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ON A CLASSIFICATION OF PARA-SASAKIAN MANIFOLDS

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Abstract. We consider para-Sasakian manifolds satisfying the curvature conditions $P \cdot R = 0$, $P \cdot Q = 0$ and $Q \cdot P = 0$, where R is the Riemannian curvature tensor, P is the projective curvature tensor and Q is the Ricci operator.

Keywords: para-Sasakian manifold, Riemannian curvature tensor, Ricci operator, Riemannian manifold

1. Introduction

In [14] Kaneyuki and Kozai defined the almost paracontact structure on a pseudo-Riemannian manifold M of dimension $(2n+1)$ and constructed the almost paracomplex structure on $M^{(2n+1)} \times \mathbb{R}$. In 2009, Zamkovoy [24] defined para-Sasakian manifolds as a normal paracontact metric manifold. Thus a para-Sasakian manifold is a subclass of paracontact metric manifolds. In [24], the author obtains a necessary and sufficient condition for a paracontact metric manifold to be a para-Sasakian manifold. Also D-homothetic transformations have been studied in para-Sasakian manifolds in [24]. In the present paper, we characterize para-Sasakian manifolds satisfying certain curvature conditions. Para-Sasakian manifolds have been studied by several authors such as I. Mihai et al. ([19], [20], [21]) and De et al. ([11], [12], [13], [16]) and many others.

In a Riemannian manifold, if there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [22]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.1}$$

for all $X, Y, Z \in \chi(M)$, where R is the curvature tensor and S is the Ricci tensor of type $(0,2)$. In fact, M is projectively flat if and only if it is of constant curvature.

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Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

The paper is organized as follows:

Section 2 is equipped with some prerequisites about para-Sasakian manifolds. In Section 3, we prove that if a para-Sasakian manifold satisfies $P \cdot R = 0$ then the manifold is an Einstein manifold. Next in Section 4, it is shown that if a para-Sasakian manifold satisfies the curvature condition $P \cdot Q = 0$, then the square of the Ricci tensor S^2 is the linear combination of the Ricci tensor S and the metric tensor g. Finally, we prove that if a para-Sasakian manifold satisfies the curvature condition $Q \cdot P = 0$, then the trace of the square of the Ricci operator of a para-Sasakian manifold is equal to $-2n$ times of the trace Ricci operator.

2. Priliminaries

Let M be an $(2n+1)$-dimensional differentiable manifold. If there exits a triplet (ϕ, ξ, η) of a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η on M^{2n+1} which satisfies the relation [1, 2, 14]

$$\phi^2 = I - \eta \otimes \xi, \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \tag{2.1}$$

then we say the triplet (ϕ, ξ, η) is an almost paracontact structure and the manifold is an almost paracontact manifold.

If an almost paracontact manifold M^{2n+1} with an almost paracontact structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that [24]

$$g(X, Y) = -g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$

then we say that M^{2n+1} is an almost paracontact metric structure (ϕ, ξ, η, g) and such a metric g is called compatible metric. Any compatible metric g is necessarily of signature $(n+1, n)$. The fundamental 2-form of M^{2n+1} is defined by

$$\Phi(X, Y) = g(X, \phi Y). \tag{2.3}$$

An almost paracontact metric structure becomes a paracontact metric structure if

$$d\eta(X, Y) = g(X, \phi Y) \tag{2.4}$$

for all vector fields X, Y, where

$$d\eta(X, Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X, Y])]. \tag{2.5}$$

Paracontact manifolds have been studied by several authors such as Kaneyuki and Williams [15], Calvaruso [4, 5], Cappelletti-Montano et al. [7, 8, 9], Martin-Molina [18], Zamkovoy et al. [25] and many others.

An almost paracontact structure is said to be normal if and only if the tensor $N_\phi - 2d\eta \otimes \xi$ vanishes identically, where N_ϕ is the Nijenhuis tensor of $\phi : N_\phi(X, Y) =$

$[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]$ [24]. A normal paracontact metric manifold is known as para-Sasakian manifold. It is known [14, 24] that an almost paracontact manifold is para-Sasakian manifold if and only if

$$(\nabla_X\phi)Y=-g(X,Y)\xi+\eta(Y)X, \tag{2.6}$$

for all vectors field X,Y, where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric. From the above equation it follows that

$$\nabla_X\xi=-\phi X. \tag{2.7}$$

Moreover, in a para-Sasakian manifold the curvature tensor R, the Ricci tensor S and the Ricci operator Q satisfy [24]

$$R(X,Y)\xi=-(\eta(Y)X-\eta(X)Y), \tag{2.8}$$

$$R(\xi,X)Y=-g(X,Y)+\eta(Y)X, \tag{2.9}$$

$$S(X,\xi)=-2n\eta(X), \tag{2.10}$$

$$Q\xi=-2n\xi, \tag{2.11}$$

$$(\nabla_X\eta)Y=g(X,\phi Y), \tag{2.12}$$

$$\eta(R(X,Y)Z)=-(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)). \tag{2.13}$$

Para-Sasakian manifolds have been studied by several authors such as Zamkovoy [24], Martin-Molina [17], Cappelletti Montano et al [6] and many others.

