}})$ if $f$ is nonnegative, and for all $t_1, t_2 \in I_{\mathbb{T}}$, and $\omega \in [0, 1]$, we have

$$f(\omega t_1 + (1-\omega)t_2) \leq f(t_1) + f(t_2). \tag{3.4}$$

Obviously, $P(I_{\mathbb{T}}) \subseteq Q(I_{\mathbb{T}})$. Also, $P(I_{\mathbb{T}})$ contains all nonnegative monotone, convex and quasi-convex functions on $I_{\mathbb{T}}$, i.e, nonnegative functions satisfying

$$f(\omega t_1 + (1-\omega)t_2) \leq max\{f(t_1) + f(t_2)\}$$

for all $t_1, t_2 \in I_{\mathbb{T}}$, and $\omega \in [0, 1]$, (see [24]).

**Definition 3.3.** A function $f : C_{I_{\mathbb{T}}} \subseteq \mathbb{T} \to [0, \infty)$ is of $s$-Godunova-Levin type on time scales, denoted $Q_s(I_{\mathbb{T}})$ with $s \in [0, 1]$ if

$$f(\omega t_1 + (1-\omega)t_2) \leq \frac{1}{\omega^s} f(t_1) + \frac{1}{(1-\omega)^s} f(t_2), \tag{3.5}$$

for all $\omega \in (0, 1)$ and $t_1, t_2 \subseteq C_{I_{\mathbb{T}}}$, where $C_{I_{\mathbb{T}}}$ is a convex subset of a time scale interval of a time scale linear space $\mathbb{T}$.

**Remark 3.3.** (i) When $s = 0$, we get a class of $P$-functions on $I_{\mathbb{T}}$.

(ii) $s = 1$ gives the class of Godunova-Levin functions on $I_{\mathbb{T}}$.

**Definition 3.4.** Let $s$ be a real number, $s \in (0,1]$. A function $f : [0,\infty) \subset \mathbb{T} \to [0,\infty)$ is said to be $s$-convex on $I_{\mathbb{T}}$ (in the second sense on time scales) or Breckner $s$-convex on $I_{\mathbb{T}}$, denoted $K_2^s(I_{\mathbb{T}})$ if

$$f(\omega t_1 + (1-\omega)t_2) \leq \omega^s f(t_1) + (1-\omega)^s f(t_2), \tag{3.6}$$

for all $t_1, t_2 \in [0,\infty) \subset \mathbb{T}$ and $\omega \in [0,1]$.

**Remark 3.4.** Definition 3.4 is a generalization of convex functions on $I_{\mathbb{T}}$ as defined in [8]. Hence, $s$-convexity on $I_{\mathbb{T}}$ just means convexity when $s = 1$ on $I_{\mathbb{T}}$.

In order to unify the concepts of Definitions 3.1 – 3.4 above for functions on time scale variables we now introduce the concept of $h$-convex functions on time scales (see [11] and [29]).
Assume that $I_{\mathbb{T}}$ and $J_{\mathbb{T}}$ are intervals in $\mathbb{T}, [0,\infty) \subseteq J_{\mathbb{T}}$ and functions $f$ and $h$ are real non-negative functions defined on $I_{\mathbb{T}}$ and $J_{\mathbb{T}}$ respectively.

**Definition 3.5.** Let $h : J_{\mathbb{T}} \to \mathbb{R}$ with $h$ not identical to zero. We say that $f : \mathbb{T} \to \mathbb{R}$ is an $h$-convex function on $I_{\mathbb{T}}$ or $f$ belongs to the class $SX(h, I_{\mathbb{T}})$ if for all $t_1, t_2 \in I_{\mathbb{T}}, f$ is non-negative, we have

$$f(\omega t_1 + (1-\omega)t_2) \leq h(\omega) f(t_1) + h(1-\omega) f(t_2), \tag{3.7}$$

for all $\omega \in (0,1)$.

**Remark 3.5.** (i) If inequality (3.7) is reversed, then $f$ is said to be $h$-concave on $I_{\mathbb{T}}$ i.e $f \in SV(h, I_{\mathbb{T}})$.

