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SOME SEMISYMMETRY CONDITIONS ON RIEMANNIAN MANIFOLDS

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Abstract. We study a Riemannian manifold M admitting a semisymmetric metric connection $\tilde{\nabla}$ such that the vector field U is a parallel unit vector field with respect to the Levi-Civita connection ∇. Firstly, we show that if M is projectively flat with respect to the semisymmetric metric connection $\tilde{\nabla}$ then M is a quasi-Einstein manifold. Also we prove that if $R\cdot\tilde{P}=0$ if and only if M is projectively semisymmetric; if $\tilde{P}\cdot R=0$ or $R\cdot\tilde{P}-\tilde{P}\cdot R=0$ then M is conformally flat and quasi-Einstein manifold. Here R, P and $\tilde{P}$ denote Riemannian curvature tensor, the projective curvature tensor of ∇ and the projective curvature tensor of $\tilde{\nabla}$, respectively.

1. Introduction

Let $\tilde{\nabla}$ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T is given by

$$T(X,Y)=\tilde{\nabla}_X Y-\tilde{\nabla}_Y X-[X,Y].$$

The connection $\tilde{\nabla}$ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. If there is a Riemannian metric g in M such that $\tilde{\nabla}g=0$, then the connection $\tilde{\nabla}$ is a metric connection, otherwise it is non-metric [24]. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

Hayden [13] introduced a metric connection $\tilde{\nabla}$ with a non-zero torsion on a Riemannian manifold. Such a connection is called a Hayden connection. In [12] and [18], Friedmann and Schouten introduced the idea of a semisymmetric linear connection in a differentiable manifold. A linear connection is said to be a *semisymmetric connection* if its torsion tensor T is of the form

$$T(X,Y)=\omega(Y)X-\omega(X)Y, \tag{1.1}$$

where the 1-form ω is defined by

$$\omega(X)=g(X,U),$$

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and U is a vector field. In [17], Pak showed that a Hayden connection with the torsion tensor of the form (1.1) is a semisymmetric metric connection. In [23], Yano considered a semisymmetric metric connection and studied some of its properties. He proved that in order that a Riemannian manifold admits a semisymmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold be conformally flat. For some properties of Riemannian manifolds with a semisymmetric metric connection (see also [1], [6], [4], [5], [7], [14], [21], [22]). Then, Murathan and Özgür [16] studied Riemannian manifolds admitting a semisymmetric metric connection $\tilde{\nabla}$ such that the vector field U is a parallel vector field with respect to the Levi-Civita connection ∇.

On the other hand, if a Riemannian manifold satisfying the condition $R \cdot R = 0$, then the manifold is called *semisymmetric* ([19], [20]). It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. A Riemannian manifold is said to be *Ricci-semisymmetric* if $R \cdot S = 0$. The class of semisymmetric manifolds includes the set of Ricci-semisymmetric manifolds ($\nabla S = 0$) as a proper subset. Evidently, the condition $R \cdot R = 0$ implies condition $R \cdot S = 0$. The converse is in general not true. Also, a Riemannian manifold satisfying the condition $R \cdot P = 0$, then the manifold is called *projectively semisymmetric*.

Motivated by the studies of the above authors, in this paper we consider Riemannian manifolds (M, g) admitting a semisymmetric metric connection such that U is a unit parallel vector field with respect to the Levi-Civita connection ∇. The paper is organized as follows: In Section 2 and Section 3, we give the necessary notions and results which will be used in the next section. In the last section, firstly we show that if M is projectively flat with respect to the semisymmetric metric connection $\tilde{\nabla}$ then M is a quasi-Einstein manifold. Then we prove that if $R \cdot \tilde{P} = 0$ if and only if M is projectively semisymmetric; if $\tilde{P} \cdot R = 0$ or $R \cdot \tilde{P} - \tilde{P} \cdot R = 0$ then M is conformally flat and quasi-Einstein manifold, where R, P and $\tilde{P}$ denote Riemannian curvature tensor, the projective curvature tensor of ∇ and the projective curvature tensor of $\tilde{\nabla}$, respectively.

