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GENERALIZED MATRIX MULTIPLICATION AND ITS SOME APPLICATION

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Abstract. In this paper, a generalized matrix multiplication is defined in $\mathbb{R}^{m,n} \times \mathbb{R}^{n,p}$ by using any scalar product in $\mathbb{R}^n$, where $\mathbb{R}^{m,n}$ denotes set of matrices of m rows and n columns. With this multiplication it has been shown that $\mathbb{R}^{n,n}$ is an algebra with unit. By considering this new multiplication we define eigenvalues and eigenvectors of square $n \times n$ matrix A. A special case is considered and generalized diagonalization is also introduced.

Keywords: Generalized matrix multiplication; inner product; eigenvector; eigenvalue.

1. Introduction

The sign matrix is required for the definition of special matrices when the Euclid inner product is used for the matrix multiplication in Lorentz space [5]. Ergin [1] obtained the rotations in Lorentz plane by using Euclidean matrix multiplication. In our former study [2], by introducing Lorentz matrix multiplication, special matrices and the motions on Lorentz plane were defined without the need of sign matrix. By this way, convenience in between the matrix multiplication and scalar product in Lorentzian was obtained. Further definition of the matrix multiplication using scalar product $\mathbb{R}^n$ of which index is ν were also given in [3].

Our aim in the present paper is to define a new matrix multiplication being compatible with any scalar product in $\mathbb{R}^n$. Some applications were also given.

Let $\mathbb{R}^{m,n}$ be the set of all $m \times n$ matrices. $\mathbb{R}^{m,n}$ with the matrix addition and the scalar-matrix multiplication is a real vector space. More properties of the ordinary matrix multiplication can be found in [4].

Let g be a scalar product on $\mathbb{R}^n$ which is nondegenerate symmetric bilinear form

$$g(x, y) = \sum_{i,j=1}^{n} g_{ij} x_i y_j,$$

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where $x = (x_1, \dots x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$. For the standard basis $\{e_i\}_{1 \le i \le n}$ of $\mathbb{R}^n$, the matrix representation of scalar product is $G = [g_{ij} = g(e_i, e_j)] \in \mathbb{R}^{n,n}$.

2. Generalized Matrix Multiplication and Properties

Let $A_1, \dots, A_m$ denote the row vectors of $A = [a_{ij}] \in \mathbb{R}^{m,n}$ and $B^1, \dots, B^p$ denote the column vectors of $B = [b_{jk}] \in \mathbb{R}^{n,p}$. Then we define a new matrix multiplication denoted by "$\bullet_g$", as

$$
\begin{aligned}
A \bullet_g B &= \begin{bmatrix} g\left(A_1, B^1\right) & g\left(A_1, B^2\right) & \cdots & g\left(A_1, B^p\right) \\ g\left(A_2, B^1\right) & g\left(A_2, B^2\right) & \cdots & g\left(A_2, B^p\right) \\ \vdots & \vdots & & \vdots \\ g\left(A_n, B^1\right) & g\left(A_n, B^2\right) & \cdots & g\left(A_n, B^p\right) \end{bmatrix} \\
&= \left[g\left(A_i, B^j\right)\right].
\end{aligned}
$$

We call this multiplication as generalized matrix multiplication and if we let A_i to be ith row of A and B^j to be jth column of B then (i, j) entry of $A \bullet_g B$ is $g\left(A_i, B^j\right)$. Note that $A \bullet_g B$ is an $m \times p$ matrix. We will denote $\mathbb{R}^{m,n}$ with generalized matrix multiplication by $\mathbb{R}_g^{m,n}$. In the special case of g we get followings:

1. For $g(x, y) = \sum\limits_{i=1}^{n} x_i y_i$, $A \bullet_g B$ coincides with usual matrix multiplication.
2. For $g(x, y) = -x_1 y_1 + \sum\limits_{i=2}^{n} x_i y_i$, $A \bullet_g B$ coincides with Lorentzian matrix multiplication defined in [2].
3. For $g(x, y) = -\sum\limits_{i=1}^{\nu} x_i y_i + \sum\limits_{i=\nu+1}^{n} x_i y_i$, $A \bullet_g B$ coincide with Pseudo matrix multiplication defined in [3].

