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RICCI SOLITONS IN α-COSYMPLECTIC MANIFOLDS *

Jay Prakash Singh and Chawngthu Lalmalsawma

Abstract. The aim of the paper is to study Ricci solitons in α-cosymplectic manifolds. Projective, pseudo projective and Weyl conformal curvatures in an α-cosymplectic manifolds admitting Ricci solitons have been studied under certain curvature conditions. Also, gradient Ricci solitons in α-cosymplectic manifolds have been studied.
Keywords: Ricci soliton, gradient Ricci soliton, α-cosymplectic manifolds, cosympletic manifolds, α-Kenmatsu manifolds

1. Introduction

The concept of Ricci soliton was introduced by Hamilton [8] while studying the Ricci flow on surfaces. It is a generalization of an Einstein metric and is defined as a triple (g, V, λ) with g a Riemannian metric, V a vector field, and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor of type $(0, 2)$ and $\pounds$ denotes the Lie derivative operator along the vector field V.

The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive, respectively [6]. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and the equation (1.1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{1.2}$$

In 2008 Sinha and Sharma [17] started the study of Ricci solitons in contact manifolds. Later Ricci solitons in contact and almost contact manifolds were studied by many authors such as: Ricci solitons in contact metric manifolds by Tripathi

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[18], Ricci solitons in manifolds with a quasi-constant curvature by Bejan [2], Ricci solitons in Lorentzian α-Sasakian manifolds by Bagewadi [1], Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds by Turan, De and Yildiz [19], Ricci solitons in Kenmotsu manifolds by Nagaraja [12], etc.

The paper is organized as follows: after the introduction and preliminaries, in Section 3 we prove that the Ricci soliton in a Ricci semi-symmetric α-cosymplectic manifold of dimension n $(n \geq 2)$, is steady. Section 4 is dedicated to the study of the pseudo-projective semi-symmetric manifold and the projective semi-symmetric manifold. In Section 5 we prove that a Weyl semi-symmetric α-Kenmotsu manifold of dimension n $(n \geq 2)$, admitting a Ricci soliton is conformally flat. In Section 6 we study the α-cosymplectic manifold satisfying $P(\xi, X) \cdot S = 0$. Finally, we prove that if a gradient Ricci soliton in an α-cosymplectic manifold of dimension n $(n \geq 2)$ is expanding, then it is an η-Einstein manifold.

2. Preliminaries

An n-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying [3]

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0,$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

On such a manifold, the fundamental form Φ of M is defined as

$$\Phi(X, Y) = g(\phi X, Y), \qquad X, Y \in \Gamma(TM).$$

In 1967 Blair [4] defined the cosymplectic structure as a quasi-Sasakian structure satisfying $d\eta = 0$. It is to be noted that the notion of cosymplectic manifold introduced by Libermann [11] is different from that of Blair [4]. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be almost cosymplectic [7] if $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. The manifold defined by $M = N \times \mathbb{R}$, where N is an almost Kählerian manifold and $\mathbb{R}$ is the real line is the simplest example of the almost cosymplectic manifold [13]. An almost contact manifold (M, ϕ, ξ, η) is said to be normal if the Nijenhuis torsion

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2(X, Y) + 2d\eta(X, Y)\xi$$

vanishes for any vector fields X and Y. A normal almost cosymplectic manifold is a cosymplectic manifold.

An almost contact metric manifold M is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant.

Kim and Pak [10] combined almost α-Kenmotsu and almost cosymplectic manifolds into a new class called almost α-cosymplectic manifolds, where α is a scalar. If we join these two classes, we obtain a new notion of an almost α-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi,$$

for any real number α. A normal almost α-cosymplectic manifold is called an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu under the condition $\alpha \neq 0$, for $\alpha \in \mathbb{R}$.