A para-Sasakian manifold is said to be Einstein if [23]

$$S(X,Y)=ag(X,Y), \tag{2.14}$$

where S is the Ricci tensor of type $(0,2)$ and a is a constant.

3. Para-Sasakian manifolds satisfying $P\cdot R=0$

In this section we characterize para-Sasakian metric manifold satisfying $P\cdot R=0$. Suppose the manifold satisfies

$$(P(X,Y)\cdot R)(U,V)W=0, \tag{3.1}$$

for all smooth vector fields X,Y,U,V and W then we have

$$\begin{aligned}&P(X,Y)R(U,V)W-R(P(X,Y)U,V)W-R(U,P(X,Y)V)W\\&-R(U,V)P(X,Y)W=0.\end{aligned} \tag{3.2}$$

Substituting $X=U=\xi$ in (3.2) implies

$$\begin{aligned}&P(\xi,Y)R(\xi,V)W-R(P(\xi,Y)\xi,V)W-R(\xi,P(\xi,Y)V)W\\&-R(\xi,V)P(\xi,Y)W=0.\end{aligned} \tag{3.3}$$

Using (1.1), (2.9) and (2.10) we have

$$P(\xi,Y)R(\xi,V)W=-\eta(W)g(Y,V)\xi-\frac{1}{2n}S(Y,V)\eta(W)\xi. \tag{3.4}$$

Again using (1.1), (2.9) and (2.10) we obtain

$$R(P(\xi,Y)\xi,V)W=0, \tag{3.5}$$

$$R(\xi,P(\xi,Y)V)W=0. \tag{3.6}$$

Finally, using (1.1), (2.9) and (2.10) we get

$$\begin{aligned} &R(\xi,V)P(\xi,Y)W\\ &=g(Y,W)\eta(V)\xi-g(Y,W)V+\frac{1}{2n}S(Y,W)\eta(V)\xi-\frac{1}{2n}S(Y,W)V. \end{aligned} \tag{3.7}$$

Substituting (3.4)-(3.7) in (3.3) yields

$$\begin{aligned} &-\eta(W)g(Y,V)\xi-\frac{1}{2n}S(Y,V)\eta(W)\xi-g(Y,W)\eta(V)\xi+g(Y,W)V\\ &-\frac{1}{2n}S(Y,W)\eta(V)\xi+\frac{1}{2n}S(Y,W)V=0. \end{aligned} \tag{3.8}$$

Replacing W by ξ in (3.8) we have

$$\begin{aligned} &-g(Y,V)\xi-\frac{1}{2n}S(Y,V)\xi-\eta(Y)\eta(V)\xi+\eta(Y)V\\ &-\frac{1}{2n}S(Y,\xi)\eta(V)\xi+\frac{1}{2n}S(Y,\xi)V=0. \end{aligned} \tag{3.9}$$

Using (2.10) in (3.9) and then taking inner product with ξ implies

$$S(Y,V)=-2ng(Y,V), \tag{3.10}$$

which implies the manifold is an Einstein manifold.

Therefore, we can state the following:

Theorem 3.1. *If a para-Sasakian manifold satisfies $P\cdot R=0$ then the manifold is an Einstein manifold.*

4. Para-Sasakian manifolds satisfying $P\cdot Q=0$

Suppose the para-Sasakian manifolds satisfies $P\cdot Q=0$, which implies

$$P(X,Y)QZ-Q(P(X,Y)Z)=0, \tag{4.1}$$

for all smooth vector fields X, Y and Z. Putting $Y=\xi$ in (4.1) we have

$$P(X,\xi)QZ-Q(P(X,\xi)Z)=0. \tag{4.2}$$

Using (1.1), (2.9) and (2.10) we have

$$P(X,\xi)QZ = g(X,QZ)\xi + \frac{1}{2n}S(X,QZ)\xi. \tag{4.3}$$

Similarly using (1.1), (2.9) and (2.10) we obtain

$$Q(P(X,\xi)Z) = -2ng(X,Z)\xi - S(X,Z)\xi. \tag{4.4}$$

Making use of (4.3) and (4.4) in (4.2) yields

$$g(X,QZ)\xi + \frac{1}{2n}S(X,QZ)\xi + 2ng(X,Z)\xi + S(X,Z)\xi = 0. \tag{4.5}$$

Taking inner product with ξ in (4.5) we have

$$S^2(X,Z) = -4nS(X,Z) - 4n^2g(X,Z), \tag{4.6}$$

where $S^2(X,Z) = S(QX,Z)$.

