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CURVATURE TENSORS AND THE THIRD TYPE ALMOST GEODESIC MAPPINGS *

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Abstract. Changes of curvature tensors of a non-symmetric affine connection space under the third type almost geodesic mappings of both the first and the second type are given in this paper. These curvature tensors are firstly presented as functions of a curvature tensor of the corresponding associated space.

Keywords: Curvature tensors; affine connection space; geodesic mappings; affine connection.

1. Introduction

Many authors have given their own contribution to the mappings between affine connection spaces theory. Some of them include J. Mikeš [1, 4, 12], I. Hinterleitner [2, 3], S. M. Minčić [7, 9, 10], N. S. Sinjukov [11], M. S. Stanković [16, 17] and many others.

An affine connection on an N-dimensional manifold M is a mapping ∇ which maps any pair (X, Y) of vector fields to a vector field $\nabla_X Y$ such that

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z; \tag{1.1}$$

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y; \tag{1.2}$$

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z, \tag{1.3}$$

for any vector field X, Y, Z and differentiable functions f, g on M.

Definition 1.1. [4] We call (M, ∇) a manifold with an affine connection, or a manifold with a linear connection.

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In local coordinates with respect to a chart $(U, \varphi), \varphi = (x^1, \dots, x^N)$,

$$\nabla_i \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^h} = L^h_{ij} \frac{\partial}{\partial x^h}, \tag{1.4}$$

where the function $L^h_{ij} = L^h_{ij}(x)$ characterizing the affine connection ∇ are components of the affine connection ∇ relative to the chart under consideration. Manifolds with an affine connection characterized by coefficients $L^h_{ij}, L^h_{ij} \neq L^h_{ji}$, will be called *non-symmetric affine connection spaces* $\mathbb{GA}_N$.

We are particularly interested in non-symmetric affine connection spaces, i.e spaces with affine connection coefficients L^h_{ij} non-symmetric by indices i and j, in this paper. For this reason the following magnitudes are necessary:

$$L^h_{\underline{ij}} = \frac{1}{2}\left(L^h_{ij} + L^h_{ji}\right) \quad \text{and} \quad L^h_{\underset{\vee}{ij}} = \frac{1}{2}\left(L^h_{ij} + L^h_{ji}\right), \tag{1.5}$$

named symmetric and anti-symmetric part of the coefficient L^h_{ij}, respectively.

Let $X^{i_1 \dots i_A}_{j_1 \dots j_B}$ be a random indexed magnitude. Anti-symmetrization without division of it by indices $j_u, j_v, 1 \leq u < v \leq B$, is

$$X^{i_1 \dots i_A}_{j_1 \dots [j_u \dots j_v] \dots j_B} := X^{i_1 \dots i_A}_{j_1 \dots j_u \dots j_v \dots j_B} - X^{i_1 \dots i_A}_{j_1 \dots j_v \dots j_u \dots j_B}. \tag{1.6}$$

A symmetric affine connection space $\mathbb{A}_N$ is *an associated space* of a space $\mathbb{GA}_N$ with affine connection coefficients L^h_{ij} if its affine connection coefficients are equal to the symmetric part of the coefficients L^h_{ij}.

Let us recall some other terms necessary in this paper. Unlike a symmetric affine connection space, a non-symmetric affine connection space causes four types of covariant differentiation (see [6]) defined as

$$\begin{aligned} T^{i_1 i_2 \dots i_A}_{j_1 j_2 \dots j_B \underset{\substack{1\\2\\3\\4}}{|} k} = {} & T^{i_1 i_2 \dots i_A}_{j_1 j_2 \dots j_B, k} + \sum_{\alpha=1}^{A} L^{i_\alpha}_{\substack{pm\\mp\\pm\\mp}} T^{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_A}_{j_1 j_2 \dots j_B} \\ & - \sum_{\alpha=1}^{B} L^{p}_{\substack{j_\alpha m\\ m j_\alpha\\ m j_\alpha\\ j_\alpha m}} T^{i_1 \dots i_A}_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_B}. \end{aligned} \tag{1.7}$$

Let $\ell : I \to M, t \mapsto \ell(t) = x(t)$ ($I \subset \mathbb{R}$ is an open interval and $\ell \subset U \subset M, (U, \varphi)$ with $\varphi = (x^i)$, is a local chart) be a differentiable curve in an N-dimensional manifold with an affine connection $\mathbb{GA}_N$, and let $\lambda = \dot{x}$ denote the corresponding tangent vector field along ℓ. A vector field X along ℓ is said to be *parallel along* ℓ if X satisfies the condition

$$\nabla_t X = 0, \tag{1.8}$$

for any t.

In case $x_0 = \ell(t_0)$ and $x_1 = \ell(t_1)$ are points on a given curve $\ell = \ell(t)$, a vector X_1 from the tangent space $T_{x_1}M$ in x_1 is a result of the *parallel transport along* ℓ from the point x_0 to the point x_1 if along ℓ, there exists a parallel vector field $X(t)$ for which $X(t_0) = x_0$ and $X(t_1) = x_1$.

1.1. Curvature tensors of a non-symmetric affine connection space

There exists only one curvature tensor

$$R^h_{ijk} = L^h_{ij,k} - L^h_{ik,j} + L^\alpha_{\underline{ij}} L^h_{\underline{\alpha k}} - L^\alpha_{\underline{ik}} L^h_{\underline{\alpha j}}, \tag{1.9}$$

of the associated space $\mathbb{A}_N$ where "," denotes a partial derivative.

Four curvature tensors and eight derived ones [7] exist in a non-symmetric affine connection space. M. Lj. Zlatanović (see [20]) listed all curvature tensors of a space $\mathbb{GA}_N$ with affine connection coefficients L^h_{ij} as:

$$\underset{1}{R}{}^h_{ijk} = L^h_{ij,k} - L^h_{ik,j} + L^\alpha_{ij} L^h_{\alpha k} - L^\alpha_{ik} L^h_{\alpha j}; \tag{1.10}$$

$$\underset{2}{R}{}^h_{ijk} = L^h_{ji,k} - L^h_{ki,j} + L^\alpha_{ji} L^h_{k\alpha} - L^\alpha_{ki} L^h_{j\alpha}; \tag{1.11}$$

$$\underset{3}{R}{}^h_{ijk} = L^h_{ij,k} - L^h_{ki,j} + L^\alpha_{ij} L^h_{k\alpha} - L^\alpha_{ki} L^h_{\alpha j} + 2L^\alpha_{kj} L^h_{\underset{\vee}{\alpha i}}; \tag{1.12}$$

$$\underset{4}{R}{}^h_{ijk} = L^h_{ij,k} - L^h_{ki,j} + L^\alpha_{ij} L^h_{k\alpha} - L^\alpha_{ki} L^h_{\alpha j} + 2L^\alpha_{jk} L^h_{\underset{\vee}{\alpha i}}, \tag{1.13}$$

and

$$\underset{1}{\widetilde{R}}{}^h_{ijk} = \frac{1}{2}\left(L^h_{ij,k} - L^h_{ik,j} + L^h_{ji,k} - L^h_{ki,j} + L^\alpha_{ij}L^h_{k\alpha} + L^\alpha_{ji}L^h_{\alpha k} - L^\alpha_{ik}L^h_{j\alpha} - L^\alpha_{ki}L^h_{\alpha j}\right); \tag{1.14}$$

$$\underset{2}{\widetilde{R}}{}^h_{ijk} = \frac{1}{2}\left(L^h_{ij,k} - L^h_{ik,j} + L^h_{ji,k} - L^h_{ki,j} + L^\alpha_{ij}L^h_{\alpha k} + L^\alpha_{ji}L^h_{k\alpha} - L^\alpha_{ik}L^h_{j\alpha} - L^\alpha_{ki}L^h_{\alpha j}\right); \tag{1.15}$$

$$\underset{3}{\widetilde{R}}{}^h_{ijk} = \frac{1}{2}\left(L^h_{ij,k} - L^h_{ik,j} + L^h_{ji,k} - L^h_{ki,j} + L^\alpha_{ji}L^h_{\alpha k} + L^\alpha_{ij}L^h_{k\alpha} - L^\alpha_{ik}L^h_{\alpha j} - L^\alpha_{ki}L^h_{j\alpha}\right); \tag{1.16}$$

$$\begin{aligned}\underset{4}{\widetilde{R}}{}^h_{ijk} &= \frac{1}{3}\left(L^h_{ij,k} - L^h_{ik,j} + L^\alpha_{ji}L^h_{\alpha k} - L^\alpha_{ki}L^h_{\alpha j}\right)\\ &+ \frac{2}{3}\left(L^h_{ji,k} - L^h_{ki,j} + L^\alpha_{\underline{ij}}L^h_{k\alpha} - L^\alpha_{\underline{ki}}L^h_{j\alpha} + L^\alpha_{\underset{\vee}{kj}}L^h_{\underset{\vee}{\alpha i}}\right);\end{aligned} \tag{1.17}$$