(ii) Obviously, if $h(\omega) = \omega$, then all non-negative functions on $I_{\mathbb{T}}$ belong to $SX(h, I_{\mathbb{T}})$ and all non-negative concave functions on $I_{\mathbb{T}}$ belong to $SV(h, I_{\mathbb{T}})$; if $h(\omega) = \frac{1}{\omega}$, then $SX(h, I_{\mathbb{T}}) = Q(I_{\mathbb{T}})$; if $h(\omega) = 1$, then $SX(h, I_{\mathbb{T}}) \supseteq P(I_{\mathbb{T}})$; and if $h(\omega) = \omega^s$, where $s \in (0,1)$, then $SX(h, I_{\mathbb{T}}) \supseteq K_s^2(I_{\mathbb{T}})$.

**Definition 3.6.** A function $f : I_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ is said to belong to the class $MT(I_{\mathbb{T}})$ if $f$ is nonnegative and for all $t_1, t_2 \in I_{\mathbb{T}}$ and $\omega \in (0,1)$ satisfies the inequality:

$$f(\omega t_1 + (1-\omega)t_2) \leq \frac{\sqrt{\omega}}{2\sqrt{1-\omega}} f(t_1) + \frac{\sqrt{1-\omega}}{2\sqrt{\omega}} f(t_2). \tag{3.8}$$

**Remark 3.6.** (i) If $\mathbb{T} = \mathbb{R}$, and $I_{\mathbb{T}} = I$, we obtain definition 2 of Tunç and Yildirim (2012), for classical $MT$-convex function (see [21], [22], [23], [27] and [28]).

(ii) If we set $\omega = \frac{1}{2}$, inequality (3.8) reduces to the inequality (3.1).

(iii) Let $f, g : [1,\infty] \subset \mathbb{T} \to \mathbb{R}$, $f(t) = t^p, g(t) = (1+t)^p, p \in (0, \frac{1}{1000})$, and $h : [1, \frac{3}{2}] \subset \mathbb{T} \to \mathbb{R}$, $h(t) = (1+t_2)^m, m \in (0, \frac{1}{100})$ are $MT$-convex functions on $I_{\mathbb{T}}$ but they are not convex on $I_{\mathbb{T}}$.

Now, we give a variant of a new class of convex functions introduced by Olanipekun et al. in [20], but in the context of time scales.

**Definition 3.7.** Let $h : J_{\mathbb{T}} \to \mathbb{R}, s \in [0,1], \omega \in (0,1)$ and $\phi$ be a given real-valued function. Then $f : I_{\mathbb{T}} \to \mathbb{R}$ is a $\phi_{h-s}, I_{\mathbb{T}}$-convex function on time scales if for all $t_1, t_2 \in I_{\mathbb{T}}$,

$$f(\omega\phi(t_1) + (1-\omega)\phi(t_2) \leq \left(\frac{h(\omega)}{\omega}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f(\phi(t_2)). \quad (3.9)$$

We observe that

(i) If $s = 0$, and $\phi(t_1) = t_1$, then $f \in P(I_{\mathbb{T}})$.

(ii) If $h(\omega) = \omega^{\frac{s}{s+1}}$ and $\phi(t_1) = t_1$, , then $f \in SX(h, I_{\mathbb{T}})$.

(iii) If $s = 1, h(\omega) = 1$ and $\phi(t_1) = t_1$, then $f \in SX(I_{\mathbb{T}})$, i.e, $f$ is convex on time scales (See, [8]).

(iv) If $h(\omega) = 1$ and $\phi(t_1) = t_1$, then $f \in K_s^2(I_{\mathbb{T}})$.

(v) If $h(\omega) = \omega^2, s = 1$ and $\phi(t_1) = t_1$, then $f \in Q(I_{\mathbb{T}})$.

(vi) If $h(\omega) = \omega^2$, and $\phi(t_1) = t_1$, then $f \in Q_s(I_{\mathbb{T}})$.

(vii) If $s = 1, h(\omega) = 2\sqrt{\omega(1-\omega)}$ and $\phi(t_1) = t_1$, then $f \in MT(I_{\mathbb{T}})$.

Moreover, suppose we denote by $Q_s(I_{\mathbb{T}})$ and $SX(\phi_{h-s}, I_{\mathbb{T}})$ the class of $s$-Godunova Levin and $\phi_{h-s}, I_{\mathbb{T}}$ convex functions on time scales respectively, then it is easy to see that: $P(I_{\mathbb{T}}) = Q_0(I_{\mathbb{T}}) = SX(\phi_{h-0}, I_{\mathbb{T}}) \subseteq SX(\phi_{h-s_1}, I_{\mathbb{T}}) \subseteq SX(\phi_{h-s_2}, I_{\mathbb{T}}) \subseteq SX(\phi_{h-1}, I_{\mathbb{T}}) = SX(\phi_h, I_{\mathbb{T}})$ for $0 \leq s_1 \leq s_2 \leq 1$ whenever $\phi$ is the identity function.
If inequality (3.9) is reversed, then $f$ is $\phi_{h-s}, I_{\mathbb{T}}$ concave, that is, $f \in SV(\phi_{h-s}, I_{\mathbb{T}})$.