2. Preliminaries

An n-dimensional Riemannian manifold (M^n, g), $n > 2$, is said to be an Einstein manifold if its Ricci tensor S satisfies the condition $S = \frac{\tau}{n} g$, where τ denotes the scalar curvature of M. If the Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{2.1}$$

where a, b are smooth functions and A is a non-zero 1-form such that

$$g(X, U) = A(X),$$

for all vector fields X. Then M is called a quasi-Einstein manifold [3].

For a $(0,k)$-tensor field, $k \geqslant 1$, on (M,g) we define the tensor $R \cdot T$ (see [9]) by

$$(R(X,Y)\cdot T)(X_1,...,X_k) = -T(R(X,Y)X_1,...,X_k) - .... - T(X_1,...X_{k-1},R(X,Y)X_k). \tag{2.2}$$

In addition, if E is a symmetric $(0,2)$-tensor field, then we define the $(0,k+2)$-tensor $Q(E,T)$ (see [9]) by

$$Q(E,T)(X_1,...,X_k;X,Y) = -T((X\wedge_E Y)X_1,...,X_k) - .... -T(X_1,...X_{k-1},(X\wedge_E Y)X_k), \tag{2.3}$$

where $X \wedge_E Y$ is defined by

$$(X\wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y.$$

The *Weyl tensor* and the *projective tensor* of a Riemannian manifold (M,g) are defined by

$$\begin{aligned} C(X,Y,Z,W) &= R(X,Y,Z,W) \\ &\quad -\frac{1}{n-2}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &\qquad + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\} \\ &\quad +\frac{\tau}{(n-1)(n-2)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \end{aligned}$$

and

$$\begin{aligned} P(X,Y,Z,W) &= R(X,Y,Z,W) \\ &\quad -\frac{1}{n-1}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\}. \end{aligned} \tag{2.4}$$

respectively, where τ denotes the scalar curvature of M. For $n \geq 4$, if $C = 0$, the manifold is called *conformally flat* [24]. If $P = 0$, the manifold is called *projectively flat*.

Now we give the Lemmas which will be used in the last section.

Lemma 2.1. [10] *Let $(M^n,g), n \geqslant 3$, be a semi-Riemannian manifold. Let at a point $x \in M$ be given a non-zero symmetric $(0,2)$-tensor E and a generalized curvature tensor B such that at x the following condition is satisfied $Q(E,B) = 0$. Moreover, let V be a vector at x such that the scalar $\rho = a(V)$ is non-zero, where a is a covector defined by $a(X) = E(X,V)$, $X \in T_X M$.*

i) If $E = \frac{1}{\rho} a \otimes a$, then at x we have $_{X,Y,Z} a(X)B(Y,Z) = 0$, where $X,Y,Z \in T_X M$.

ii) If $E - \frac{1}{\rho} a \otimes a$ is non-zero, then at x we have $B = \frac{\gamma}{2} E \wedge E$, $\gamma \in \mathbb{R}$. Moreover, in both cases, at x we have $B \cdot B = Q(Ric(B), B)$.

Lemma 2.2. [11] *Let (M^n,g), $n \geqslant 4$, be a semi-Riemannian manifold and E be the symmetric $(0,2)$-tensor at $x \in M$ defined by $E = \alpha g + \beta \omega \otimes \omega$, $\omega \in T^*_X M$, $\alpha, \beta \in \mathbb{R}$. If at x the curvature tensor R is expressed by $R = \frac{\gamma}{2} E \wedge E$, $\gamma \in \mathbb{R}$, then the Weyl tensor vanishes at x.*

3. Semisymmetric metric connection

Let ∇ is the Levi-Civita connection of a Riemannian manifold M. It is known [23] that if $\tilde{\nabla}$ is a semisymmetric metric connection then

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X, Y)U,$$

where

$$\omega(X) = g(X, U),$$

and X, Y, U are vector fields on M. Let R and $\tilde{R}$ denote the Riemannian curvature tensor of ∇ and $\tilde{\nabla}$, respectively. Then we know [23] that