$$\underset{5}{\widetilde{R}}{}^h_{ijk} = L^h_{ji,k} - L^h_{ki,j} + 2L^\alpha_{ij}L^h_{\underset{\vee}{\alpha k}} - L^\alpha_{ki}L^h_{j\alpha} + L^\alpha_{ji}L^h_{\alpha k}; \tag{1.18}$$

$$\underset{6}{\widetilde{R}}{}^h_{ijk} = L^h_{ji,k} - L^h_{ki,j} + L^\alpha_{ji}L^h_{k\alpha} - 2L^\alpha_{\underset{\vee}{ik}}L^h_{\alpha j} - L^\alpha_{ik}L^h_{j\alpha}; \tag{1.19}$$

$$\underset{7}{\widetilde{R}}{}^h_{ijk} = L^h_{ij,k} - L^h_{ik,j} + L^\alpha_{ij}L^h_{\alpha k} - 2L^\alpha_{\underset{\vee}{ik}}L^h_{j\alpha} - L^\alpha_{ki}L^h_{\alpha j}; \tag{1.20}$$

$$\underset{8}{\widetilde{R}}{}^h_{ijk} = L^h_{ij,k} - L^h_{ik,j} + 2L^\alpha_{ji}L^h_{\underset{\vee}{\alpha k}} - L^\alpha_{ik}L^h_{\alpha j} - L^\alpha_{ij}L^h_{k\alpha}. \tag{1.21}$$

1.2. Geodesic and almost geodesic mappings

Let us remember what geodesic and almost geodesic lines of a symmetric affine connection space are. Recall also what geodesic mappings between two such spaces are.

Definition 1.2. [4, 11] A curve ℓ in space $\mathbb{A}_N$ is geodesic when its tangent vector field remains in tangent distribution of ℓ during parallel transport along the curve.

Definition 1.3. [4, 11] Let $\mathbb{A}_N$ and $\overline{\mathbb{A}}_N$ be manifolds with a symmetric affine connection. A diffeomorphism $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is called geodesic mapping of $\mathbb{A}_N$ onto $\overline{\mathbb{A}}_N$ if it maps any geodesic curve in $\mathbb{A}_N$ onto a geodesic curve in $\overline{\mathbb{A}}_N$.

Trying to generalize the concept of a geodesic mapping for Riemannian and spaces of symmetric affine connection, N. S. Sinjukov introduced [11] the following terms:

A curve $l : x^h = x^h(t)$ is an almost geodesic line if its tangential vector $\lambda^h = dx^h/dt \neq 0$ satisfies the equations

$$\overline{\lambda}^h_{(2)} = \overline{a}(t)\lambda^h + \overline{b}(t)\overline{\lambda}^h_{(1)}, \quad \overline{\lambda}^h_{(1)} = \lambda^h_{\|p}\lambda^p, \quad \overline{\lambda}^h_{(2)} = \overline{\lambda}^h_{(1)\|p}\lambda^p, \tag{1.22}$$

where $\overline{a}(t)$ and $\overline{b}(t)$ are functions of a parameter t, and $\|$ denotes a covariant derivation with respect to the connection in $\overline{A}_N$.

Definition 1.4. [4, 11] A mapping f of a symmetric affine connection space $\mathbb{A}_N$ onto a space $\overline{\mathbb{A}}_N$ is called an *almost geodesic mapping* if any geodesic line of the space $\mathbb{A}_N$ is mapping into an almost geodesic line of the space $\mathbb{A}_N$.

Sinjukov (see [11]) singled out three types of almost geodesic mappings, π_1, π_2, π_3 for spaces without torsion ($L^h_{ij} - L^h_{ji} = 0$). Furthermore, he obtained curvature tensors R^h_{ijk} and $\overline{R}^h_{ijk}$ of symmetric affine connection spaces $\mathbb{A}_N$ and $\overline{\mathbb{A}}_N$ functionally connected with an almost geodesic mapping f of the third type satisfy the equation

$$\overline{R}^h_{ijk} = R^h_{ijk} + \psi_{ij}\delta^h_k - \psi_{ik}\delta^h_j + \sigma_{ijk}\varphi^h, \tag{1.23}$$

where

$$\psi_{ij} = \psi_{i;j} - \psi_i\psi_j - \sigma_{ij}(\nu + \varphi^\alpha\psi_\alpha); \tag{1.24}$$

$$\sigma_{ijk} = \sigma_{ij;k} - \sigma_{ik;j} + \psi_k\sigma_{ij} - \psi_j\sigma_{ik} + \sigma_{ij}\sigma_{kp}\varphi^p - \sigma_{ik}\sigma_{jp}\varphi^p. \tag{1.25}$$

M. Stanković [13, 14, 15] started an advancement of almost geodesic mappings theory into non-symmetric affine connection spaces (spaces with torsion) theory. The third type almost geodesic mappings of the first kind is determined by a condition for the function $\underset{1}{b}(x;\lambda)$:

$$\underset{1}{b} = \frac{\underset{1}{b}_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma}{\sigma_{\epsilon\delta}\lambda^\epsilon\lambda^\delta}, \tag{1.26}$$

where $\sigma_{\epsilon\delta}\lambda^\epsilon\lambda^\delta \neq 0$. The third type almost geodesic mappings of the second kind is determined by the condition

$$\underset{2}{b} = \frac{\underset{2}{b}_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma}{\sigma_{\epsilon\delta}\lambda^\epsilon\lambda^\delta}, \tag{1.27}$$

Let almost geodesic mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be the third type one of the θ-th kind, $\theta = 1, 2$, which inverse map f^{-1} is of the same type. This mapping has [18] *the property of reciprocity*. This special class of the third type almost mappings is denoted by $\underset{\theta}{\widetilde{\pi}}_3$.

The basic equations which give the characterization of mappings of the first kind $\underset{1}{\widetilde{\pi}}_3$ have the form

$$\overline{L}^h_{ij} = L^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i + \sigma_{ij}\varphi^h + \theta_j\delta^h_i - \theta_i\delta^h_j \tag{1.28}$$

$$\varphi^h_{\underset{1}{|}k} = \psi_k\varphi^h + \nu\delta^h_k, \tag{1.29}$$

where ψ_i, θ_i are covariant vectors, φ^h is a contravariant one, ν is an invariant and σ_{ij} is a symmetric tensor of the type $(0,2)$.

In the case of the class of almost geodesic mappings of the second kind $\underset{2}{\widetilde{\pi}}_3$, the equation (1.29) is changed with the following one

$$\varphi^h_{\underset{2}{|}k} = \psi_k \varphi^h + \nu \delta^h_k. \tag{1.30}$$

Stanković obtained affine connection coefficients L^h_{ij} of a space $\mathbb{GA}_N$ and $\overline{L}^h_{ij} = f(L^h_{ij})$ of a non-symmetric affine connection space $\mathbb{G}\overline{\mathbb{A}}_N$ in the case the third type almost geodesic mapping of the first kind $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$, satisfy the equation

$$\overline{L}^h_{ij} = L^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i + \sigma_{ij} \varphi^h + \theta_j \delta^h_i - \theta_i \delta^h_j. \tag{1.31}$$

1.3. Motivation

There exist five linearly independent curvature tensors into the set of twelve ones [8]. In an attempt to simplify the calculation processes, different authors take different 5-tuples of independent curvature tensors. This is what Minčić, Stanković and Velimirović [10] and Zlatanović [20] have done in their research.

Research has started into the changes of curvature tensors of a non-symmetric affine connection space $\mathbb{GA}_N$ under the third type almost geodesic mappings. Basic equations of the second type almost geodesic mappings of both the first and the second kind are some of the results presented in [19]. Changes of independent curvature tensors [20] $\underset{1}{K}{}^h_{ijk} = \underset{1}{R}{}^h_{ijk}$, $\underset{2}{K}{}^h_{ijk} = \underset{1}{\widetilde{R}}{}^h_{ijk}$, $\underset{3}{K}{}^h_{ijk} = \underset{3}{R}{}^h_{ijk}$, $\underset{4}{K}{}^h_{ijk} = \underset{3}{\widetilde{R}}{}^h_{ijk}$ and $\underset{5}{K}{}^h_{ijk} = \frac{1}{4}\left(3\underset{4}{\widetilde{R}}{}^h_{ijk} + \underset{1}{R}{}^h_{ijk}\right)$ are analyzed in [18].