Theorem 2.1. *The following statements are satisfied.*

1. *For every* $A \in \mathbb{R}_g^{m,n}, B \in \mathbb{R}_g^{n,p}, C \in \mathbb{R}_h^{p,r}$, $A \bullet_g (B \bullet_h C) = (A \bullet_g B) \bullet_h C$.
2. *For every* $A \in \mathbb{R}_g^{m,n}, B, C \in \mathbb{R}_g^{n,p}$, $A \bullet_g (B + C) = A \bullet_g B + A \bullet_g C$
3. *For every* $A, B \in \mathbb{R}_g^{m,n}, C \in \mathbb{R}_g^{n,p}$, $(A + B) \bullet_g C = A \bullet_g C + B \bullet_g C$
4. *For every* $k \in \mathbb{R}$, $A \in \mathbb{R}_g^{m,n}, B \in \mathbb{R}_g^{n,p}$, $k(A \bullet_g B) = (kA) \bullet_g B = A \bullet_g (kB)$

where g and h are nondegenerate symmetric bilinear forms on $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively.

Definition 2.1. The determinant of a matrix $A = [a_{ij}] \in \mathbb{R}_g^{n,n}$ is denoted by $\det A$ and defined as

$$\det\, A = \sum_{\sigma \in S_n} s(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n},$$

where S_n is set of all permutations of the set $\{1, 2, \cdots, n\}$ and $s(\sigma)$ is sign of the permutation σ.

Let $\{E_{ij}\}$ be the standard basis of $\mathbb{R}^{n,n}$ and $X = [x_{ij}] \in \mathbb{R}^{n,n}$. Consider the linear equation system $X \bullet_g E_{ij} = E_{ij}$ for $1 \leq i, j \leq n$. If we handle the equations having the variables $\{x_{11}, x_{12}, \ldots, x_{1n}\}$, we obviously get

$$\begin{cases} \sum\limits_{i=1}^{n} g_{i1} x_{1i} = 1, \\ \sum\limits_{i=1}^{n} g_{ij} x_{ji} = 0, \;\; (2 \leq j \leq n). \end{cases}$$

Denoting the first column matrix of X by X^1, the ordinary matrix form of above system can be stated as

$$G^T X^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since G is symmetric and regular matrix, we deduce

$$X^1 = G^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If we proceed the same way for the equations having the variables $\{x_{j1}, x_{j2}, \ldots, x_{jn}\}$, $2 \leq j \leq n$, we have

$$X = \frac{1}{\det G} \left[g^{ij} \right],$$

where g^{ij} is (i, j) cofactor of G. Now we are ready to give the definition of identity matrix.

Definition 2.2. Let g be a nondegenerate symmetric bilinear form on $\mathbb{R}^n$, then $n \times n$ identity matrix according to generalized matrix multiplication, is denoted by I_n and defined by

$$I_n = \frac{1}{\det G} \left[g^{ij} \right].$$

Note that for every $A=[a_{ij}]\in\mathbb{R}_{\nu}^{n,n},\ I_n\bullet_g A=A\bullet_g I_n=A$. Indeed,

$$\begin{aligned} I_n\bullet_g A &= I_n\bullet_g\left(\sum_{i,j=1}^{n}a_{ij}E_{ij}\right)\\ &= \sum_{i,j=1}^{n}a_{ij}\left(I_n\bullet_g E_{ij}\right)\\ &= \sum_{i,j=1}^{n}a_{ij}E_{ij}\\ &= A. \end{aligned}$$

Corollary 2.1. $\mathbb{R}_g^{n,n}$ *with generalized matrix multiplication is an algebra with unit.*

Definition 2.3. An $n\times n$ matrix A is g-invertible, if there exists an $n\times n$ matrix B such that $A\bullet_g B=B\bullet_g A=I_n$. Then B is called g-inverse of A and is shown by A_g^{-1}.