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) - \theta(Y, Z)g(X, W) \\ &\quad +\theta(X, Z)g(Y, W) - g(Y, Z)\theta(X, W) \\ &\quad +g(X, Z)\theta(Y, W), \end{aligned} \tag{3.1}$$

where

$$\theta(X, Y) = g(AX, Y) = (\nabla_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}g(X, Y).$$

Now assume that U is a parallel unit vector field with respect to the Levi-Civita connection ∇, i.e., $\nabla U = 0$ and $\|U\| = 1$. Then

$$(\nabla_X \omega)Y = \nabla_X \omega(Y) - \omega(\nabla_X Y) = 0. \tag{3.2}$$

So θ is a symmetric $(0, 2)$-tensor field. Hence equation (3.1) can be written as

$$\tilde{R} = R - g \barwedge \theta, \tag{3.3}$$

where $\barwedge$ is Kulkarni-Nomizu product, which is defined by

$$\begin{aligned} (g \barwedge \theta)(X, Y, Z, W) &= \theta(Y, Z)g(X, W) - \theta(X, Z)g(Y, W) \\ &\quad + g(Y, Z)\theta(X, W) - g(X, Z)\theta(Y, W). \end{aligned} \tag{3.4}$$

Since U is a parallel unit vector field, it is easy to see that $\tilde{R}$ is a generalized curvature tensor and it is trivial that $R(X, Y)U = 0$. Hence by a contraction we find $S(Y, U) = w(\mathcal{S}Y)$, where S denotes the Ricci tensor of ∇ and $\mathcal{S}$ is the Ricci operator defined by $g(\mathcal{S}X, Y) = S(X, Y)$. It is easy to see that we also have the following relations [16]:

$$\tilde{\nabla}_X U = X - \omega(X)U,$$

$$\tilde{R}(X, Y)U = 0, \; R \cdot \theta = 0, \tag{3.5}$$

$$\theta^2(X, Y) := g(AX, AY) = \frac{1}{4}g(X, Y),$$

and

$$\tilde{S} = S - (n - 2)(g - \omega \otimes \omega), \tag{3.6}$$

$$\tilde{\tau} = \tau - (n-2)(n-1). \tag{3.7}$$

Using (2.4), (3.1), (3.6) and (3.7), we get

$$\tilde{C} = C,$$

and

$$\tilde{P} = P - \frac{1}{n-1} g \barwedge \theta + G, \tag{3.8}$$

where $\tilde{P}$ denotes the projective curvature tensor with respect to semisymmetric metric tensor $\tilde{\nabla}$ and G is defined by

$$G(X, Y, Z, W) = \frac{n-2}{n-1}\{g(Y, Z)\omega(X)\omega(W) - g(X, Z)\omega(Y)\omega(W)\}. \tag{3.9}$$

We also have the followings:

$$\tilde{P}(X, Y)U = 0, \tag{3.10}$$

$$R \cdot G = 0, \quad G \cdot R = 0. \tag{3.11}$$

4. Main results

In this section, the tensors $\tilde{P}$, $\tilde{P} \cdot R$ and $Q(\theta, T)$ are defined in the same way with (3.8), (2.2) and (2.3). Let $\tilde{P}_{hijk}$, $(R \cdot \tilde{P})_{hijklm}$, $(\tilde{P} \cdot R)_{hijklm}$ denote the local components of the tensors $\tilde{P}$, $R \cdot \tilde{P}$ and $\tilde{P} \cdot R$, respectively.