On such an α-cosymplectic manifold, we have

$$(\nabla_X \phi)Y = \alpha\big[g(\phi X, Y)\xi - \eta(Y)\phi X\big] \tag{2.1}$$

and

$$\nabla_X \xi = -\alpha\phi^2 X = \alpha[X - \eta(X)\xi]. \tag{2.2}$$

On an α-cosymplectic manifold M, the following relations are held ([14], [15])

$$R(\xi, X)Y = \alpha^2\big[\eta(Y)X - g(X,Y)\xi\big], \tag{2.3}$$

$$R(X,Y)\xi = \alpha^2\big[\eta(X)Y - \eta(Y)X\big], \tag{2.4}$$

$$S(\xi, X) = -\alpha^2(n-1)\eta(X), \tag{2.5}$$

$$\eta\big(R(X,Y)Z\big) = \alpha^2\big[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\big]. \tag{2.6}$$

Using (2.2) we have

$$\pounds_\xi g(X,Y) = 2\alpha g(X,Y) - 2\alpha\eta(X)\eta(Y). \tag{2.7}$$

From (1.1) and (2.7) we get

$$S(X,Y) = \alpha\eta(X)\eta(Y) - (\lambda + \alpha)g(X,Y). \tag{2.8}$$

Equation (2.8) yields

$$QX = \alpha\eta(X)\xi - (\lambda + \alpha)X, \tag{2.9}$$

$$S(X,\xi) = -\lambda\eta(X), \tag{2.10}$$

$$r = (1-n)\alpha - \lambda n. \tag{2.11}$$

Comparing (2.5) and (2.10) we get

$$\lambda = \alpha^2(n-1). \tag{2.12}$$

Since $\alpha^2 \geq 0$, for $\alpha \in \mathbb{R}$, from Equation (2.12) we get $\lambda \geq 0$, for all $n \geq 2$. Thus we can state the following:

Lemma 2.1. *A Ricci soliton in an n-dimensional α-cosymplectic manifold, $n \geq 2$, is either steady or expanding.*

We have already stated that an α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu under the condition $\alpha \neq 0$, for $\alpha \in \mathbb{R}$. Thus we can state the following lemmas:

Lemma 2.2. *A Ricci soliton in an n-dimensional α-cosymplectic manifold, $n \geq 2$, is steady if and only if it is a cosymplectic manifold.*

Lemma 2.3. *A Ricci soliton in an n-dimensional α-cosymplectic manifold, $n \geq 2$, is expanding if and only if it is an α-Kenmotsu manifold.*

3. Ricci semi-symmetric α-cosymplectic manifold, $n \geq 2$

Consider an α-cosymplectic manifold which is Ricci semi-symmetric. Then we have [5]

$$R(X,Y) \cdot S = 0.$$

Now we assume that the condition

$$R(\xi, X) \cdot S(Y,Z) = 0 \tag{3.1}$$

holds in M.

From (3.1) it follows that

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \tag{3.2}$$

Using (2.3), (2.8) and (2.10), we get from (3.2)

$$\alpha^2 \Big[2\alpha\eta(X)\eta(Y)\eta(Z) - \alpha\eta(Y)g(X,Z) - \alpha\eta(Z)g(X,Y) \Big] = 0,$$

or

$$\alpha^3 \Big[2\eta(X)\eta(Y)\eta(Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \Big] = 0. \tag{3.3}$$

Contracting (3.3) over X and Y we get

$$\alpha^3(n-1)\eta(Z) = 0. \tag{3.4}$$

In general, $\eta(Z) \neq 0$. Therefore, $\alpha = 0$. Thus we can state the following:

Theorem 3.1. *A Ricci semi-symmetric α-cosymplectic manifold, $n \geq 2$, admitting Ricci soliton is a cosymplectic manifold.*

By virtue of Lemma 2.2 we have

Corollary 3.1. *A Ricci soliton in a Ricci semi-symmetric α-cosymplectic manifold, $n \geq 2$, is steady.*