The permanence properties of diamond-$\phi_h, I_{\mathbb{T}}$ derivative and convexity operations on time scales constitute an important source of examples in this area.

**Proposition 3.1.** *Let $f, g \in SX(\phi_{h-s}, I_{\mathbb{T}})$, i.e, these are $\phi_{h-s}, I_{\mathbb{T}}$ convex, $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-$\phi_{h-s}, I_{\mathbb{T}}$ differentiable at $t \in \mathbb{T}$, and $c$ be any constant. Then $f+g, cf, fg, \frac{1}{g}(g \neq 0), \frac{f}{g}(g \neq 0)$ are all diamond-$\phi_{h-s}, I_{\mathbb{T}}$ differentiable at $t \in \mathbb{T}$.*

*Proof.* Since $f$ and $g$ are $\diamond_{\phi_{h-s}, I_{\mathbb{T}}}$ differentiable, then let
$f^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t) = \left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\nabla}(\phi(t))$
and
$g^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t) = \left(\frac{h(\omega)}{\omega}\right)^{-s} g^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} g^{\nabla}(\phi(t))$, then

(i). $f+g : \mathbb{T} \to \mathbb{R}$ is diamond-$(\phi_{h-s}, I_{\mathbb{T}})$ differentiable at $t \in \mathbb{T}$, and

$$\begin{aligned}(f,g)^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t) &= \left(\frac{h(\omega)}{\omega}\right)^{-s} (f,g)^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} (f,g)^{\nabla}(\phi(t)) \\ &= f^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t) + g^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t).\end{aligned}$$

(ii). For any constant $c \in \mathbb{R}, cf : \mathbb{T} \to \mathbb{R}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}$ and $(cf)^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) = c\left[\left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\nabla}(\phi(t))\right]$

$$= cf^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t).$$

(iii). $fg : \mathbb{T} \to \mathbb{R}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}$ and

$$\begin{aligned}(fg)^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) &= \left(\frac{h(\omega)}{\omega}\right)^{-s}(fg)^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s}(fg)^{\nabla}(\phi(t)) \\ &= f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)(\phi(t))g(\phi(t)) + \left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\sigma(t)}(\phi(t))g^{\Delta}(\phi(t)) \\ &\quad + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\rho(t)}(\phi(t))g^{\nabla}(\phi(t)).\end{aligned}$$

(iv). For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{1}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}$ with

$$\begin{aligned}\left(\frac{1}{g}\right)^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) &= -\frac{1}{g(\phi(t))g^{\sigma}(\phi(t))g^{\rho}(\phi(t))}((g^{\sigma}(\phi(t)) + g^{\rho}(\phi(t)))\, g^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) \\ &= -\left(\frac{h(\omega)}{\omega}\right)^{-s} g^{\Delta}(\phi(t))g^{\sigma}(\phi(t)) \\ &\quad -\left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} g^{\nabla}(\phi(t))g^{\rho}(\phi(t)).\end{aligned}$$

(v). For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}$ with

$$\begin{aligned}\left(\frac{f}{g}\right)^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) &= \frac{1}{g(\phi(t))g^{\sigma}(\phi(t))g^{\rho}(\phi(t))}(f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)g^{\sigma}(\phi(t))g^{\rho}(\phi(t)) \\ &= -\left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\sigma}(\phi(t))g^{\rho}(\phi(t))g^{\Delta}(\phi(t)) - \\ &\quad - \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\rho}(\phi(t))g^{\sigma}(\phi(t))g^{\nabla}(\phi(t))).\end{aligned}$$

□

The following proposition easily follows and so we will omit the proof.