Theorem 4.1. *Let (M, g) be a Riemannian manifold admitting a semisymmetric metric connection. If M is projectively flat with respect to semisymmetric metric tensor $\tilde{\nabla}$, then M is a quasi-Einstein manifold.*

Proof. *Let (M, g) be a Riemannian manifold admitting a semisymmetric metric connection. Then using (2.4) and (3.1) we have*

$$\begin{aligned} \tilde{P}_{hijk} &= R_{hijk} - (g \barwedge \theta)_{hijk} \\ &\quad - \frac{1}{n-1}\{S_{ij}g_{hk} - (n-2)[g_{ij}g_{hk} - g_{hk}(\omega \otimes \omega)_{ij}] \\ &\qquad - S_{hj}g_{ik} + (n-2)[g_{hj}g_{ik} - g_{ik}(\omega \otimes \omega)_{hj}]\} \end{aligned} \tag{4.1}$$

Now if M is projectively flat with respect to semisymmetric metric tensor $\tilde{\nabla}$, then from (4.1) we have

$$\begin{aligned} R_{hijk} &= (g \barwedge \theta)_{hijk} \\ &\quad + \frac{1}{n-1}\left\{S_{ij}g_{hk} - S_{hj}g_{ik}\right\} \\ &\quad + \frac{n-2}{n-1}\left\{g_{hk}(\omega \otimes \omega)_{ij} - g_{ik}(\omega \otimes \omega)_{hj} - g_{ij}g_{hk} + g_{hj}g_{ik}\right\}. \end{aligned}$$

Help of (3.4), we get

$$
\begin{aligned}
R_{hijk} &= \left\{g_{ij}g_{hk} - g_{ik}g_{hj}\right\} \\
&+\frac{1}{n-1}\left\{S_{ij}g_{hk} - S_{hj}g_{ik}\right\} \\
&-\frac{n-2}{n-1}\left\{g_{ij}g_{hk} - g_{hj}g_{ik} - g_{hk}(\omega\otimes\omega)_{ij} + g_{ik}(\omega\otimes\omega)_{hj}\right\} \\
&+g_{ik}(\omega\otimes\omega)_{hj} - g_{hk}(\omega\otimes\omega)_{ij} \\
&+g_{hj}(\omega\otimes\omega)_{ik} - g_{ij}(\omega\otimes\omega)_{hk}.
\end{aligned} \tag{4.2}
$$

Contracting (4.2) with g^{hj}*, we obtain*

$$S_{ik} = \frac{n+\tau-2}{n}g_{ik} + (2-n)(\omega\otimes\omega)_{ik},$$

which gives us that M *is a quasi-Einstein manifold.* □

Proposition 4.1. *Let* (M, g) *be a Riemannian manifold admitting a semisymmetric metric connection* $\tilde{\nabla}$*. If* U *is a parallel unit vector field with respect to the Levi-Civita connection* ∇*, then*

$$(R\cdot\tilde{P})_{hijklm} = (R\cdot P)_{hijklm} \tag{4.3}$$

$$(\tilde{P}\cdot R)_{hijklm} = (P\cdot R)_{hijklm} - \frac{1}{n-1}Q(g - \omega\otimes\omega, R)_{hijklm}. \tag{4.4}$$

Proof. Since U is parallel, we have $R\cdot\theta = 0$ and $R\cdot G = 0$. So from (3.8), we obtain

$$R\cdot\tilde{P} = R\cdot P - \frac{1}{n-1}g\,\overline{\wedge}\,R\cdot\theta + R\cdot G = R\cdot P. \tag{4.5}$$

Applying (3.1) in (2.2) and using (2.3) and (3.11), we get

$$
\begin{aligned}
(\tilde{P}\cdot R)_{hijiklm} &= (P\cdot R)_{hijklm} - \frac{1}{n-1}Q(\theta, R)_{hijklm} \\
&\quad -\frac{1}{2(n-1)}(g_{hl}R_{mijk} - g_{hm}R_{lijk} - g_{il}R_{mhjk} \\
&\quad +g_{im}R_{lhjk} - g_{jl}R_{mkhi} - g_{jm}R_{lkhi} \\
&\quad -g_{kl}R_{mjhi} + g_{km}R_{lihi}) + (G\cdot R)_{hijklm} \\
&= (P\cdot R)_{hijklm} - \frac{1}{n-1}Q(\theta + \frac{1}{2}g, R)_{hijklm} \\
&= (P\cdot R)_{hijklm} - \frac{1}{n-1}Q(g - \omega\otimes\omega, R)_{hijklm}.
\end{aligned} \tag{4.6}
$$