4. Pseudo projective semi-symmetric α-cosymplectic manifold, $n \geq 2$

We consider the pseudo projective curvature tensor P of type $(1,3)$ which is defined by [16]

$$\begin{aligned} P(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}\Big(\frac{a}{n-1} + b\Big)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{4.1}$$

where R is a Riemannian curvature tensor of type $(1,3)$, r is the scalar curvature and a and b are a non-zero constant. From (4.1) we can define a $(0,4)$ type pseudo-projective curvature tensor $\hat{P}$ as follows

$$\begin{aligned} \hat{P}(X,Y,Z,W) &= a\hat{R}(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ &- \frac{r}{n}\Big(\frac{a}{n-1} + b\Big)[g(Y,Z)g(X,W) - g(Y,U)g(Y,W)]. \end{aligned}$$

where $\hat{R}$ is a Riemannian curvature tensor of type $(0,4)$, from which it follows that

$$\sum_{i=1}^{n} \hat{P}(e_i, Y, Z, e_i) = [a + (n-1)b]\big[S(Y,Z) - \frac{r}{n} g(Y,Z)\big]. \tag{4.2}$$

Again from (4.1) we obtain

$$\eta\big(P(X,Y)Z\big) = \big[a\alpha^2 + \frac{r}{n}\Big(\frac{a}{n-1} + b\Big) + (\lambda + \alpha)b\big] \times \big[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\big],$$

or

$$\eta\big(P(X,Y)Z\big) = \beta[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{4.3}$$

where $\beta = \big[a\alpha^2 + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + (\lambda + \alpha)b\big]$.

Now we assume that the condition

$$R(\xi, X) \cdot P(Y,Z)W = 0 \tag{4.4}$$

holds in M.

From (4.4) it follows that

$$\begin{aligned} R(\xi,X)P(Y,Z)W - P(R(\xi,X)Y,Z)W - P(Y,R(\xi,X)Z)W \\ -P(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{4.5}$$

Using (2.3) in (4.5) we find

$$\begin{aligned} \alpha^2\Big[\eta\big(P(Y,Z)W\big)X - \hat{P}(Y,Z,W,X)\xi - \eta(Y)P(X,Z)W \\ +g(X,Y)P(\xi,Z)W - \eta(Z)P(Y,X)W + g(X,Z)P(Y,\xi)W \\ -\eta(W)P(Y,Z)X + g(X,W)P(Y,Z)\xi\Big] = 0, \end{aligned} \tag{4.6}$$

where $\hat{P}(Y,Z,W,X) = g\big(X, P(Y,Z)W\big)$.

Taking the inner product of (4.5) with ξ we get

$$\alpha^2\Big[\eta\big(P(Y,Z)W\big)\eta(X) - \hat{P}(Y,Z,W,X) - \eta(Y)\eta\big(P(X,Z)W\big) + g(X,Y)\eta\big(P(\xi,Z)W\big) - \eta(Z)\eta\big(P(Y,X)W\big) + g(X,Z)\eta\big(P(Y,\xi)W\big) - \eta(W)\eta\big(P(Y,Z)X\big) + g(X,W)\eta\big(P(Y,Z)\xi\big)\Big] = 0. \tag{4.7}$$

By virtue of (4.3), (4.7) yields

$$\alpha^2\Big[\hat{P}(Y,Z,W,X) + \beta\big\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\big\}\Big] = 0. \tag{4.8}$$

Contracting (4.8) over X and Y and using (4.2) we get

$$\alpha^2\Big[[a+(n-1)b]\big\{S(Z,W) - \frac{r}{n}g(Z,W)\big\} + \beta(n-1)g(Z,W)\Big] = 0. \tag{4.9}$$

We suppose that the α-cosymplectic manifold is an α-Kenmotsu manifold i.e., $\alpha \neq 0$. Thus (4.9) can be written as

$$S(Z,W) = \Big[\frac{r}{n} - \frac{\beta(n-1)}{a+(n-1)b}\Big]g(Z,W),$$

or

$$S(Z,W) = \rho g(Z,W), \tag{4.10}$$

where $\rho = \Big[\frac{r}{n} - \frac{\beta(n-1)}{a+(n-1)b}\Big]$.