The purpose of this paper is twofold. The first is an expression of curvature tensors of a space $\mathbb{GA}_N$ as linear functions of the corresponding curvature tensor R^h_{ijk} of the associated space $\mathbb{A}_N$. The second - and the main one - is a presentation of changes of all twelve curvature tensors of the space $\mathbb{GA}_N$ under the third type almost geodesic mappings of both the first and the second kind.

2. Change of curvature tensors

Before we present the results of our study, some predications should be presented. The first is a proposition which connects covariant derivatives of a vector φ^h with respect to connections of a non-symmetric space and the associated affine connection one. We should emphasize that there exist only two covariant derivatives of a vector φ^h with respect to a non-symmetric affine connection.

Proposition 2.1. [18] *Covariant derivative $\varphi^h_{;k}$ of a vector φ^h from the space $\mathbb{GA}_N$ based on the connection of the associated space $\mathbb{A}_N$ and on the $\varphi^h_{\underset{1}{|k}}$ is*

$$\varphi^h_{;k} = \nu\delta^h_k + \psi_k\varphi^h - L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha. \tag{2.1}$$

Covariant derivative $\varphi^h_{;k}$ of a vector φ^h from the space $\mathbb{GA}_N$ based on the connection of the associated space and on the $\varphi^h_{\underset{2}{|k}}$, is

$$\varphi^h_{;k} = \nu\delta^h_k + \psi_k\varphi^h + L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha. \tag{2.2}$$

Magnitudes ν, ψ_i and φ^i in the equations (2.1, 2.2) are an invariant, a covariant vector and a contravariant one, respectively. □

The following proposition analyzes the covariant derivative of affine connection coefficients under our aimed almost geodesic mappings.

Proposition 2.2. [18] *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be an almost geodesic mapping of the third type between spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$. Affine connection coefficients L^h_{ij} and $\overline{L}^h_{ij}$ of these spaces satisfy the equation*

$$\begin{aligned}\overline{L}^h_{ij;k} = L^h_{ij;k} + \sigma_{ij;k}\varphi^h + \sigma_{ij}\psi_k\varphi^h - \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha \\ + \psi_{i;k}\delta^h_j + \psi_{j;k}\delta^h_i + \sigma_{ij}\nu\delta^h_k + \theta_{j;k}\delta^h_i - \theta_{i;k}\delta^h_j,\end{aligned} \tag{2.3}$$

where ";" denotes a covariant derivative with regard to the associated space connection. □

The following propositions and lemma are necessary in the following research.

Proposition 2.3. [18] *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be an almost geodesic mapping of the third type. Affine connection coefficients L^h_{ij} and $\overline{L}^h_{ij}$ of these spaces satisfy the equation*

$$\overline{L}^\alpha_{\underset{\vee}{ij}}\overline{L}^h_{\underset{\vee}{\alpha k}} = L^\alpha_{\underset{\vee}{ij}}L^h_{\underset{\vee}{\alpha k}} + L^h_{\underset{\vee}{ij}}\theta_k + L^h_{\underset{\vee}{ik}}\theta_j + \theta_j\theta_k\delta^h_i - L^h_{\underset{\vee}{jk}}\theta_i - L^\alpha_{\underset{\vee}{ij}}\theta_\alpha\delta^h_k - \theta_i\theta_k\delta^h_j. \tag{2.4}$$

□

The next lemma is crucial for the main results of this paper.

Lemma 2.1. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be an almost geodesic mapping of the type $\underset{1}{\widetilde{\pi}}_3$. Riemann-Christoffel curvature tensor defined with respect to the connection coefficients $L^h_{\underline{ij}} = \frac{1}{2}\left(L^h_{ij} + L^h_{ji}\right)$*

$$R^h_{ijk} = L^h_{\underline{ij},k} - L^h_{\underline{ik},j} + L^\alpha_{\underline{ij}} L^h_{\underline{\alpha k}} - L^\alpha_{\underline{ik}} L^h_{\underline{\alpha j}} \tag{2.5}$$

of the associated space $\mathbb{A}_N$ and the corresponding one $\overline{R}^h_{ijk}$ of the space $\overline{\mathbb{A}}_N$ satisfy the equation

$$\overline{R}^h_{ijk} = R^h_{ijk} + \psi_{ij}\delta^h_k - \psi_{ik}\delta^h_j + \psi_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h - \sigma_{ij} L^h_{\underset{\vee}{pk}}\varphi^p + \sigma_{ik} L^h_{\underset{\vee}{pj}}\varphi^p. \tag{2.6}$$

In case f is the third class almost geodesic mapping of the second type, the equation (2.6) becomes

$$\overline{R}^h_{ijk} = R^h_{ijk} + \psi_{ij}\delta^h_k - \psi_{ik}\delta^h_j + \psi_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij} L^h_{\underset{\vee}{pk}}\varphi^p - \sigma_{ik} L^h_{\underset{\vee}{pj}}\varphi^p. \tag{2.7}$$

for ψ_{ij} and σ_{ijk} defined in the equations (1.24, 1.25). □

We can start with the presentation of our results now. Based on the equations (1.10 - 1.13), the following proposition holds:

Proposition 2.4. *Curvature tensors (1.10 - 1.13) of a non-symmetric affine connection space $\mathbb{GA}_N$ satisfy the equations*

$$\underset{1}{R}{}^h_{ijk} = R^h_{ijk} + L^h_{\underset{\vee}{ij};k} - L^h_{\underset{\vee}{ik};j} + L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} - L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}}; \tag{2.8}$$

$$\underset{2}{R}{}^h_{ijk} = R^h_{ijk} - L^h_{\underset{\vee}{ij};k} + L^h_{\underset{\vee}{ik};j} + L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} - L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}}; \tag{2.9}$$

$$\underset{3}{R}{}^h_{ijk} = R^h_{ijk} + L^h_{\underset{\vee}{ij};k} + L^h_{\underset{\vee}{ik};j} - L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} + L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}} - 2L^\alpha_{\underset{\vee}{jk}} L^h_{\underset{\vee}{\alpha i}}; \tag{2.10}$$

$$\underset{4}{R}{}^h_{ijk} = R^h_{ijk} + L^h_{\underset{\vee}{ij};k} + L^h_{\underset{\vee}{ik};j} - L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} + L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}} + 2L^\alpha_{\underset{\vee}{jk}} L^h_{\underset{\vee}{\alpha i}}, \tag{2.11}$$

where R^h_{ijk} is a curvature tensor of the associated space $\mathbb{A}_N$ and ";" denotes a covariant derivative with regard to the connection of the associated space. □

Derived curvature tensors $\underset{v}{\widetilde{R}}{}^h_{ijk}$, $v = 1, \dots, 8$, can be expressed as linear functions of curvature tensor R^h_{ijk} defined with (1.9).

Proposition 2.5. *Derived curvature tensors* $\underset{v}{\widetilde{R}}{}^h_{ijk}$, $v = 1, \ldots, 8$, *satisfy the equations*

$$\underset{1}{\widetilde{R}}{}^h_{ijk} = R^h_{ijk} - L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} + L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}}; \tag{2.12}$$

$$\underset{2}{\widetilde{R}}{}^h_{ijk} = R^h_{ijk} + L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} + L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}}; \tag{2.13}$$

$$\underset{3}{\widetilde{R}}{}^h_{ijk} = R^h_{ijk} - L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} - L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}}; \tag{2.14}$$

$$\underset{4}{\widetilde{R}}{}^h_{ijk} = R^h_{ijk} + \frac{1}{3}\left(-L^h_{\underset{\vee}{ij};k} + L^h_{\underset{\vee}{ik};j} + L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{k\alpha}} + 2L^\alpha_{\underset{\vee}{jk}} L^h_{\underset{\vee}{i\alpha}} + 3L^\alpha_{\underset{\vee}{ki}} L^h_{\underset{\vee}{\alpha j}}\right) \tag{2.15}$$

$$\underset{5}{\widetilde{R}}{}^h_{ijk} = R^h_{ijk} - L^h_{\underset{\vee}{ij};k} + L^h_{\underset{\vee}{ik};j} + L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} - L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}} + 4L^\alpha_{\underline{ij}} L^h_{\underset{\vee}{\alpha k}}; \tag{2.16}$$

$$\underset{6}{\widetilde{R}}{}^h_{ijk} = R^h_{ijk} - L^h_{\underset{\vee}{ij};k} + L^h_{\underset{\vee}{ik};j} + L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} - L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}} - 4L^\alpha_{\underset{\vee}{ik}} L^h_{\underline{j\alpha}}; \tag{2.17}$$

$$\underset{7}{\widetilde{R}}{}^h_{ijk} = R^h_{ijk} + L^h_{\underset{\vee}{ij};k} - L^h_{\underset{\vee}{ik};j} + L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} - 5L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}}; \tag{2.18}$$

$$\underset{8}{\widetilde{R}}{}^h_{ijk} = R^h_{ijk} + L^h_{\underset{\vee}{ij};k} - L^h_{\underset{\vee}{ik};j} + L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}} - L^\alpha_{\underset{\vee}{ik}} L^h_{\underset{\vee}{\alpha j}} - 2L^\alpha_{\underset{\vee}{ij}} L^h_{\alpha k} - 2L^\alpha_{\underline{ij}} L^h_{k\alpha}. \tag{2.19}$$

Proof. The equations (2.12, 2.14) hold directly from the definition of covariant derivative with regard to the associated space connection and these results are used in [18].

