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SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR $h-$CONVEX FUNCTIONS ON THE CO-ORDINATES VIA FRACTIONAL INTEGRALS

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Abstract. By making use of the identity obtained by Sarıkaya, some new Hermite-Hadamard type inequalities for h-convex functions on the co-ordinates via fractional integrals are established. Our results have some relationships with the results of Sarıkaya ([20]).
Keywords: Hermite-Hadamard inequality; h-convex functions; co-ordinated convex function; Riemann-Liouville integral.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

holds, the double inequality is well known in the literature as Hermite-Hadamard inequality (see [7]).

Both inequalities hold in the reversed direction if f is concave. For recent results, generalizations and new inequalities related to the Hermite-Hadamard inequality see ([5],[15],[18],[22]).

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

Let us now consider a bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \le t f(x, y) + (1-t) f(z, w)$$

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holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$. If the inequality reversed then f is said to be concave on Δ.

A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$ and $y \in [c, d]$ (see [6]).

A formal definition for coordinated convex function may be stated as follows:

Definition 1.1. A function $f : \Delta \to \mathbb{R}$ will be called coordinated convex on Δ, for all $t, k \in [0, 1]$ and $(x, u), (y, w) \in \Delta$, if the following inequality holds:

$$\begin{aligned} & f(tx + (1-t)y, ku + (1-k)w) \\ \leq\; & tkf(x, u) + k(1-t) f(y, u) + t(1-k) f(x, w) + (1-t)(1-k) f(y, w). \end{aligned}$$

Clearly, every convex mapping is convex on the co-ordinates, but the converse is not generally true ([6]). Some interesting and important inequalities for convex functions on the co-ordinates can be found in ([10, 11, 13, 14, 21]).

In [1], Alomari and Darus established the following definition of s–convex function in the second sense on co-ordinates.

Definition 1.2. Consider the bidimensional interval $\Delta := [a, b] \times [c, d]$ in $[0, \infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbb{R}$ is s−convex on Δ if

$$f(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w) \leq \lambda^s f(x, y) + (1-\lambda)^s f(z, w)$$

holds for all $(x, y), (z, w) \in \Delta$ with $\lambda \in [0, 1]$, and for some fixed $s \in (0, 1]$.

A function $f : \Delta \to \mathbb{R}$ is s-convex on Δ is called co-ordinated s-convex on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are s-convex for all $x \in [a, b]$ and $y \in [c, d]$ with some fixed $s \in (0, 1]$.

In [12], Latif and Alomari gave the notion of h-convexity of a function f on a rectangle from the plane $\mathbb{R}^2$ and h-convexity on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$ as follows:

Definition 1.3. Let us consider a bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. A mapping $f : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ is said to be h-convex on Δ, if f is non-negative and if the following inequality:

$$f(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w) \leq h(\lambda) f(x, y) + h(1-\lambda) f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ with $\lambda \in (0, 1)$. Let us denote this class of functions by $SX(h, \Delta)$. The function f is said to be h-concave if the inequality reversed. We denote this class of functions by $SV(h, \Delta)$.

A function $f : \Delta \to \mathbb{R}$ is said to be h-convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are h-convex where defined for all $x \in [a, b]$ and $y \in [c, d]$. A formal definition of h-convex functions may also be stated as follows:

Definition 1.4. [12] A function $f : \Delta \to \mathbb{R}$ is said to be $h-$convex on the co-ordinates on Δ, if the following inequality:

$$\begin{aligned} & f\left(tx+(1-t)\,y, ku+(1-k)\,w\right) \\ & \leq h(t)\,h(k)\,f(x,u)+h(k)\,h(1-t)\,f(y,u) \\ & +h(t)\,h(1-k)\,f(x,w)+h(1-t)\,h(1-k)\,f(y,w) \end{aligned} \tag{1.1}$$

holds for all $t, k \in [0,1]$ and $(x,u), (x,w), (y,u), (y,w) \in \Delta$.

Obviously, if $h(\alpha) = \alpha$, then all the non-negative convex (concave) functions on Δ belong to the class $SX(h,\Delta)$ $(SV(h,\Delta))$ and if $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then the class of s-convex on Δ belong to the class $SX(h,\Delta)$. Similarly we can say that if $h(\alpha) = \alpha$, then the class of non-negative convex (concave) functions on the co-ordinates on Δ is contained in the class of h-convex (concave) functions on the co-ordinates on Δ and if $h(\alpha) = \alpha^s$,where $s \in (0,1)$, then the class of s-convex functions on the co-ordinates on Δ is contained in the class of h-convex functions on the co-ordinates on Δ.

In the following we will give some necessary definitions which are used further in this paper. For more details, one can consult ([8],[9],[17]).

Definition 1.5. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ order $\alpha > 0$ and $a \geq 0$ are defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt,\quad x > a$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt,\quad x < b$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{a^+}f(x) = J^{0}_{b^-}f(x) = f(x)$.