**Proposition 3.2.** *Let $f$ be a non negative function on $I_{\mathbb{T}}$. Let $h$ be a non negative function on $I_{\mathbb{T}}$.*

*(i). If $h(\omega) \leq \omega^{1-\frac{1}{s}}$ where $s \in (0,1]$, $\omega \in (0,1)$, then $f \in SX(\phi_{h-s}, I_{\mathbb{T}})$.*

*(ii). If $h(\omega) \geq \omega^{1-\frac{1}{s}}$ for any $\omega \in (0,1)$ and $s \in (0,1]$, then $f \in SV(\phi_{h-s}, I_{\mathbb{T}})$.*

It is clear that Proposition 3.2 implies that all convex functions on $I_{\mathbb{T}}$ are the examples of our newly defined class of convex function on $I_{\mathbb{T}}$. An example of such particularly is $h(\omega) = \omega^k$ for $k > 1 - \frac{1}{s}$ where $s \in (0,1]$.

**Remark 3.7.** For $t_1, t_2 \in I_{\mathbb{T}}, p, q > 0$, the inequality (3.9) is equivalent to

$$f\left(\frac{p\phi(t_1) + q\phi(t_2)}{p+q}\right) \leq \frac{\left(\frac{h(p)}{p}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(q)}{q}\right)^{-s} f(\phi(t_2))}{p+q}.$$

The following definition is useful in defining another form of inequality (3.9) on time scales.

**Definition 3.8.** A function $h : J_{\mathbb{T}} \to \mathbb{R}$ is said to be a *supermultiplicative* function on $J_{\mathbb{T}} \subset \mathbb{T}$ if for all $m, n \in J_{\mathbb{T}}$,

$$h(mn) \geq h(m)h(n), \tag{3.10}$$

$h$ is said to be a *submultiplicative* function on time scales if the inequality (3.10) is reversed and respectively a *multiplicative* function on time scales if the equality holds in (3.10).

We now prove some time scales analogue for $\phi_{h-s}, I_{\mathbb{T}}$-convex function where $h$ is either supermultiplicative or submultiplicative. Some results in [20] are useful in the sequel.

**Proposition 3.3.** *Let $h : J_{\mathbb{T}} \to \mathbb{R}$ be a non negative function on $J_{\mathbb{T}} \subset \mathbb{T}$ and let $f : \mathbb{T} \to \mathbb{R}$ be a function such that $f \in SX(\phi_{h-s}, I_{\mathbb{T}})$, where $\phi(t) = t$. Then for all $t_1, t_2, t_3 \in \mathbb{T}$ such that $t_1 < t_2 < t_3$ and $t_3 - t_1, t_3 - t_2, t_2 - t_1, \in J_{\mathbb{T}}$, the following inequality holds:*

$$[(t_3 - t_1), (t_2 - t_1), h(t_3 - t_2)]^{-s} f(t_1) - [(t_3 - t_2), (t_2 - t_1), h(t_3 - t_1)]^{-s} f(t_2)$$
$$+[(t_3 - t_1), (t_3 - t_2), h(t_2 - t_1)]^{-s} f(t_3) \geq 0. \tag{3.11}$$

*If the function $h$ is submultiplicative and $f \in SX(\phi_{h-s}, I_{\mathbb{T}})$, then the inequality (3.11) is reversed.*

*Proof.* Since $f \in SX(\phi_{h-s}, I_{\mathbb{T}})$, and $t_1, t_2, t_3 \in \mathbb{T}$ are points which satisfy assumptions of the proposition, then

$$\frac{t_3 - t_2}{t_3 - t_1}, \frac{t_2 - t_1}{t_3 - t_1} \in J_{\mathbb{T}} \text{ and } \frac{t_3 - t_2}{t_3 - t_1} + \frac{t_2 - t_1}{t_3 - t_1} = 1.$$

Also,

$$h(t_3 - t_2)^{-s} = \left(h\left(\frac{t_3 - t_2}{t_3 - t_1}(t_3 \dot{-} t_1)\right)\right)^{-s} \geq \left(h\left(\frac{t_3 - t_2}{t_3 - t_1}\right) h(t_3 - t_1)\right)^{-s}$$

and

$$h(t_2-t_1)^{-s} = \left(h\left(\frac{t_2-t_1}{t_3-t_1}(t_3 \cdot - t_1)\right)\right)^{-s} \geq \left(h\left(\frac{t_2-t_1}{t_3-t_1}\right)h(t_3-t_1)\right)^{-s}.$$