Example 2.1. Let $g(x,y)=x_1y_1+x_1y_2+x_2y_1+2x_2y_2$ be a scalar product on $\mathbb{R}^2$ and $A=\begin{bmatrix}2 & 4\\ -2 & 3\end{bmatrix}\in\mathbb{R}_2^2$. Then $I_2=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$ and $A_g^{-1}=\begin{bmatrix}\frac{9}{7} & -1\\ -\frac{4}{7} & \frac{1}{2}\end{bmatrix}$.

Definition 2.4. Let $A=[a_{ij}]$ be an $m\times n$ matrix. Then the transpose of A is the $n\times m$ matrix A^T obtained by interchanging the rows and columns of A, so that the $(i,j)\,th$ entry of A^T is a_{ji}.

Theorem 2.2. *Let A and B be matrices of the appropriate sizes so that the following operations make sense, and c be a scalar. Then*

1. $(A+B)^T=A^T+B^T$
2. $(A\bullet_g B)^T=B^T\bullet_g A^T$
3. $(cA)^T=cA^T$
4. $\left(A^T\right)^T=A.$

Proof. We only prove the property (2). The others can be easily done.

$$\begin{aligned} (A\bullet_g B)^T &= \left[g\left(A_i,B^j\right)\right]^T\\ &= \left[g\left(A_j,B^i\right)\right]\\ &= \left[g\left(B^i,A_j\right)\right]\\ &= \left[g\left(\left(B^T\right)_i,\left(A^T\right)^j\right)\right]\\ &= B^T\bullet_g A^T. \end{aligned}$$

□

Definition 2.5. Let $A \in \mathbb{R}_g^{n,n}$. If $A^T = A$ and $A^T = -A$ then A is said to be symmetric and skew-symmetric matrix, respectively.

Definition 2.6. Let A be an $m \times n$ matrix and denote the rows of A by A_i, $i = 1, \ldots, m$. The elementary row operations on matrix A are:

1. R_{ij}, the interchange of rows A_i and A_j,
2. $R_i(c)$, the multiplication of row A_i by the nonzero scalar c,
3. $R_{ij}(a)$, the addition of a times row A_j to row A_i, $i \neq j$.

Definition 2.7. Let I_n be identity matrix according to generalized matrix multiplication obtained from g nondegenerate symmetric bilinear form on $\mathbb{R}^n$. An elementary matrix is an $n \times n$ matrix obtained from the identity matrix I_n by performing on I_n a single elementary row operation. For the operations $R_{ij}, R_i(c)$ and $R_{ij}(a)$, elementary matrices denoted by $E_{ij}, E_i(c)$ and $E_{ij}(a)$, respectively.

Let I_n be identity matrix according to generalized matrix multiplication obtained from $g(x,y) = x_1y_1 + x_2y_2 - x_2y_3 - x_3y_2 - 2x_3y_3$ nondegenerate symmetric bilinear form on $\mathbb{R}^3$. Then

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

and the following are examples of elementary matrices:

$$E_{13} = \begin{bmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & 0 & 0 \end{bmatrix}, E_3(-6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 2 & 2 \end{bmatrix}, E_{31}(2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 2 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

Theorem 2.3. *Let R be an elementary row operation and let E be the corresponding $m \times m$ elementary matrix. Then ,*

$$R(A) = E \bullet_g A$$

where A is any $m \times n$ matrix.