Definition 1.6. Let $f \in L_1([a,b]\times[c,d])$. The Riemann-Liouville integrals $J^{\alpha,\beta}_{a^+,c^+}$, $J^{\alpha,\beta}_{a^+,d^-}$, $J^{\alpha,\beta}_{b^-,c^+}$ and $J^{\alpha,\beta}_{b^-,d^-}$ of order $\alpha,\beta > 0$ with $a, c \geq 0$ are defined by

$$\begin{aligned} & J^{\alpha,\beta}_{a^+,c^+}f(x,y) \\ = & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} f(t,s)\,dsdt,\ x > a, y > c \end{aligned}$$

$$\begin{aligned} & J^{\alpha,\beta}_{a^+,d^-}f(x,y) \\ = & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1} f(t,s)\,dsdt,\ x > a, y < d \end{aligned}$$

$$J_{b^-,c^+}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s)\,dsdt, \; x<b, y>c$$

$$J_{b^-,d^-}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t,s)\,dsdt, \; x<b, y<d$$

respectively. Here, Γ is the Gamma function,

$$J_{a^+,c^+}^{0,0} f(x,y) = J_{a^+,d^-}^{0,0} f(x,y) = J_{b^-,c^+}^{0,0} f(x,y) = J_{b^-,d^-}^{0,0} f(x,y) = f(x,y)$$

and

$$J_{a^+,c^+}^{1,1} f(x,y) = \int_a^x \int_c^y f(t,s)\,dsdt.$$

For some recent results connected with fractional integral inequalities see ([2, 3, 4, 16, 19, 23]).

In [20], Sarıkaya established the following inequalities of Hadamard's type for coordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$:

Theorem 1.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be coordinated convex on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \le a < b$, $0 \le c < d$ and $f \in L_1(\Delta)$. Then one has the inequalities:*

(1.2)

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \le\; & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\ & \times \left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] \\ \le\; & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned}$$

Theorem 1.2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \le a < b$, $0 \le c < d$. If $\left|\frac{\partial^2 f}{\partial t \partial k}\right|$ is a convex function on the co-ordinates on*

Δ, *then one has the inequalities:*

(1.3)
$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\
&\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-A\right|\\
\leq\ &\frac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)}\\
&\times\left(\left|\frac{\partial^2 f}{\partial k\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial k\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial k\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial k\partial t}(b,d)\right|\right)
\end{aligned}$$

where

$$\begin{aligned}
A\ =\ &\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J^{\beta}_{c^+}f(a,d)+J^{\beta}_{c^+}f(b,d)+J^{\beta}_{d^-}f(a,c)+J^{\beta}_{d^-}f(b,c)\right]\\
&+\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b,c)+J^{\alpha}_{a^+}f(b,d)+J^{\alpha}_{b^-}f(a,c)+J^{\alpha}_{b^-}f(a,d)\right].
\end{aligned}$$

Theorem 1.3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0\leq a<b$, $0\leq c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial k}\right|^q$, $q>1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

(1.4)
$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\
&+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.\\
&\left.\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\right|\\
\leq\ &\frac{(b-a)(d-c)}{\left[(\alpha p+1)(\beta p+1)\right]^{\frac{1}{p}}}\left(\frac{1}{4}\right)^{\frac{1}{q}}\\
&\times\left(\left|\frac{\partial^2 f}{\partial k\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial k\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial k\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial k\partial t}(b,d)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$A = \frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{c^+}^{\beta} f(a,d) + J_{c^+}^{\beta} f(b,d) + J_{d^-}^{\beta} f(a,c) + J_{d^-}^{\beta} f(b,c)\right] + \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b,c) + J_{a^+}^{\alpha} f(b,d) + J_{b^-}^{\alpha} f(a,c) + J_{b^-}^{\alpha} f(a,d)\right]$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

In order to prove our main results we need the following lemma (see [20]).

Lemma 1.1. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $0 \le a < b$, $0 \le c < d$. *If* $\frac{\partial^2 f}{\partial t \partial k} \in L(\Delta)$ *, then the following equality holds:*

(1.5)
$$\begin{aligned}
&\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \\
&+ \left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \right. \\
&\left. \times \left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c)\right]\right\} \\
&- \frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{c^+}^{\beta} f(a,d) + J_{c^+}^{\beta} f(b,d) + J_{d^-}^{\beta} f(a,c) + J_{d^-}^{\beta} f(b,c)\right] \\
&- \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b,c) + J_{a^+}^{\alpha} f(b,d) + J_{b^-}^{\alpha} f(a,c) + J_{b^-}^{\alpha} f(a,d)\right] \\
= &\frac{(b-a)(d-c)}{4}\left\{\int_0^1\int_0^1 t^{\alpha}k^{\beta}\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b, kc+(1-k)d)\,dkdt \right. \\
&- \int_0^1\int_0^1 (1-t)^{\alpha}k^{\beta}\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b, kc+(1-k)d)\,dkdt \\
&- \int_0^1\int_0^1 t^{\alpha}(1-k)^{\beta}\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b, kc+(1-k)d)\,dkdt \\
&\left. + \int_0^1\int_0^1 (1-t)^{\alpha}(1-k)^{\beta}\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b, kc+(1-k)d)\,dkdt\right\}.
\end{aligned}$$