Let $h(t_3-t_1)>0$. If in inequality (3.9), we set $\omega = \frac{t_3-t_2}{t_3-t_1}, 1-\omega = \frac{t_2-t_1}{t_3-t_1}$, $a=t_1, b=t_3$, we have $t_2 = \omega a + (1-\omega)b$ and so

$$\begin{aligned} f(t_2) &\leq \left(\frac{h\left(\frac{t_3-t_2}{t_3-t_1}\right)}{\frac{t_3-t_2}{t_3-t_1}}\right)^{-s} f(t_1) + \left(\frac{h\left(\frac{t_3-t_2}{t_3-t_1}\right)}{\frac{t_3-t_2}{t_3-t_1}}\right)^{-s} f(t_3) \\ &\leq \left(\frac{\frac{h(t_3-t_2)}{h(t_3-t_1)}}{\frac{t_3-t_2}{t_3-t_1}}\right)^{-s} f(t_1) + \left(\frac{\frac{h(t_3-t_2)}{h(t_3-t_1)}}{\frac{t_3-t_2}{t_3-t_1}}\right)^{-s} f(t_3). \qquad (3.12) \end{aligned}$$

Multiplying inequality (3.12) by $(\frac{t_3-t_2}{t_3-t_1})^{-s}(h(t_3-t_1))^{-s}$ and further multiplication by $(t_3-t_1)^{-s}(t_2-t_1)^{-s}$ with rearrangement yields (3.11). $\square$

**Remark 3.8.** (i) Inequality (3.11) can alternatively be used in Definition 3.7 since inequalities (3.9) and (3.11) are equivalent.

(ii) If we consider inequality (3.11) with $h(t) = t^2$ where $s = 1$, we will obtain an alternate definition of Godunova-Levin function on time scales, that is, inequality (3.3).

(iii) Inequality (3.11) is equivalent to Definition 14 of [18] with $h(t) = 1, s = 1$ by considering points $t_1, t_2 \in I_{\mathbb{T}}$ with $t_1 < t_2$ and $t \in I_{\mathbb{T}}$ such that $t_1 < t < t_2$ and $t = \omega t_1 + (1-\omega)t_2$.

(iv) Another way of writing (3.12) is:

$$\frac{f(t_1)-f(t_2)}{\left(\frac{h\left(\frac{t_1-t_2}{t_1-t_3}\right)}{\frac{t_1-t_2}{t_1-t_3}}\right)^{-s}} \leq \frac{f(t_2)-f(t_3)}{\left(\frac{h\left(\frac{t_2-t_3}{t_1-t_3}\right)}{\frac{t_2-t_3}{t_1-t_3}}\right)^{-s}},$$

where $t_1 < t_3$ and $t_1, t_3 \neq t_2$.

**Theorem 3.2.** *Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be defined and $\Delta_{\phi_{h-s,I_{\mathbb{T}}}}$ differentiable function on $I_{\mathbb{T}}^k$. If $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}$ is nondecreasing (nonincreasing) on $I_{\mathbb{T}}^k$, then $f$ is $\phi_{h-s,I_{\mathbb{T}}}$ convex (concave) on $I_{\mathbb{T}}$.*

*Proof.* By Remark 3.8(iv), it suffices to prove that

$$\frac{f(\phi(t))-f(\phi(t_1))}{t-t_1} \leq \frac{f(\phi(t_2))-f(\phi(t))}{t_2-t}. \qquad (3.14)$$

Let $t_1 \leq \gamma_1 < \xi_2$. From the mean value Theorem (see [5]), there exists points $\xi_1, \gamma_1 \in [t_1, t)_{\mathbb{T}}$ and $\xi_2, \gamma_2 \in [t, t_2)_{\mathbb{T}}$ such that

$$f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_1) \leq \frac{f(\phi(t))-f(\phi(t_1))}{t-t_1} \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_1)$$

and

$$f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_2) \leq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_2). \tag{3.15}$$

Since $t_1 < \gamma_1 < \xi_2$ and from the assumption that $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_1) \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_2)$, inequality (3.14) holds, then

$$\frac{f(\phi(t)) - f(\phi(t_1))}{t - t_1} \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_1) \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_2) \leq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \tag{3.16}$$

for nondecreasing $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}$ and

$$\frac{f(\phi(t)) - f(\phi(t_1))}{t - t_1} \geq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_1) \geq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_2) \geq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \tag{3.17}$$

for nonincreasing $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}$. □

The inequality (3.16) is equivalent to the inequality (3.9) and with the $\phi_{h-s,I_{\mathbb{T}}}$-convexity of $f$, while the inequality (3.17) is equivalent with the $\phi_{h-s,I_{\mathbb{T}}}$-concavity of $f$. It is obvious that the nabla version of the Theorem (3.1) holds for nondecreasing (nonincreasing) $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}$.