Hence we have the following theorem:

Theorem 4.1. *A pseudo-projective semi-symmetric α-Kenmotsu manifold, $n \geq 2$, admitting a Ricci soliton is an Einstein manifold.*

Again, contracting (4.9) over Z and W, we get

$$n(n-1)\alpha^2\beta = 0. \tag{4.11}$$

From (4.11) it follows that

$$\alpha^2\beta = 0,$$

or

$$\alpha^2\big[a\alpha^2 + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + (\lambda+\alpha)b\big] = 0. \tag{4.12}$$

If we put $a = 1$ and $b = -\frac{1}{(n-1)}$ then (4.1) takes the form

$$\begin{aligned} P(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y] \\ &= \tilde{P}(X,Y)Z, \end{aligned} \tag{4.13}$$

where $\tilde{P}(X,Y)Z$ is the projective curvature tensor and is a particular case of P.

Now putting $a = 1$ and $b = -\frac{1}{(n-1)}$ in (4.12) and making use of (2.12) we get

$$\alpha^3 = 0,$$

or

$$\alpha = 0. \tag{4.14}$$

Thus we can state the following:

Theorem 4.2. *A projective semi-symmetric α-cosymplectic manifold, $n \geq 2$, admitting a Ricci soliton is a cosymplectic manifold.*

By virtue of Lemma 2.2 we have

Corollary 4.1. *A Ricci soliton in a projective semi-symmetric α-cosymplectic manifold, $n \geq 2$, is steady.*

5. Weyl semi-symmetric α-cosymplectic manifold, $n > 2$

We consider the Weyl conformal curvature tensor C of type $(1,3)$ which is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\big[&g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &-S(X,Z)Y\big] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{5.1}$$

where R is a Riemannian curvature tensor of type $(1,3)$. From (4.1) we can define a $(0,4)$ type Weyl conformal curvature tensor $\hat{C}$ as follows:

$$\begin{aligned} \hat{C}(X,Y,Z,W) &= \hat{R}(X,Y,Z,W) - \frac{1}{n-2}\big[g(Y,Z)S(X,W) \\ &- g(X,Z)S(Y,W) + S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\big] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned}$$

where $\hat{R}$ is a Riemannian curvature tensor of type $(0,4)$. From which it follows that

$$\sum_{i=1}^{n} \hat{C}(e_i,Y,Z,e_i) = 0. \tag{5.2}$$

Again, from (5.1) we obtain

$$\eta\big(C(X,Y)Z\big) = 0. \tag{5.3}$$

Now we assume that the condition

$$R(\xi, X) \cdot C(Y,Z)W = 0 \tag{5.4}$$

holds in M.

From (5.4) it follows that

$$\begin{aligned} R(\xi,X)C(Y,Z)W - C(R(\xi,X)Y,Z)W - C(Y,R(\xi,X)Z)W \\ -C(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{5.5}$$

Using (2.3) in (5.5) we find

$$\begin{aligned} \alpha^2\Big[\eta\big(C(Y,Z)W\big)X - \hat{C}(Y,Z,W,X)\xi - \eta(Y)C(X,Z)W \\ +g(X,Y)C(\xi,Z)W - \eta(Z)C(Y,X)W + g(X,Z)C(Y,\xi)W \\ -\eta(W)C(Y,Z)X + g(X,W)C(Y,Z)\xi\Big] = 0, \end{aligned} \tag{5.6}$$

where $\hat{C}(Y,Z,W,X) = g\big(X, C(Y,Z)W\big)$.

Taking the inner product of (5.6) with ξ we get

$$\begin{aligned} \alpha^2\Big[\eta\big(C(Y,Z)W\big)\eta(X) - \hat{C}(Y,Z,W,X) - \eta(Y)\eta\big(C(X,Z)W\big) \\ +g(X,Y)\eta\big(C(\xi,Z)W\big) - \eta(Z)\eta\big(C(Y,X)W\big) + g(X,Z)\eta\big(C(Y,\xi)W\big) \\ -\eta(W)\eta\big(C(Y,Z)X\big) + g(X,W)\eta\big(C(Y,Z)\xi\big)\Big] = 0. \end{aligned} \tag{5.7}$$