For this reason we are going to prove the equation about the curvature tensor $\underset{4}{\widetilde{R}}{}^h_{ijk}$ because it is the most complicated case in this proof. All other equations can be proved by using the analogous processes with respect to the equations (2.8-2.11).

If we use the facts that $L^h_{ij} = L^h_{\underline{ij}} + L^h_{\underset{\vee}{ij}}$ and $L^h_{\underline{ij}} = \frac{1}{2}\left(L^h_{ij} + L^h_{ji}\right)$, the bracketed expression in the definition of the fourth derived curvature tensor becomes

$$\begin{aligned} &L^h_{ij,k} - L^h_{ik,j} + 2L^h_{ji,k} - 2L^h_{ki,j} + L^\alpha_{ij} L^h_{k\alpha} - L^\alpha_{ik} L^h_{j\alpha} + L^\alpha_{ji} L^h_{\alpha k} - L^\alpha_{ki} L^h_{\alpha j} + 2L^\alpha_{\underset{\vee}{kj}} L^h_{\underset{\vee}{\alpha i}} = \\ &\underset{2}{R}{}^h_{ijk} + L^h_{ij,k} - L^h_{ik,j} + L^h_{ji,k} - L^h_{ki,j} + L^\alpha_{ij} L^h_{k\alpha} - L^\alpha_{ik} L^h_{j\alpha} + L^\alpha_{ji} L^h_{\alpha k} - L^\alpha_{ki} L^h_{\alpha j} + 2L^\alpha_{\underset{\vee}{kj}} L^h_{\underset{\vee}{\alpha i}}. \end{aligned}$$

We also have it that the following equation is valid:

$$L^\alpha_{ij} L^h_{k\alpha} + L^\alpha_{ji} L^h_{\alpha k} = (L^\alpha_{\underline{ij}} + L^\alpha_{\underset{\vee}{ij}})(L^h_{\underline{k\alpha}} + L^h_{\underset{\vee}{k\alpha}}) + (L^\alpha_{\underline{ji}} + L^\alpha_{\underset{\vee}{ji}})(L^h_{\underline{\alpha k}} + L^h_{\underset{\vee}{\alpha k}}) = 2L^\alpha_{\underline{ij}} L^h_{\underline{\alpha k}} - 2L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{\alpha k}}.$$

Finally, we obtain that the following holds:

$$\begin{aligned} &\underset{2}{R}{}^h_{ijk} + L^h_{ij,k} - L^h_{ik,j} + L^h_{ji,k} - L^h_{ki,j} + L^\alpha_{ij} L^h_{k\alpha} - L^\alpha_{ik} L^h_{j\alpha} + L^\alpha_{ji} L^h_{\alpha k} - L^\alpha_{ki} L^h_{\alpha j} + 2L^\alpha_{\underset{\vee}{kj}} L^h_{\underset{\vee}{\alpha i}} = \\ &2R^h_{ijk} + \underset{2}{R}{}^h_{ijk} + 2L^\alpha_{\underset{\vee}{ij}} L^h_{\underset{\vee}{k\alpha}} + 2L^\alpha_{\underset{\vee}{jk}} L^h_{\underset{\vee}{i\alpha}} + 2L^\alpha_{\underset{\vee}{ki}} L^h_{\underset{\vee}{\alpha j}}, \end{aligned}$$

which after interchange of the corresponding result for $\underset{2}{R}{}^h_{ijk}$ proves this equation. □

Based on the results (2.3, 2.12-2.19) together with the equations (2.6, 2.7), we are able to prove the following theorems.

Theorem 2.1. *Let* $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ *be the third type almost geodesic mapping of the first kind and let there be*

$$\underset{1}{\Psi}_{ij} = \psi_{ij} + \theta_{i;j} + \theta_i\theta_j; \tag{2.20}$$

$$\underset{2}{\Psi}_{ij} = \psi_{ij} - \theta_{i;j} + \theta_i\theta_j; \tag{2.21}$$

$$\underset{3}{\Psi}_{ij} = \psi_{ij} + \theta_{i;j} + \theta_i\theta_j; \tag{2.22}$$

$$\underset{4}{\Psi}_{ij} = \psi_{ij} + \theta_{i;j} - 3\theta_i\theta_j. \tag{2.23}$$

Curvature tensors $\underset{u}{R}{}^h_{ijk}$ *and* $\underset{u}{\overline{R}}{}^h_{ijk} = f(\underset{u}{R}{}^h_{ijk})$, $u = 1, \dots, 4$, *satisfy the equations*

$$\begin{aligned}\underset{1}{\overline{R}}{}^h_{ijk} &= \underset{1}{R}{}^h_{ijk} + \underset{1}{\Psi}_{ij}\delta^h_k - \underset{1}{\Psi}_{ik}\delta^h_j + \underset{1}{\Psi}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h\\ &- \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha + \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha - L^\alpha_{\underset{\vee}{ij}}\theta_\alpha\delta^h_k + L^\alpha_{\underset{\vee}{ik}}\theta_\alpha\delta^h_j - 2L^h_{\underset{\vee}{jk}}\theta_i;\end{aligned} \tag{2.24}$$

$$\begin{aligned}\underset{2}{\overline{R}}{}^h_{ijk} &= \underset{2}{R}{}^h_{ijk} + \underset{2}{\Psi}_{ij}\delta^h_k - \underset{2}{\Psi}_{ik}\delta^h_j + \underset{2}{\Psi}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h\\ &- \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha + \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha - L^\alpha_{\underset{\vee}{ij}}\theta_\alpha\delta^h_k + L^\alpha_{\underset{\vee}{ik}}\theta_\alpha\delta^h_j - 2L^h_{\underset{\vee}{jk}}\theta_i;\end{aligned} \tag{2.25}$$

$$\begin{aligned}\underset{3}{\overline{R}}{}^h_{ijk} &= \underset{3}{R}{}^h_{ijk} + \underset{3}{\Psi}_{ij}\delta^h_k - \underset{3}{\Psi}_{ik}\delta^h_j + \underset{3}{\Psi}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h\\ &- \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha + \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha - 2\theta_{i;j}\delta^h_k + 2L^\alpha_{\underset{\vee}{jk}}\theta_\alpha\delta^h_i - 2L^h_{\underset{\vee}{ik}}\theta_j + 2L^h_{\underset{\vee}{ij}}\theta_k;\end{aligned} \tag{2.26}$$

$$\begin{aligned}\underset{4}{\overline{R}}{}^h_{ijk} &= \underset{4}{R}{}^h_{ijk} + \underset{4}{\Psi}_{ij}\delta^h_k - \underset{4}{\Psi}_{ik}\delta^h_j + \underset{4}{\Psi}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h\\ &- \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha + \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha - 2\theta_{i;j}\delta^h_k - 2L^\alpha_{\underset{\vee}{jk}}\theta_\alpha\delta^h_i + 2L^h_{\underset{\vee}{ik}}\theta_j - 2L^h_{\underset{\vee}{ij}}\theta_k,\end{aligned} \tag{2.27}$$

for ψ_{ij} *and* σ_{ijk} *defined in the equations* (1.24, 1.25).