We now ask a question of interest: *Can the generalized class of convex function (3.19) be continuous on time scales?* The answer to this is affirmative. We shall first discuss the geometrical interpretation of $\phi_{h-s,I_{\mathbb{T}}}$ convexity on time scales in order to justify this claim.

The $\phi_{h-s,I_{\mathbb{T}}}$ convexity of a function $f : I_{\mathbb{T}} \to \mathbb{R}$ on time scales geometrically means that the points of the graph of $f(\phi(t))|[\phi(t_1), \phi(t_2)]$ are under the chord(or on the chord) joining the endpoints $(\phi(t_1), f(\phi(t_1)))$ and $(\phi(t_2), f(\phi(t_2)))$ for every $t_1, t_2 \in I_{\mathbb{T}}$. Thus

$$f(\phi(t)) \leq f(\phi(t_1)) + \frac{f(\phi(t_2)) - f(\phi(t_1))}{\phi(t_2) - \phi(t_1)}(\phi(t) - \phi(t_1))$$

for all $\phi(t) \in [\phi(t_1), \phi(t_2)]$ and all $\phi(t_1), \phi(t_2) \in I_{\mathbb{T}}$.
This shows that convex functions are majorized locally (i.e, on any compact subinterval) by affine functions.

**Theorem 3.3.** *Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be a continuous function on $I_{\mathbb{T}}$. Then $f$ is $\phi_{h-s,I_{\mathbb{T}}}$ convex on $I_{\mathbb{T}}$ if and only if $f$ satisfies inequality (3.1), i.e, $f$ is midpoint convex on $I_{\mathbb{T}}$.*

*Proof.* Sufficiently assume for contradiction that $f$ is not $\phi_{h-s,I_{\mathbb{T}}}$ convex on $I_{\mathbb{T}}$. Thus, there would exist subinterval $(\phi(a), \phi(b)]$ such that $f(\phi(t))|[\phi(a), \phi(b)]$ is not under the chord (or on the chord) joining $(\phi(a), f(\phi(a)))$ and $(\phi(b), f(\phi(b)))$ so that for $\phi(t) \in [\phi(a), \phi(b)]$

$$f(\psi(t)) = f(\phi(t)) - \frac{f(\phi(b)) - f(\phi(a))}{\phi(b) - \phi(a)}(\phi(t) - \phi(a)) - f(\phi(a)).$$

Thus $\xi = \sup\{f(\psi(\phi(t)))|\phi(t) \in [\phi(a), \phi(b)]\} > 0$ and $\psi(\phi(a)) = \psi(\phi(b)) = 0$ since $\psi$ is continuous.

Also, let $K = \inf\{\phi(t) \in [\phi(a), \phi(b)]|\psi(\phi(t)) = \xi\}$, then we necessarily prove that $\phi(k) = \xi$ and $k \in (\phi(a), \phi(b))$.
For every $c > 0$ for which $k + c \in (\phi(a), \phi(b))$, we have by the definition of $k$, $\psi(k - c) < \psi(k)$ and $\psi(k + c) \leq \psi(k)$.
So that

$$\psi(k) > \frac{\psi(k - c) + \psi(k + c)}{2}.$$

This contradicts the fact that $\psi$ is midpoint convex on $I_{\mathbb{T}}$. $\square$

**Remark 3.9.** (i) Theorem 3.1 remains true if the condition of midpoint convexity on time scales is replaced by

$$f((1 - \omega)\phi(t_1) + \omega\phi(t_2)) \leq \left(\frac{h(1 - \omega)}{1 - \omega}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(\omega)}{\omega}\right)^{-s} f(\phi(t_2)),$$

for some $0 \leq s \leq 1, h(\omega) = 1, \phi(t_1) = t_1$ and $\omega = \frac{1}{2}$.

(ii) If we replace the condition of continuity in Theorem 3.2 by boundedness from above on every compact subinterval of time scales, Theorem 3.2 still holds.

## 4. Conclusion

The concepts of $\phi_{h-s,I_{\mathbb{T}}}$-convex function on time scales generalizes the time scale version of many known classes of convex functions. This implies that $\phi_{h-s,I_{\mathbb{T}}}$-convex functions on time scales readily provides many inequalities which generalize and extend the Hermite-Hadamard-type inequalities and several other inequalities for some classes of convex functions defined on time scales.

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