This leads to the following:

Theorem 4.1. *If a para-Sasakian manifold satisfies the curvature condition $P \cdot Q = 0$, then the square of the Ricci tensor S^2 is the linear combination of the Ricci tensor S and the metric tensor g.*

For symmetric $(0,2)$-tensor fields A and B on M define Kulkarni-Nomizu product $A\bar{\wedge}B$ of A and B by ([3], p-47)

$$A \bar{\wedge} B(X_1,...,X_4) = A(X_1,X_4)B(X_2,X_3) - A(X_1,X_3)B(X_2,X_4)$$
$$+A(X_2,X_3)B(X_1,X_4) - A(X_2,X_4)B(X_1,X_3).$$

Here we recall the following lemma:

Lemma 4.1. *[10] Let A be symmetric $(0,2)$-tensor at point x of a semi Riemannian manifold (M,g), dim $M \geq 3$, and let $T = g \bar{\wedge} A$ be the Kulkarni-Nomizu product of g and A. Then the relation*

$$T \cdot T = \alpha Q(g,T), \alpha \in \mathbb{R} \tag{4.7}$$

is satisfied at x if and only if the condition

$$A^2 = \alpha A + \lambda g, \alpha \in \mathbb{R}$$

holds at x.

Hence we have the following corollary:

Corollary 4.1. *Let M be a para-Sasakian manifold satisfying the condition $P{\cdot}Q = 0$ then $T \cdot T = \alpha Q(g,T)$, where $T = g \bar{\wedge} S$ and $\alpha = -4n$.*

5. Para-Sasakian manifolds satisfying $Q \cdot P = 0$

Suppose para-Sasakian manifolds satisfying $Q \cdot P = 0$. Therefore

$$(Q \cdot P)(X,Y)Z = 0, \tag{5.1}$$

which implies

$$Q(P(X,Y)Z) - P(QX,Y)Z - P(X,QY)Z - P(X,Y)QZ = 0, \tag{5.2}$$

for all vector fields X, Y and Z.

Substituting $Y = \xi$ in (5.2) implies

$$Q(P(X,\xi)Z) - P(QX,\xi)Z - P(X,Q\xi)Z - P(X,\xi)QZ = 0. \tag{5.3}$$

Using (1.1) and (2.10) in (2.11) we have

$$Q(P(X,\xi)Z) = [\frac{1}{2n}S(X,Z) + g(X,Z)]Q\xi. \tag{5.4}$$

Using (1.1) and (2.9) we obtain

$$P(QX,\xi)Z = [\frac{1}{2n}S(QX,Z) + g(QX,Z)]\xi \tag{5.5}$$

Again using (1.1), (2.9) and (2.11) yields

$$P(X,Q\xi)Z = -[S(X,Z) + 2ng(X,Z)]\xi. \tag{5.6}$$

Finally, using (1.1) and (2.9) we have

$$P(X,\xi)QZ = [\frac{1}{2n}S(X,QZ) + g(X,QZ)]\xi. \tag{5.7}$$

Substituting (5.4)-(5.7) in (5.3) yields

$$\begin{aligned} &[\frac{1}{2n}S(X,Z) + g(X,Z)]Q\xi - [\frac{1}{2n}S(QX,Z) + g(QX,Z)]\xi \\ &+[S(X,Z) + 2ng(X,Z)]\xi - [\frac{1}{2n}S(X,QZ) + g(X,QZ)]\xi = 0. \end{aligned} \tag{5.8}$$

Taking inner product with ξ in (5.8) and using (2.11) we have

$$g(Q^2X,Z) = -2ng(QX,Z). \tag{5.9}$$

Let $\{e_1, e_2, e_3, ..., e_{2n+1}\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = Z = e_i$ and taking summation over i, $1 \leqslant i \leqslant (2n+1)$ we have

$$Tr(Q^2) = \sum_{i=1}^{2n+1} g(Q^2e_i,e_i) = -2n\sum_{i=1}^{2n+1} g(Qe_i,e_i) = -2nTr(Q). \tag{5.10}$$

This leads to the following:

Theorem 5.1. *If a para-Sasakian manifold satisfies the curvature condition $Q \cdot P = 0$, then the trace of square of the Ricci operator of a para-Sasakian manifold is equal to $-2n$ times trace of the Ricci operator.*

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