Proof. The equations (2.24, 2.26) are proved in [18]. The other two curvature tensors from this theorem satisfy the equations

$$\underset{2}{R}{}^h_{ijk} = \underset{1}{R}{}^h_{ijk} - 2\left(L^h_{\underset{\vee}{ij;k}} - L^h_{\underset{\vee}{ik;j}}\right) \quad \text{and} \quad \underset{4}{R}{}^h_{ijk} = \underset{3}{R}{}^h_{ijk} + 4L^\alpha_{\underset{\vee}{jk}}L^h_{\underset{\vee}{\alpha i}},$$

which, together with the equation (1.31), implies the validity of the equations (2.25, 2.27). □

Corollary 2.1. *Let* $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ *be the third type almost geodesic mapping of the* *second kind. Curvature tensors* $\underset{u}{R}{}^h_{ijk}$ *and* $\overline{\underset{u}{R}}{}^h_{ijk} = f(\underset{u}{R}{}^h_{ijk})$, $u = 1, \dots, 4$, *satisfy the equations*

$$\overline{\underset{1}{R}}{}^h_{ijk} = \underset{1}{R}{}^h_{ijk} + \underset{1}{\Psi}_{ij}\delta^h_k - \underset{1}{\Psi}_{ik}\delta^h_j + \underset{1}{\Psi}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha - L^\alpha_{\underset{\vee}{ij}}\theta_\alpha\delta^h_k + L^\alpha_{\underset{\vee}{ik}}\theta_\alpha\delta^h_j - 2L^h_{\underset{\vee}{jk}}\theta_i; \tag{2.28}$$

$$\overline{\underset{2}{R}}{}^h_{ijk} = \underset{2}{R}{}^h_{ijk} + \underset{2}{\Psi}_{ij}\delta^h_k - \underset{2}{\Psi}_{ik}\delta^h_j + \underset{2}{\Psi}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha - L^\alpha_{\underset{\vee}{ij}}\theta_\alpha\delta^h_k + L^\alpha_{\underset{\vee}{ik}}\theta_\alpha\delta^h_j - 2L^h_{\underset{\vee}{jk}}\theta_i; \tag{2.29}$$

$$\overline{\underset{3}{R}}{}^h_{ijk} = \underset{3}{R}{}^h_{ijk} + \underset{3}{\Psi}_{ij}\delta^h_k - \underset{3}{\Psi}_{ik}\delta^h_j + \underset{3}{\Psi}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha - 2\theta_{i;j}\delta^h_k + 2L^\alpha_{\underset{\vee}{jk}}\theta_\alpha\delta^h_i - 2L^h_{\underset{\vee}{ik}}\theta_j + 2L^h_{\underset{\vee}{ij}}\theta_k; \tag{2.30}$$

$$\overline{\underset{4}{R}}{}^h_{ijk} = \underset{4}{R}{}^h_{ijk} + \underset{4}{\Psi}_{ij}\delta^h_k - \underset{4}{\Psi}_{ik}\delta^h_j + \underset{4}{\Psi}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha - 2\theta_{i;j}\delta^h_k - 2L^\alpha_{\underset{\vee}{jk}}\theta_\alpha\delta^h_i + 2L^h_{\underset{\vee}{ik}}\theta_j - 2L^h_{\underset{\vee}{ij}}\theta_k, \tag{2.31}$$

for $\psi_{ij}, \sigma_{ijk}, \underset{u}{\Psi}_{ij}$, $u = 1, \dots, 4$, *defined as above.* □

The following theorem also holds, which is going to be proved.

Theorem 2.2. *Let* $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ *be the third type almost geodesic mapping of the second kind and let there be*

$$\underset{1}{\widetilde{\Psi}}_{ij} = \psi_{ij} - \theta_i\theta_j; \tag{2.32}$$

$$\underset{2}{\widetilde{\Psi}}_{ij} = \psi_{ij} - \theta_i\theta_j; \tag{2.33}$$

$$\underset{3}{\widetilde{\Psi}}_{ij} = \psi_{ij} + \theta_i\theta_j; \tag{2.34}$$

$$\underset{4}{\widetilde{\Psi}}_{ij} = \psi_{ij} - \frac{1}{3}\theta_{i;j} - \frac{5}{3}\theta_i\theta_j - \frac{1}{3}L^\alpha_{\underset{\vee}{ij}}\theta_\alpha; \tag{2.35}$$

$$\underset{5}{\widetilde{\Psi}}_{ij} = \psi_{ij} - \theta_{i;j} + \theta_i\theta_j - 4\psi_i\theta_j; \tag{2.36}$$

$$\underset{6}{\widetilde{\Psi}}_{ij} = \psi_{ij} - \theta_{i;j} + \theta_i\theta_j + 4\theta_i\psi_j; \tag{2.37}$$

$$\underset{7}{\widetilde{\Psi}}_{ij} = \psi_{ij} + \theta_{i;j} + \theta_i\theta_j; \tag{2.38}$$

$$\underset{8}{\widetilde{\Psi}}_{ij} = \psi_{ij} - \theta_i\theta_j. \tag{2.39}$$

Curvature tensors $\underset{v}{\widetilde{R}}{}^h_{ijk}$ *and* $\widetilde{\overline{\underset{v}{R}}}{}^h_{ijk} = f(\underset{v}{\widetilde{R}}{}^h_{ijk})$, $v = 1, \dots, 8$, *satisfy the equations*

$$\begin{aligned}\underset{1}{\widetilde{\overline{R}}}{}^{h}_{ijk} &= \underset{1}{\widetilde{R}}{}^{h}_{ijk} + \underset{1}{\widetilde{\Psi}}{}_{ij}\delta^{h}_{k} - \underset{1}{\widetilde{\Psi}}{}_{ik}\delta^{h}_{j} + \underset{1}{\widetilde{\Psi}}{}_{[jk]}\delta^{h}_{i} + \sigma_{ijk}\varphi^{h} - \sigma_{ij}\underset{\vee}{L^{h}_{\alpha k}}\varphi^{\alpha} + \sigma_{ik}\underset{\vee}{L^{h}_{\alpha j}}\varphi^{\alpha}\\ &+ \underset{\vee}{L^{\alpha}_{ij}}\theta_{\alpha}\delta^{h}_{k} - \underset{\vee}{L^{\alpha}_{ik}}\theta_{\alpha}\delta^{h}_{j} + 2\underset{\vee}{L^{h}_{jk}}\theta_{i};\end{aligned}\tag{2.40}$$

$$\begin{aligned}\underset{2}{\widetilde{\overline{R}}}{}^{h}_{ijk} &= \underset{2}{\widetilde{R}}{}^{h}_{ijk} + \underset{2}{\widetilde{\Psi}}{}_{ij}\delta^{h}_{k} - \underset{2}{\widetilde{\Psi}}{}_{ik}\delta^{h}_{j} + \underset{2}{\widetilde{\Psi}}{}_{[jk]}\delta^{h}_{i} + \sigma_{ijk}\varphi^{h} - \sigma_{ij}\underset{\vee}{L^{h}_{\alpha k}}\varphi^{\alpha} + \sigma_{ik}\underset{\vee}{L^{h}_{\alpha j}}\varphi^{\alpha}\\ &- \underset{\vee}{L^{\alpha}_{ij}}\theta_{\alpha}\delta^{h}_{k} - \left(\underset{\vee}{L^{\alpha}_{ik}}\theta_{\alpha} + 2\theta_{i}\theta_{k}\right)\delta^{h}_{j} + 2\theta_{j}\theta_{k}\delta^{h}_{i} + 2\underset{\vee}{L^{h}_{ij}}\theta_{k} + 2\underset{\vee}{L^{h}_{ik}}\theta_{j};\end{aligned}\tag{2.41}$$

$$\begin{aligned}\underset{3}{\widetilde{\overline{R}}}{}^{h}_{ijk} &= \underset{3}{\widetilde{R}}{}^{h}_{ijk} + \underset{3}{\widetilde{\Psi}}{}_{ij}\delta^{h}_{k} - \underset{3}{\widetilde{\Psi}}{}_{ik}\delta^{h}_{j} + \underset{3}{\widetilde{\Psi}}{}_{[jk]}\delta^{h}_{i} + \sigma_{ijk}\varphi^{h} - \sigma_{ij}\underset{\vee}{L^{h}_{\alpha k}}\varphi^{\alpha} + \sigma_{ik}\underset{\vee}{L^{h}_{\alpha j}}\varphi^{\alpha}\\ &+ \underset{\vee}{L^{\alpha}_{ij}}\theta_{\alpha}\delta^{h}_{k} + 2\left(\underset{\vee}{L^{\alpha}_{ik}}\theta_{\alpha} + \theta_{i}\theta_{k}\right)\delta^{h}_{j} - 2\theta_{j}\theta_{k}\delta^{h}_{i} - 2\underset{\vee}{L^{h}_{ij}}\theta_{k} - 2\underset{\vee}{L^{h}_{ik}}\theta_{j};\end{aligned}\tag{2.42}$$