Proof. The proof is similar to standard one. □

Theorem 2.4. *The determinants of the elementary matrices are given as*

1. $\det E_{ij} = -\det I_n$
2. $\det E_i(c) = c \det I_n$

3. $\det E_{ij}(a) = \det I_n$

Proof. We omit the details since it is clear from the properties of determinant function. $\square$

Theorem 2.5. *Let E be an elementary matrix and A be an $n \times n$ matrix. Then the following holds*

$$\det(E \bullet_g A) = (\det I_n)^{-1} \cdot \det E \cdot \det A.$$

Proof. Let A be a $n \times n$ matrix and E be an elementary matrix. Firstly, by taking $E = E_i(c)$, we get $\det(E \bullet_f A) = \det(E_i(c) \bullet_f A)$. The properties of determinant function, Theorem 2.3 and Theorem 2.4 leads to

$$\begin{aligned} \det(E_i(c) \bullet_g A) &= c \cdot \det A \\ &= (\det I_n)^{-1} \cdot \det E_i(c) \cdot \det A. \end{aligned}$$

For the other cases $E = E_{ij}(a)$ and $E = E_{ij}$, the proof can be done similarly. $\square$

Theorem 2.6. *For every $A, B \in \mathbb{R}_g^{n,n}$,* $\det(A \bullet_g B) = (\det I_n)^{-1} \det A \cdot \det B$.

Proof. Let A be a matrix obtained by multiplying elementary matrices E_1 and E_2. From above theorem, we get

$$\begin{aligned} \det(A \bullet_g B) &= \det((E_1 \bullet_g E_2) \bullet_g B) \\ &= \det(E_1 \bullet_g (E_2 \bullet_g B)) \\ &= (\det I_n)^{-1} \det E_1 \det(E_2 \bullet_g B) \\ &= \left((\det I_n)^{-1}\right)^2 \det E_1 \det E_2 \det B \\ &= (\det I_n)^{-1} \det(E_1 \bullet_g E_2) \det B \\ &= (\det I_n)^{-1} \det A \det B. \end{aligned}$$

For $A = E_1 \bullet_g \ldots \bullet_g E_k$, where E_i is elementary matrix $(1 \le i \le k)$, the proof can be done similarly. On the other hand, since every regular matrix can be written as the multiplication of elementary matrices, for every regular matrix A

$$\det(A \bullet_g B) = (\det I_n)^{-1} \det A \cdot \det B.$$

If A is a singular matrix, then $\det A = 0$ and $\det(A \bullet_g B) = 0$. Thus the proof is completed. $\square$

Corollary 2.2. *If A is a regular matrix then* $\det A \neq 0$.

Eigenvalues and eigenvectors play an important role in matrix theory because of its application in the areas of mathematics, physics and engineering. By this aim, we define the eigenvalues and eigenvectors of square $n \times n$ matrix A by generalized matrix multiplication.

Definition 2.8. Let $A \in \mathbb{R}_g^{n,n}$. An eigenvector of A is a nonzero vector x in $\mathbb{R}^n$ such that

$$A \bullet_g x = \lambda x$$

for some scalar λ. The scalar λ is called an eigenvalue of the matrix A, and we say that the vector x is an eigenvector belonging to the eigenvalue λ.

Example 2.2. Let $g(x, y) = 2x_1y_1 + 2x_2y_2 + x_3y_2 + x_2y_3 + 3x_3y_3$ be a scalar product on $\mathbb{R}^3$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \in \mathbb{R}_g^{3,3}.$$

Then eigenvalues of A are $\lambda_1 = 2, \lambda_2 = 3$ and $\lambda_3 = 5$. Some eigenvectors of A corresponding to λ_1, λ_2 and λ_3 are

$$u_1 = \begin{bmatrix} \frac{3}{16} \\ -\frac{5}{4} \\ 1 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

respectively.