This completes the proof of the Proposition. □

As an immediate consequence of Proposition 4.1, we have the followings:

Theorem 4.2. *Let (M, g) be a Riemannian manifold admitting a semisymmetric metric connection $\tilde{\nabla}$ and U be a parallel unit vector field with respect to the Levi-Civita connection ∇. Then $R \cdot \tilde{P} = 0$ if and only if M is projectively semisymmetric.*

Theorem 4.3. *Let (M^n, g) be a semisymmetric $n > 3$ dimensional Riemannian manifold admitting a semisymmetric metric connection $\tilde{\nabla}$ and S be the symmetric $(0, 2)$-tensor defined by $S = \alpha g + \beta\omega \otimes \omega$. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ and $\tilde{P} \cdot R = 0$, then M is a conformally flat quasi-Einstein manifold.*

Proof. Since the condition $\tilde{P} \cdot R = 0$ holds on M, from (4.4),we have

$$(4.7) \qquad (P \cdot R)_{hijklm} = \frac{1}{n-1} Q(g - \omega \otimes \omega, R)_{hijklm}$$

After some calculations from (4.7), we get

$$(4.8) \qquad (R \cdot R)_{hijklm} - \frac{1}{n-1} Q(S, R)_{hijklm} = \frac{1}{n-1} Q(g - \omega \otimes \omega, R)_{hijklm}.$$

Since M is a semisymmetric Riemannian manifold, then from (4.8), we have

$$(4.9) \qquad Q(S + g - \omega \otimes \omega, R)_{hijklm} = 0.$$

Now let $S = \alpha g + \beta\omega \otimes \omega$, $\alpha, \beta \in \mathbb{R}$. Then from (4.9), we get

$$(4.10) \qquad Q(\lambda_1 g - \lambda_2 \omega \otimes \omega, R)_{hijklm} = 0,$$

where $\lambda_1 = \alpha + 1$, $\lambda_2 = \beta + 1$. So we have two possibilities:

$$(4.11) \qquad rank(\lambda_1 g - \lambda_2 \omega \otimes \omega) = 1$$

or

$$(4.12) \qquad rank(\lambda_1 g - \lambda_2 \omega \otimes \omega) > 1.$$

Suppose that(4.11) holds at a point x. Thus we have

$$\lambda_1 g - \lambda_2 \omega \otimes \omega = \rho z \otimes z,$$

where $z \in T^*_X M$ and $\rho \in \mathbb{R}$. Because of non-zero coefficient of g, this relation does not occur. Thus the case (4.12) must be fullfilled at x. By virtue of Lemma 2.1, (4.10) gives us

$$R = \frac{\gamma}{2}((g - \omega \otimes \omega) \wedge (g - \omega \otimes \omega)), \quad \gamma \neq 0, \quad \gamma \in \mathbb{R}.$$

So again from Lemma 2.2, we obtain $C = 0$, which give us that M is conformally flat. Moreover, contracting (4.10) with g^{ij}, we get

$$Q(\lambda_1 g - \lambda_2 \omega \otimes \omega, S)_{hklm} = 0,$$

which gives us

$$S = \lambda_1 g - \lambda_2 \omega \otimes \omega,$$

where $\lambda_1, \lambda_2 : M \longrightarrow \mathbb{R}$ are functions. So by virtue of (2.1), M is a quasi-Einstein manifold. Thus the proof of the Theorem is completed. □

Theorem 4.4. *Let (M^n, g) be a Ricci-semisymmetric $n > 3$ dimensional Riemannian manifold admitting a semisymmetric metric connection $\tilde{\nabla}$ and U be a parallel unit vector field with respect to the Levi-Civita connection ∇ and $R \cdot \tilde{P} - \tilde{P} \cdot R = 0$, then M is a conformally flat quasi-Einstein manifold.*