$$\begin{aligned}\underset{4}{\widetilde{\overline{R}}}{}^{h}_{ijk} &= \underset{4}{\widetilde{R}}{}^{h}_{ijk} + \underset{4}{\widetilde{\Psi}}{}_{ij}\delta^{h}_{k} - \underset{4}{\widetilde{\Psi}}{}_{ik}\delta^{h}_{j} + \underset{4}{\widetilde{\Psi}}{}_{[jk]}\delta^{h}_{i} + \sigma_{ijk}\varphi^{h} - \sigma_{ij}\underset{\vee}{L^{h}_{\alpha k}}\varphi^{\alpha} + \sigma_{ik}\underset{\vee}{L^{h}_{\alpha j}}\varphi^{\alpha}\\ &- \frac{2}{3}\left(5\underset{\vee}{L^{\alpha}_{ik}}\theta_{\alpha} + \theta_{i}\theta_{k}\right)\delta^{h}_{j} + \frac{4}{3}\theta_{j}\theta_{k}\delta^{h}_{i} + \frac{2}{3}\underset{\vee}{L^{h}_{ij}}\theta_{k} + \frac{4}{3}\underset{\vee}{L^{h}_{jk}}\theta_{i} + 2\underset{\vee}{L^{h}_{ik}}\theta_{j};\end{aligned}\tag{2.43}$$

$$\begin{aligned}\underset{5}{\widetilde{\overline{R}}}{}^{h}_{ijk} &= \underset{5}{\widetilde{R}}{}^{h}_{ijk} + \underset{5}{\widetilde{\Psi}}{}_{ij}\delta^{h}_{k} - \underset{5}{\widetilde{\Psi}}{}_{ik}\delta^{h}_{j} + \underset{5}{\widetilde{\Psi}}{}_{[jk]}\delta^{h}_{i} + \sigma_{ijk}\varphi^{h} - \sigma_{ij}\underset{\vee}{L^{h}_{\alpha k}}\varphi^{\alpha} + \sigma_{ik}\underset{\vee}{L^{h}_{\alpha j}}\varphi^{\alpha}\\ &- \left(\underset{\vee}{L^{\alpha}_{ij}}\theta_{\alpha} + 4\psi_{j}\theta_{i} + L^{\alpha}_{\underline{ij}}\theta_{\alpha} + 4\sigma_{ij}\varphi^{\alpha}\theta_{\alpha}\right)\delta^{h}_{k} + \underset{\vee}{L^{\alpha}_{ik}}\theta_{\alpha}\delta^{h}_{j} + \left(8\psi_{j}\theta_{k} - 4\psi_{k}\theta_{j}\right)\delta^{h}_{i}\\ &- 2\underset{\vee}{L^{h}_{jk}}\left(\theta_{i} + 2\psi_{i}\right) + 4\underset{\vee}{L^{h}_{ik}}\psi_{j} + 4L^{h}_{\underline{ij}}\theta_{k} + 4\sigma_{ij}\theta_{k}\varphi^{h} + 4\underset{\vee}{L^{h}_{\alpha k}}\sigma_{ij}\varphi^{\alpha};\end{aligned}\tag{2.44}$$

$$\begin{aligned}\underset{6}{\widetilde{\overline{R}}}{}^{h}_{ijk} &= \underset{6}{\widetilde{R}}{}^{h}_{ijk} + \underset{6}{\widetilde{\Psi}}{}_{ij}\delta^{h}_{k} - \underset{6}{\widetilde{\Psi}}{}_{ik}\delta^{h}_{j} + \underset{6}{\widetilde{\Psi}}{}_{[jk]}\delta^{h}_{i} + \sigma_{ijk}\varphi^{h} - \sigma_{ij}\underset{\vee}{L^{h}_{\alpha k}}\varphi^{\alpha} + \sigma_{ik}\underset{\vee}{L^{h}_{\alpha j}}\varphi^{\alpha}\\ &- \underset{\vee}{L^{\alpha}_{ij}}\theta_{\alpha}\delta^{h}_{k} + \left(\underset{\vee}{L^{\alpha}_{ik}}\theta_{\alpha} + 4\theta_{i}\psi_{k} - 4\underset{\vee}{L^{\alpha}_{ik}}\psi_{\alpha} - 4\psi_{i}\theta_{k} + 4\theta_{i}\psi_{k}\right)\delta^{h}_{j}\\ &- \left(\theta_{j}\psi_{k} - \theta_{k}\psi_{j} - 4\psi_{j}\theta_{k}\right)\delta^{h}_{i} - 2\underset{\vee}{L^{h}_{jk}}\theta_{i} - 4\underset{\vee}{L^{h}_{ik}}\theta_{j} - 4L^{h}_{\underline{ij}}\theta_{k} + 4L^{h}_{\underline{jk}}\theta_{i}\\ &- 4\underset{\vee}{L^{\alpha}_{ik}}\sigma_{j\alpha}\varphi^{h} - 4\sigma_{ij}\theta_{k}\varphi^{h} + 4\theta_{i}\sigma_{jk}\varphi^{h};\end{aligned}\tag{2.45}$$

$$\begin{aligned}\underset{7}{\widetilde{\overline{R}}}{}^{h}_{ijk} &= \underset{7}{\widetilde{R}}{}^{h}_{ijk} + \underset{7}{\widetilde{\Psi}}{}_{ij}\delta^{h}_{k} - \underset{7}{\widetilde{\Psi}}{}_{ik}\delta^{h}_{j} + \underset{7}{\widetilde{\Psi}}{}_{[jk]}\delta^{h}_{i} + \sigma_{ijk}\varphi^{h} - \sigma_{ij}\underset{\vee}{L^{h}_{\alpha k}}\varphi^{\alpha} + \sigma_{ik}\underset{\vee}{L^{h}_{\alpha j}}\varphi^{\alpha}\\ &- \left(\underset{\vee}{L^{\alpha}_{ij}}\theta_{\alpha} - \theta_{i}\theta_{j}\right)\delta^{h}_{k} + 5\underset{\vee}{L^{\alpha}_{ik}}\theta_{\alpha}\delta^{h}_{j} - 4\theta_{j}\theta_{k}\delta^{h}_{i} - 6\underset{\vee}{L^{h}_{jk}}\theta_{i} - 4\underset{\vee}{L^{h}_{ij}}\theta_{k} - 4\underset{\vee}{L^{h}_{ik}}\theta_{j};\end{aligned}\tag{2.46}$$

$$\begin{aligned}\underset{8}{\widetilde{\overline{R}}}{}^{h}_{ijk} &= \underset{8}{\widetilde{R}}{}^{h}_{ijk} + \underset{8}{\widetilde{\Psi}}{}_{ij}\delta^{h}_{k} - \underset{8}{\widetilde{\Psi}}{}_{ik}\delta^{h}_{j} + \underset{8}{\widetilde{\Psi}}{}_{[jk]}\delta^{h}_{i} + \sigma_{ijk}\varphi^{h} - \sigma_{ij}\underset{\vee}{L^{h}_{\alpha k}}\varphi^{\alpha} + \sigma_{ik}\underset{\vee}{L^{h}_{\alpha j}}\varphi^{\alpha}\\ &- 2\left(L^{\alpha}_{ij}\psi_{\alpha} + L^{\alpha}_{ji}\theta_{\alpha} + 2\psi_{i}\psi_{j} + 2\psi_{i}\theta_{j} + \sigma_{ij}\varphi^{\alpha}\psi_{\alpha} + \sigma_{ij}\varphi^{\alpha}\theta_{\alpha} + \frac{1}{2}\underset{\vee}{L^{\alpha}_{ij}}\theta_{\alpha}\right)\delta^{h}_{k}\\ &- 2\left(\psi_{i}\psi_{k} + \psi_{i}\theta_{k} - \theta_{i}\psi_{k} - \theta_{i}\theta_{k} - \frac{1}{2}\underset{\vee}{L^{\alpha}_{ik}}\theta_{\alpha}\right)\delta^{h}_{j}\\ &- 2\left(\psi_{j}\psi_{k} - \psi_{j}\theta_{k} + \theta_{j}\psi_{k} + \theta_{j}\theta_{k}\right)\delta^{h}_{i} + 2\left(\theta_{i}\sigma_{jk} + \sigma_{ij}\theta_{k}\right)\varphi^{h}\\ &- 2\left(L^{\alpha}_{ij}\sigma_{k\alpha} + \psi_{i}\sigma_{jk} + \sigma_{ik}\psi_{j} + \sigma_{ik}\theta_{j} + \sigma_{ij}\psi_{k} + \sigma_{ij}\sigma_{k\alpha}\varphi^{\alpha}\right)\varphi^{h}\\ &- 2\underset{\vee}{L^{h}_{jk}}\theta_{i} - 4\underset{\vee}{L^{h}_{ij}}\psi_{k} - 2L^{h}_{kj}\psi_{i} - 2L^{h}_{ik}\theta_{j} - 2L^{h}_{k\alpha}\sigma_{ij}\varphi^{\alpha} + 2L^{h}_{jk}\theta_{i},\end{aligned}\tag{2.47}$$

for ψ_{ij} *and* σ_{ijk} *defined in the equations* (1.24, 1.25).