By virtue of Equation (5.3), (5.7) yields

$$\alpha^2\hat{C}(Y,Z,W,X) = 0. \tag{5.8}$$

We suppose that the α-cosymplectic manifold is an α-Kenmotsu manifold i.e., $\alpha \neq 0$. Then we have

$$\hat{C}(Y,Z,W,X) = 0. \tag{5.9}$$

Thus we can state the following:

Theorem 5.1. *A Weyl semi-symmetric α-Kenmotsu manifold, $n > 2$, admitting a Ricci soliton is conformally flat.*

6. α-cosymplectic manifold, $n \geq 2$ satisfying $P(\xi, X) \cdot S = 0$

Making use of (2.3), (2.8) and (2.10) in (4.1) we get

$$P(\xi, Y)Z = \big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + \lambda b\big][\eta(Z)Y - g(Y,Z)\xi] \\ +\alpha b[\eta(Y)\eta(Z)\xi - g(Y,Z)\xi],$$

or

$$P(\xi, Y)Z = \beta[\eta(Z)Y - g(Y,Z)\xi] + \gamma[\eta(Y)\eta(Z)\xi - g(Y,Z)\xi], \tag{6.1}$$

where $\beta = \big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + \lambda b\big]$ and $\gamma = \alpha b$.

Now we consider that a given manifold satisfies

$$P(\xi, X) \cdot S(Y, Z) = 0,$$

from which it follows that

$$S(P(\xi, X)Y, Z) + S(Y, P(\xi, X)Z) = 0. \tag{6.2}$$

Using (6.1) in (6.2) yields

$$\beta\eta(Y)S(X,Z) - \beta g(X,Y)S(\xi,Z) + \gamma\eta(X)\eta(Y)S(\xi,Z) \\ -\gamma g(X,Y)S(\xi,Z) + \beta\eta(Z)S(X,Y) - \beta g(X,Z)S(\xi,Y) \\ +\gamma\eta(X)\eta(Z)S(\xi,Y) - \gamma g(X,Z)S(\xi,Y) = 0. \tag{6.3}$$

Making use of (2.8) and (2.10) in (6.3) we get

$$\big(\alpha\beta - \lambda\gamma\big)[2\eta(X)\eta(Y)\eta(Z) - g(X,Z)\eta(Y) \\ -g(X,Y)\eta(Z)] = 0. \tag{6.4}$$

Contracting (6.4) over X and Y we get

$$\big(\alpha\beta - \lambda\gamma\big)(1-n)\eta(Z) = 0. \tag{6.5}$$

Putting $Z = \xi$ in (6.5) yields

$$\big(\alpha\beta - \lambda\gamma\big)(1-n) = 0, \tag{6.6}$$

from which it follows that

$$\big(\alpha\beta - \lambda\gamma\big) = 0,$$

or

$$\alpha\big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big)\big] = 0. \tag{6.7}$$

We suppose that the α-cosymplectic manifold is an α-Kenmotsu manifold i.e., $\alpha \neq 0$. Then (6.7) yields

$$\big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big)\big] = 0,$$

or

$$\alpha^2 = -\frac{r}{n}\big(\frac{1}{n-1} + \frac{b}{a}\big). \tag{6.8}$$

Thus we can state the following:

Theorem 6.1. *If an α-cosymplectic manifold, $n \geq 2$, admitting a Ricci soliton and satisfying $P(\xi, X) \cdot S = 0$ is an α-Kenmotsu manifold, then it satisfies $\alpha^2 = -\frac{r}{n}\big(\frac{1}{n-1} + \frac{b}{a}\big)$.*

By virtue of Lemma 2.3 we have

Corollary 6.1. *If a Ricci soliton in an α-cosymplectic manifold, $n \geq 2$, satisfying $P(\xi, X) \cdot S = 0$ is expanding, then $\alpha^2 = -\frac{r}{n}\big(\frac{1}{n-1} + \frac{b}{a}\big)$.*

For $a = 1$ and $b = -\frac{1}{(n-1)}$, from (6.6)

$$\alpha^3 = 0,$$

or

$$\alpha = 0. \tag{6.9}$$

Thus we can state the following:

Theorem 6.2. *An α-cosymplectic manifold, $n \geq 2$, admitting a Ricci soliton and satisfying $\tilde{P}(\xi, X) \cdot S = 0$ is a cosymplectic manifold.*

By virtue of Lemma 2.3 we have

Corollary 6.2. *A Ricci solitons in an α-cosymplectic manifold, $n \geq 2$, satisfying $\tilde{P}(\xi, X) \cdot S = 0$ is steady.*

7. Gradient Ricci soliton in α-cosymplectic manifolds

From Equation (1.2) we have

$$\nabla\nabla f = S + \lambda g. \tag{7.1}$$

This can be written as

$$\nabla_Y Df = QY + \lambda Y, \tag{7.2}$$

where D is the gradient operator of g. Using (7.2) we can obtain

$$R(X,Y)Df = (\nabla_X Q)Y + (\nabla_Y Q)X. \tag{7.3}$$

Taking the inner product of (7.3) with ξ we get

$$g\big(R(X,Y)Df,\xi\big) = g\big((\nabla_X Q)Y,\xi\big) + g\big((\nabla_Y Q)X,\xi\big). \tag{7.4}$$

Using (2.2) and (2.9) we have

$$g\big((\nabla_\xi Q)Y,\xi\big) = 0, \tag{7.5}$$

and

$$g\big((\nabla_Y Q)\xi,\xi\big) = 0. \tag{7.6}$$

By virtue of (7.5) and (7.6), Equation (7.4) yields

$$g\big(R(\xi,Y)Df,\xi\big) = 0. \tag{7.7}$$

Again, using (2.3) in (7.7) we get

$$g\big(R(\xi,Y)Df,\xi\big) = \alpha^2\big[\eta(Y)\eta(Df) - g(Y,Df)\big]. \tag{7.8}$$

From (7.7) and (7.8) we have

$$\alpha^2\big[\eta(Y)\eta(Df) - g(Y,Df)\big] = 0. \tag{7.9}$$

Now we suppose that $\alpha \neq 0$, i.e., the given manifold is an α-Kenmotsu manifold. Equation (7.9) yields

$$\eta(Y)\eta(Df) = g(Y,Df). \tag{7.10}$$

From (7.10) we obtain

$$Df = (\xi f)\xi. \tag{7.11}$$

Using (7.11) in (7.2)

$$Y(\xi f)\xi + \alpha(\xi f)\big[Y - \eta(Y)\xi\big] = QY + \lambda Y. \tag{7.12}$$

Taking the inner product of (7.12) with X, we obtain

$$Y(\xi f)\eta(X) + \alpha(\xi f)\big[g(X,Y) - \eta(X)\eta(Y)\big] = S(X,Y) + \lambda g(X,Y). \tag{7.13}$$

Putting $X = \xi$ and using (2.10) in (7.13) we get

$$Y(\xi f) = S(\xi,Y) + \lambda\eta(Y) = 0. \tag{7.14}$$

From (7.14) it is clear that ξf is constant. Thus (7.13) in (7.14) yields

$$\alpha(\xi f)\big[g(X,Y)-\eta(X)\eta(Y)\big]=S(X,Y)+\lambda g(X,Y),$$

or

$$S(X,Y)=\big[\alpha(\xi f)-\lambda\big]g(X,Y)-\alpha(\xi f)\eta(X)\eta(Y). \tag{7.15}$$

Hence we can state the following:

Theorem 7.1. *If an α-cosymplectic manifold, $n \geq 2$, admitting a gradient Ricci soliton is an α-Ketmotsu manifold, then it is an η-Einstein manifold.*

By virtue of Lemma 2.2 we have

Corollary 7.1. *If a gradient Ricci soliton in an α-cosymplectic manifold, $n \geq 2$, is expanding, then it is an η-Einstein manifold.*

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JP Singh
Department of Mathematics and Computer Science
Mizoram University
Tanhril, Aizawl-796004, India
jpsmaths@gmail.com

C. Lalmalsawma
Department of Mathematics and Computer Science
Mizoram University
Tanhril, Aizawl-796004, India
sweezychawngthu@gmail.com