Proof. *Using (4.3) and (4.4), we obtain*

$$\begin{aligned} 0 &= R \cdot \tilde{P} - \tilde{P} \cdot R = R \cdot P - P \cdot R \\ &\quad + \frac{1}{n-1} Q(g - \omega \otimes \omega, R)_{hijklm} \\ &= \frac{1}{n-1} g_{ik} (R \cdot S)_{hjlm} - \frac{1}{n-1} g_{hk} (R \cdot S)_{ijlm} \\ &\quad + \frac{1}{n-1} Q(S + g - \omega \otimes \omega, R)_{hijklm}. \end{aligned}$$

Since M is a Ricci-semisymmetric Riemannian manifold, (i.e. $R \cdot S = 0$), then from the above equation, we get

$$Q(S + g - \omega \otimes \omega, R)_{hijklm} = 0. \tag{4.13}$$

Using the same method in the proof of Theorem 4.3, we obtain M is a conformally flat quasi Einstein manifold. □

Example 4.1. Let M^{2n+1} be a $(2n+1)$-dimensional almost contact manifold endowed with an almost contact structure (ϕ, ξ, η), that is, ϕ is a $(1,1)$-tensor field, ξ is a vector field, and η is a 1-form such that

$$\phi^2 = I - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1$$

Then

$$\phi(\xi) = 0 \quad \text{and} \quad \eta \circ \xi = 0.$$

Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

or, equivalently

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } g(X, \xi) = \eta(X),$$

for all $X, Y \in \chi(M)$. Then M^{2n+1} becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). An almost contact metric manifold is cosymplectic [15] , if $\nabla_X \phi = 0$. From the formula $\nabla_X \phi = 0$ it follows that

$$\nabla_X \xi = 0, \quad \nabla_X \eta = 0 \text{ and } R(X, Y)\xi = 0.$$

Then we have

$$P(X, Y)\xi = 0.$$

So we have the following relations:

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi,$$

and

$$\theta = \frac{1}{2}g - \eta \otimes \eta.$$

Hence $\nabla\theta = 0$ and $R \cdot \theta = 0$, which gives us $R \cdot \tilde{P} = R \cdot P$.

A cosymplectic manifold M is said to be a cosymplectic space form if the ϕ-sectional curvature tensor is constant c along M. A cosymplectic space form will be denoted by $M(c)$. Then the Riemannian curvature tensor R on $M(c)$ is given by [15]

$$\begin{aligned} R(X,Y,Z,W) &= \frac{c}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &\quad + g(X,\phi W)g(Y,\phi Z) - g(X,\phi Z)g(Y,\phi W) \\ &\quad - 2g(X,\phi Y)g(Z,\phi W) - g(X,W)\eta(Y)\eta(Z) \\ &\quad + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) + g(Y,W)\eta(X)\eta(Z)\}. \end{aligned}$$

From direct calculation we get

$$S(X,W) = \frac{nc}{2}\left\{g(X,W) - \eta(X)\eta(W)\right\},$$

which gives us that M is a quasi-Einstein manifold.

5. Conclusions

Hayden [13] introduced a metric connection $\tilde{\nabla}$ with a non-zero torsion on a Riemannian manifold. Then, Friedmann and Schouten introduced the idea of a semisymmetric linear connection in a differentiable manifold ([12], [18]). In [23], Yano proved that a Riemannian manifold admits a semisymmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold be conformally flat. Recently, Murathan and Özgür [16] studied Riemannian manifolds admitting a semisymmetric metric connection $\tilde{\nabla}$ such that the vector field U is a parallel vector field with respect to the Levi-Civita connection ∇. On the other hand, if a Riemannian manifold satisfying the condition $R \cdot R = 0$ ($R.S = 0$), then the manifold is called *semisymmetric* (*Ricci semisymmetric*) ([19], [20]). In this paper, firstly we show that if M is projectively flat with respect to the semisymmetric metric connection $\tilde{\nabla}$ then M is a quasi-Einstein manifold. Then we prove that if $R \cdot \tilde{P} = 0$ if and only if M is projectively semisymmetric; if $\tilde{P} \cdot R = 0$ or $R \cdot \tilde{P} - \tilde{P} \cdot R = 0$ then M is conformally flat and quasi-Einstein manifold.

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