Proof. The equations (2.40, 2.42) are proved in [18]. The equation (2.41) holds directly from the

$$\underset{2}{\widetilde{R}}{}^{h}_{ijk} = \underset{1}{\widetilde{R}}{}^{h}_{ijk} + 2L^{\alpha}_{\underset{\vee}{ij}}L^{h}_{\underset{\vee}{\alpha k}}.$$

The equation (2.43) is a direct corollary of the following one:

$$\begin{aligned}
\overline{L}^{\alpha}_{\underset{\vee}{ij}}\overline{L}^{h}_{\underset{\vee}{k\alpha}} \quad &+ \quad 2\overline{L}^{\alpha}_{\underset{\vee}{jk}}\overline{L}^{h}_{\underset{\vee}{i\alpha}} + 3\overline{L}^{\alpha}_{\underset{\vee}{ki}}\overline{L}^{h}_{\underset{\vee}{\alpha j}} = L^{\alpha}_{\underset{\vee}{ij}}L^{h}_{\underset{\vee}{k\alpha}} + 2L^{\alpha}_{\underset{\vee}{jk}}L^{h}_{\underset{\vee}{i\alpha}} + 3L^{\alpha}_{\underset{\vee}{ki}}L^{h}_{\underset{\vee}{\alpha j}} \\
&+ \quad \left(-L^{\alpha}_{\underset{\vee}{ij}}\theta_{\alpha} - 5\theta_i\theta_j\right)\delta^h_k - \left(3L^{\alpha}_{\underset{\vee}{ik}}\theta_{\alpha} - \theta_i\theta_k\right)\delta^h_j + \left(-2L^{\alpha}_{\underset{\vee}{jk}}\theta_{\alpha} + 4\theta_j\theta_k\right)\delta^h_i \\
&+ \quad 2L^{h}_{\underset{\vee}{ij}}\theta_k + 4L^{h}_{\underset{\vee}{jk}}\theta_i + 6L^{h}_{\underset{\vee}{ik}}\theta_j.
\end{aligned}$$

We can see that the following holds:

$$\underset{5}{\widetilde{R}}{}^{h}_{ijk} = \underset{2}{R}{}^{h}_{ijk} + 4L^{\alpha}_{\underline{ij}}L^{h}_{\underset{\vee}{\alpha k}}.$$

The following equation is also valid:

$$\begin{aligned}
\overline{L}^{\alpha}_{\underline{ij}}\overline{L}^{h}_{\underset{\vee}{\alpha k}} \quad &= \quad L^{\alpha}_{\underline{ij}}L^{h}_{\underset{\vee}{\alpha k}} + L^{h}_{\underset{\vee}{jk}}\psi_i + L^{h}_{\underset{\vee}{ik}}\psi_j + L^{h}_{\underset{\vee}{\alpha k}}\sigma_{ij}\varphi^{\alpha} + L^{h}_{\underline{ij}}\theta_k + \psi_i\theta_k\delta^h_j + \psi_j\theta_k\delta^h_i \\
&+ \quad \sigma_{ij}\theta_k\varphi^h - L^{\alpha}_{\underline{ij}}\theta_{\alpha}\delta^h_k - \psi_i\theta_j\delta^h_k - \psi_j\theta_i\delta^h_k - \sigma_{ij}\varphi^{\alpha}\theta_{\alpha}\delta^h_k,
\end{aligned}$$

which proves the equation about the fifth derived curvature tensor.

Analogously, from the equation

$$\underset{6}{\widetilde{R}}{}^{h}_{ijk} = \underset{2}{\widetilde{R}}{}^{h}_{ijk} - 4L^{\alpha}_{\underset{\vee}{ik}}L^{h}_{\underline{j\alpha}},$$

we obtain that the equation (2.45) is valid.

It holds that

$$\underset{7}{\widetilde{R}}{}^{h}_{ijk} = \underset{1}{R}{}^{h}_{ijk} - 4L^{\alpha}_{\underset{\vee}{ik}}L^{h}_{\underset{\vee}{\alpha j}}$$

Based on the first of the results presented in Theorem 2.1 and Proposition 2.3, we obtain that the equation (2.46) is correct.

Finally, we have that the following equation holds:

$$\underset{8}{\widetilde{R}}{}^{h}_{ijk} = \underset{1}{R}{}^{h}_{ijk} - 2L^{\alpha}_{\underset{\vee}{ij}}L^{h}_{\alpha k} - 2L^{\alpha}_{\underline{ij}}L^{h}_{k\alpha}.$$

Furthermore, it is also valid

$$\begin{aligned}
\overline{L}^{\alpha}_{\underset{\vee}{ij}}\overline{L}^{h}_{\alpha k} \quad &+ \quad \overline{L}^{\alpha}_{\underline{ij}}\overline{L}^{h}_{k\alpha} = L^{\alpha}_{\underset{\vee}{ij}}L^{h}_{\alpha k} + L^{\alpha}_{\underline{ij}}L^{h}_{k\alpha} \\
&+ \left(L^{\alpha}_{ij}\psi_{\alpha} + L^{\alpha}_{ji}\theta_{\alpha} + 2\psi_i\psi_j + 2\psi_i\theta_j + \sigma_{ij}\varphi^{\alpha}\psi_{\alpha} + \sigma_{ij}\varphi^{\alpha}\theta_{\alpha}\right)\delta^h_k \\
&+ \left(\psi_i\psi_k + \psi_i\theta_k - \theta_i\psi_k - \theta_i\theta_k\right)\delta^h_j + \left(\psi_j\psi_k - \psi_j\theta_k + \theta_j\psi_k + \theta_j\theta_k\right)\delta^h_i \\
&+ \left(L^{\alpha}_{ij}\sigma_{k\alpha} + \psi_i\sigma_{jk} + \sigma_{ik}\psi_j + \sigma_{ik}\theta_j + \sigma_{ij}\psi_k - \theta_i\sigma_{jk} + \sigma_{ij}\sigma_{k\alpha}\varphi^{\alpha} - \sigma_{ij}\theta_k\right)\varphi^h \\
&+ L^h_{ij}\psi_k - L^h_{ji}\psi_k + L^h_{kj}\psi_i + L^h_{ik}\theta_j + L^h_{k\alpha}\sigma_{ij}\varphi^{\alpha} - L^h_{jk}\theta_i,
\end{aligned}$$

which proves the equation (2.47). □

Corollary 2.2. *Let $f : \mathbb{GA}_N \to \overline{\mathbb{GA}}_N$ be the third type almost geodesic mapping of the second kind. Curvature tensors $\underset{u}{R}{}^h_{ijk}$ and $\underset{u}{\overline{R}}{}^h_{ijk} = f(\underset{u}{R}{}^h_{ijk})$, $u = 1, \dots, 4$, satisfy the equations*

$$\begin{aligned}
\underset{1}{\widetilde{\overline{R}}}{}^h_{ijk} &= \underset{1}{\widetilde{R}}{}^h_{ijk} + \underset{1}{\widetilde{\Psi}}_{ij}\delta^h_k - \underset{1}{\widetilde{\Psi}}_{ik}\delta^h_j + \underset{1}{\widetilde{\Psi}}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^{\alpha} - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^{\alpha} \\
&+ L^{\alpha}_{\underset{\vee}{ij}}\theta_{\alpha}\delta^h_k - L^{\alpha}_{\underset{\vee}{ik}}\theta_{\alpha}\delta^h_j + 2L^h_{\underset{\vee}{jk}}\theta_i;
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
\underset{2}{\widetilde{\overline{R}}}{}^h_{ijk} &= \underset{2}{\widetilde{R}}{}^h_{ijk} + \underset{2}{\widetilde{\Psi}}_{ij}\delta^h_k - \underset{2}{\widetilde{\Psi}}_{ik}\delta^h_j + \underset{2}{\widetilde{\Psi}}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^{\alpha} - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^{\alpha} \\
&- L^{\alpha}_{\underset{\vee}{ij}}\theta_{\alpha}\delta^h_k - \left(L^{\alpha}_{\underset{\vee}{ik}}\theta_{\alpha} + 2\theta_i\theta_k\right)\delta^h_j + 2\theta_j\theta_k\delta^h_i + 2L^h_{\underset{\vee}{ij}}\theta_k + 2L^h_{\underset{\vee}{ik}}\theta_j;
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
\underset{3}{\widetilde{\overline{R}}}{}^h_{ijk} &= \underset{3}{\widetilde{R}}{}^h_{ijk} + \underset{3}{\widetilde{\Psi}}_{ij}\delta^h_k - \underset{3}{\widetilde{\Psi}}_{ik}\delta^h_j + \underset{3}{\widetilde{\Psi}}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^{\alpha} - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^{\alpha} \\
&+ L^{\alpha}_{\underset{\vee}{ij}}\theta_{\alpha}\delta^h_k + 2\left(L^{\alpha}_{\underset{\vee}{ik}}\theta_{\alpha} + \theta_i\theta_k\right)\delta^h_j - 2\theta_j\theta_k\delta^h_i - 2L^h_{\underset{\vee}{ij}}\theta_k - 2L^h_{\underset{\vee}{ik}}\theta_j;
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
\underset{4}{\widetilde{\overline{R}}}{}^h_{ijk} &= \underset{4}{\widetilde{R}}{}^h_{ijk} + \underset{4}{\widetilde{\Psi}}_{ij}\delta^h_k - \underset{4}{\widetilde{\Psi}}_{ik}\delta^h_j + \underset{4}{\widetilde{\Psi}}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^{\alpha} - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^{\alpha} \\
&- \frac{2}{3}\left(5L^{\alpha}_{\underset{\vee}{ik}}\theta_{\alpha} + \theta_i\theta_k\right)\delta^h_j + \frac{4}{3}\theta_j\theta_k\delta^h_i + \frac{2}{3}L^h_{\underset{\vee}{ij}}\theta_k + \frac{4}{3}L^h_{\underset{\vee}{jk}}\theta_i + 2L^h_{\underset{\vee}{ik}}\theta_j;
\end{aligned} \tag{2.51}$$