Theorem 2.7. *The eigenvectors of a symmetric matrix $A \in \mathbb{R}_g^{n,n}$ corresponding to different eigenvalues are orthogonal to each other.*

Proof. For the eigenvectors x, y corresponding to two different eigenvalues λ, μ of the matrix A, we can say that $A \bullet_g x = \lambda x$ and $A \bullet_g y = \mu y$, so

$$y^T \bullet_g A \bullet_g x = \lambda y^T \bullet_g x = \lambda g(x, y). \tag{2.1}$$

But numbers are always their own transpose, so

$$\begin{aligned} y^T \bullet_g A \bullet_g x &= x^T \bullet_g A \bullet_g y \\ &= x^T \bullet_g \mu y \\ &= \mu \left(x^T \bullet_g y \right) \\ y^T \bullet_g A \bullet_g x &= \mu g(x, y). \qquad (2.2) \end{aligned}$$

From (2.1) and (2.2), we get

$$(\lambda - \mu)\, g(x, y) = 0.$$

So $\lambda = \mu$ or $g(x, y) = 0$, and it isn't the former, so x and y are orthogonal. $\square$

Example 2.3. Let $g(x, y) = 2x_1y_1 + 2x_2y_2 + x_3y_2 + x_2y_3 + 3x_3y_3$ and

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}_g^{3,3}.$$

A is a symmetric matrix. Then the eigenvalues of A are $\lambda_1 = \frac{\sqrt{5}+7}{2}, \lambda_2 = \frac{7-\sqrt{5}}{2}$ and $\lambda_3 = 0$. Some eigenvectors of A corresponding to λ_1, λ_2 and λ_3 are

$$u_1 = \begin{bmatrix} -\frac{1}{2}\sqrt{5}-\frac{1}{2} \\ -\frac{1}{2}\sqrt{5}-\frac{3}{2} \\ 1 \end{bmatrix} \quad , \quad u_2 = \begin{bmatrix} \frac{1}{2}\sqrt{5}-\frac{1}{2} \\ \frac{1}{2}\sqrt{5}-\frac{3}{2} \\ 1 \end{bmatrix} \quad , \quad u_3 = \begin{bmatrix} \frac{-5}{2} \\ 2 \\ 1 \end{bmatrix}$$

respectively. For $i \neq j$, we get

$$g(u_i, u_j) = u_i^T \bullet_g u_j = 0.$$

Then $\{u_1, u_2, u_3\}$ form an orthogonal basis of $\mathbb{R}_g^3$.

Throughout the rest of the paper, g will denote the scalar multiplication function $g(x,y) = \sum_{i=1}^{n} g_i x_i y_i$, for the convenience of readers, where $g_i \neq 0$ for $1 \leq i \leq n$. Hence, the unit matrix corresponding to the generalized matrix can be stated as

$$I_n = \begin{bmatrix} \frac{1}{g_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{g_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{g_n} \end{bmatrix}.$$

Definition 2.9. Let $A \in \mathbb{R}_g^{n,n}$. If $A^{-1} = A^T$ $\left(i.e. AA^T = A^T A = I_n\right)$ then A is said to be g-orthogonal matrix.

Theorem 2.8. *Let $A \in \mathbb{R}_g^{n,n}$. Then*

1. *A is g-orthogonal if and only if the row vectors of A form an orthogonal basis of $\mathbb{R}_g^n$ under the scalar product; and*
2. *A is g-orthogonal if and only if the column vectors of A form an orthogonal basis of $\mathbb{R}_g^n$ under the scalar product.*

Proof. We will only prove (1), since the proof of (2) is almost identical. Let $A_1, \ldots, A_n$ denote the row vectors of A. Then

$$A \bullet_g A^T = \begin{bmatrix} g(A_1, A_1) & \cdots & g(A_1, A_n) \\ \vdots & & \vdots \\ g(A_n, A_1) & \cdots & g(A_n, A_n) \end{bmatrix}.$$