$$\begin{aligned}
\underset{5}{\widetilde{\overline{R}}}{}^h_{ijk} &= \underset{5}{\widetilde{R}}{}^h_{ijk} + \underset{5}{\widetilde{\Psi}}_{ij}\delta^h_k - \underset{5}{\widetilde{\Psi}}_{ik}\delta^h_j + \underset{5}{\widetilde{\Psi}}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^{\alpha} - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^{\alpha} \\
&- \left(L^{\alpha}_{\underset{\vee}{ij}}\theta_{\alpha} + 4\psi_j\theta_i + L^{\alpha}_{\underline{ij}}\theta_{\alpha} + 4\sigma_{ij}\varphi^{\alpha}\theta_{\alpha}\right)\delta^h_k + L^{\alpha}_{\underset{\vee}{ik}}\theta_{\alpha}\delta^h_j + \left(8\psi_j\theta_k - 4\psi_k\theta_j\right)\delta^h_i \\
&- 2L^h_{\underset{\vee}{jk}}\left(\theta_i + 2\psi_i\right) + 4L^h_{\underset{\vee}{ik}}\psi_j + 4L^h_{\underline{ij}}\theta_k + 4\sigma_{ij}\theta_k\varphi^h + 4L^h_{\underset{\vee}{\alpha k}}\sigma_{ij}\varphi^{\alpha};
\end{aligned} \tag{2.52}$$

$$\begin{aligned}
\underset{6}{\widetilde{\overline{R}}}{}^h_{ijk} &= \underset{6}{\widetilde{R}}{}^h_{ijk} + \underset{6}{\widetilde{\Psi}}_{ij}\delta^h_k - \underset{6}{\widetilde{\Psi}}_{ik}\delta^h_j + \underset{6}{\widetilde{\Psi}}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^{\alpha} - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^{\alpha} \\
&- L^{\alpha}_{\underset{\vee}{ij}}\theta_{\alpha}\delta^h_k + \left(L^{\alpha}_{\underset{\vee}{ik}}\theta_{\alpha} + 4\theta_i\psi_k - 4L^{\alpha}_{\underset{\vee}{ik}}\psi_{\alpha} - 4\psi_i\theta_k + 4\theta_i\psi_k\right)\delta^h_j \\
&- \left(\theta_j\psi_k - \theta_k\psi_j - 4\psi_j\theta_k\right)\delta^h_i - 2L^h_{\underset{\vee}{jk}}\theta_i - 4L^h_{\underset{\vee}{ik}}\theta_j - 4L^h_{\underline{ij}}\theta_k + 4L^h_{\underline{jk}}\theta_i \\
&- 4L^{\alpha}_{\underset{\vee}{ik}}\sigma_{j\alpha}\varphi^h - 4\sigma_{ij}\theta_k\varphi^h + 4\theta_i\sigma_{jk}\varphi^h;
\end{aligned} \tag{2.53}$$

$$\begin{aligned}
\underset{7}{\widetilde{\overline{R}}}{}^h_{ijk} &= \underset{7}{\widetilde{R}}{}^h_{ijk} + \underset{7}{\widetilde{\Psi}}_{ij}\delta^h_k - \underset{7}{\widetilde{\Psi}}_{ik}\delta^h_j + \underset{7}{\widetilde{\Psi}}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^{\alpha} - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^{\alpha} \\
&- \left(L^{\alpha}_{\underset{\vee}{ij}}\theta_{\alpha} - \theta_i\theta_j\right)\delta^h_k + 5L^{\alpha}_{\underset{\vee}{ik}}\theta_{\alpha}\delta^h_j - 4\theta_j\theta_k\delta^h_i - 6L^h_{\underset{\vee}{jk}}\theta_i - 4L^h_{\underset{\vee}{ij}}\theta_k - 4L^h_{\underset{\vee}{ik}}\theta_j;
\end{aligned} \tag{2.54}$$

$$\begin{aligned}
\widetilde{\overline{R}}{}^{h}_{\underset{8}{}ijk} &= \widetilde{R}{}^{h}_{\underset{8}{}ijk} + \underset{8}{\widetilde{\Psi}}_{ij}\delta^h_k - \underset{8}{\widetilde{\Psi}}_{ik}\delta^h_j + \underset{8}{\widetilde{\Psi}}_{[jk]}\delta^h_i + \sigma_{ijk}\varphi^h + \sigma_{ij}L^h_{\underset{\vee}{\alpha k}}\varphi^\alpha - \sigma_{ik}L^h_{\underset{\vee}{\alpha j}}\varphi^\alpha \\
&- 2\Big(L^\alpha_{ij}\psi_\alpha + L^\alpha_{ji}\theta_\alpha + 2\psi_i\psi_j + 2\psi_i\theta_j + \sigma_{ij}\varphi^\alpha\psi_\alpha + \sigma_{ij}\varphi^\alpha\theta_\alpha + \frac{1}{2}L^\alpha_{\underset{\vee}{ij}}\theta_\alpha\Big)\delta^h_k \\
&- 2\Big(\psi_i\psi_k + \psi_i\theta_k - \theta_i\psi_k - \theta_i\theta_k - \frac{1}{2}L^\alpha_{\underset{\vee}{ik}}\theta_\alpha\Big)\delta^h_j - 2\big(\psi_j\psi_k - \psi_j\theta_k + \theta_j\psi_k + \theta_j\theta_k\big)\delta^h_i \\
&- 2\big(L^\alpha_{ij}\sigma_{k\alpha} + \psi_i\sigma_{jk} + \sigma_{ik}\psi_j + \sigma_{ik}\theta_j + \sigma_{ij}\psi_k - \theta_i\sigma_{jk} + \sigma_{ij}\sigma_{k\alpha}\varphi^\alpha - \sigma_{ij}\theta_k\big)\varphi^h \\
&- 2L^h_{\underset{\vee}{jk}}\theta_i - 4L^h_{\underset{\vee}{ij}}\psi_k - 2L^h_{kj}\psi_i - 2L^h_{ik}\theta_j - 2L^h_{k\alpha}\sigma_{ij}\varphi^\alpha + 2L^h_{jk}\theta_i,
\end{aligned} \tag{2.55}$$

for σ_{ijk}, $\underset{v}{\widetilde{\Psi}}_{ij}$, $v = 1, \dots, 8$, *defined as above.* □

3. Conclusion

The first result of this paper is representation of all curvature tensors and derived tensors of a non-symmetric affine connection space $\mathbb{GA}_N$ as linear functions of the curvature tensor of the associated space $\mathbb{A}_N$. As a result of this, we connected the curvature and derived curvature tensors of spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N = f(\mathbb{GA}_N)$, where f is the third type almost geodesic mapping of the first kind (Theorems 2.1 and 2.2). Corollaries of these theorems analyze the case when f is the third type almost geodesic mapping of the second kind.

These results may help researchers interested in the third type almost geodesic mapping theory. Furthermore, researches who need curvature tensors of non-symmetric affine connection spaces and spaces associated to them may be interested in the results about connections of the corresponding tensors.

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