It follows that $A \bullet_g A^T = I_n$ if and only if for every $i, j = 1, \ldots, n$

$$\langle A_i, A_j \rangle = \begin{cases} \frac{1}{g_i} & , \quad i = j \\ 0 & , \quad i \neq j \end{cases}.$$

Then $\{A_1, \ldots, A_n\}$ is an orthogonal basis of $\mathbb{R}_g^n$. $\square$

Definition 2.10. A matrix A is diagonalizable if there exists a nonsingular matrix P and a diagonal matrix D such that

$$D = P^{-1} \bullet_g A \bullet_g P.$$

Theorem 2.9. *Let all the eigenvalues of $A \in \mathbb{R}_g^{n,n}$ are real. Then A is diagonalizable if and only if it has n linearly independent eigenvectors.*

Proof. Let $x_1, \ldots, x_n$ be n linearly independent eigenvectors of A associated with the eigenvalues $\lambda_1, \ldots, \lambda_n$. That is,

$$A \bullet_g x_i = \lambda_i x_i \quad , \quad i = 1, \ldots, n \ .$$

Now, we denote $P = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$. Since the columns of P are linearly independent, P is invertible. Let D be $diag\left[\frac{\lambda_1}{g_1}, \ldots, \frac{\lambda_n}{g_n}\right]$. Then

$$\begin{aligned} A \bullet_g P &= A \bullet_g \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \bullet_g diag\left[\frac{\lambda_1}{g_1}, \ldots, \frac{\lambda_n}{g_n}\right] \\ &= P \bullet_g D. \end{aligned}$$

Since $A \bullet_g P = P \bullet_g D$, it follows that $D = P^{-1} \bullet_g A \bullet_g P$ which shows that A is diagonalizable.

To prove the other direction we assume that A is diagonalizable. Then there exists a nonsingular matrix P and a diagonal matrix $D = diag\left[\frac{\lambda_1}{g_1}, \ldots, \frac{\lambda_n}{g_n}\right]$ such that

$$D = P^{-1} \bullet_g A \bullet_g P.$$

If we multiply above equation with P from the left, we get

$$A \bullet_g P = P \bullet_g D$$

which implies

$$A \bullet_g v_i = \lambda_i v_i \quad , \quad i = 1, \ldots, n \tag{2.3}$$

where v_i are columns of P. The equations (2.3) show that $v_1, \ldots, v_n$ are eigenvectors of A corresponding to eigenvalues $t_1, \ldots, t_n$. Furthermore, since P is invertible, $\{v_1, \ldots, v_n\}$ are linearly independent. □

Example 2.4. Let $g(x, y) = -x_1 y_1 - x_2 y_2 + x_3 y_3$ and

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & -2 \end{bmatrix} \in \mathbb{R}_g^{3.3}.$$

The eigenvalues of A are $\lambda_1 = -2, \lambda_2 = 0, \lambda_3 = -1$ and eigenvectors corresponding these eigenvalues are

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad , \quad u_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad , \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

respectively. Therefore

$$P = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}] \quad and \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & -\frac{1}{2} \\ 1 & -2 & -2 \end{bmatrix}.$$

Finally

$$P^{-1} \bullet_g A \bullet_g P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

REFERENCES

1. A. A. Ergin: *On the 1-parameter Lorentzian motions.* Comm. Fac. Sci. Univ. Ankara Ser. A_1 Math. Statist. **40** (1991), 59–66.
2. H. Gündoğan and O. Keçilioğlu: *Lorentzian matrix multiplication and the motions on Lorentzian plane.* Glas. Mat. Ser. III **41(61)**, no. 2 (2006), 329–334.
3. H. Gündoğan and O. Keçilioğlu: *Pseudo Matrix Multiplication.* Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **66**, no. 2, (2017), 37–43.
4. S. Lang: *Linear Algebra.* Addison-Wesley Publishing Co., London, 1971.
5. B. O'Neill: *Semi-Riemannian Geometry With Applications to Relativity.* Academic Press, New York, 1983.

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