2. Main Results

Theorem 2.1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be h-convex function on the co-ordinates on Δ in $\mathbb{R}^2$ and $f\in L_2(\Delta)$. Then one has the inequalities:*

(2.1)
$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \left[h\left(\frac{1}{2}\right)\right]^2 \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}} \\
& \times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right] \\
\leq\ & \left[h\left(\frac{1}{2}\right)\right]^2 \alpha\beta\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] \\
& \times\left[\int_0^1\int_0^1 t^{\alpha-1}k^{\alpha-1}\right. \\
& \left.\times\left[h(t)h(k)+h(t)h(1-k)+h(1-t)h(k)+h(1-t)h(1-k)\right]dkdt\right].
\end{aligned}$$

Proof. According to (1.1) with $x=t_1a+(1-t_1)b, y=(1-t_1)a+t_1b, u=k_1c+(1-k_1)d, w=(1-k_1)c+k_1d$ and $t=k=\frac{1}{2}$, we find that

(2.2)
$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \left[h\left(\frac{1}{2}\right)\right]^2 \\
& \times\left[f(t_1a+(1-t_1)b,k_1c+(1-k_1)d)+f(t_1a+(1-t_1)b,(1-k_1)c+k_1d)\right. \\
& \left.+f((1-t_1)a+t_1b,k_1c+(1-k_1)d)+f((1-t_1)a+t_1b,(1-k_1)c+k_1d)\right].
\end{aligned}$$

Thus, multiplying both sides of (2.2) by $t_1^{\alpha-1}k_1^{\beta-1}$, then by integrating with respect to (t_1,k_1) on $[0,1]\times[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\alpha\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\; &\left[h\left(\frac{1}{2}\right)\right]^2 \\
&\times \left[\begin{array}{c}
\int_0^1 \int_0^1 t_1^{\alpha-1} k_1^{\beta-1} \\
\times \left[f\left(t_1 a+(1-t_1) b, k_1 c+(1-k_1) d\right)\right. \\
+f\left(t_1 a+(1-t_1) b,(1-k_1) c+k_1 d\right) \\
+f\left((1-t_1) a+t_1 b, k_1 c+(1-k_1) d\right) \\
\left.+f\left((1-t_1) a+t_1 b,(1-k_1) c+k_1 d\right)\right] dk_1 dt_1
\end{array}\right].
\end{aligned}$$

On the right side of the above inequality, using the changes of the variable

$$\begin{gathered}
\left\{x=t_1 a+(1-t_1) b, y=k_1 c+(1-k_1) d\right\}, \\
\left\{x=t_1 a+(1-t_1) b, y=(1-k_1) c+k_1 d\right\} \\
\left\{x=(1-t_1) a+t_1 b, y=k_1 c+(1-k_1) d\right\}, \\
\left\{x=(1-t_1) a+t_1 b, y=(1-k_1) c+k_1 d\right\}
\end{gathered}$$

respectively, we get

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\; & \left[h\left(\frac{1}{2}\right)\right]^2 \frac{\alpha\beta}{(b-a)^\alpha (d-c)^\beta}\left\{\int_a^b \int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} f(x, y)\, dy dx\right. \\
& +\int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} f(x, y)\, dy dx \\
& +\int_a^b \int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} f(x, y)\, dy dx \\
& \left.+\int_a^b \int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} f(x, y)\, dy dx\right\}
\end{aligned}$$

from which the first inequality is proved.

For the proof of the second inequality (2.1), we first note that if f is a h-convex function on Δ, then, by using (1.1) with $x=a, y=b, u=c, w=d$, it yields

$$\begin{aligned}
& f(ta+(1-t) b, kc+(1-k) d) \\
\leq\; & h(t) h(k) f(a, c)+h(t) h(1-k) f(a, d) \\
& +h(1-t) h(k) f(b, c)+h(1-t) h(1-k) f(b, d)
\end{aligned}$$

$$\begin{aligned} & f((1-t)a+tb, kc+(1-k)d) \\ \leq\ & h(1-t)h(k)f(a,c)+h(1-t)h(1-k)f(a,d) \\ & +h(t)h(k)f(b,c)+h(t)h(1-k)f(b,d) \end{aligned}$$

$$\begin{aligned} & f(ta+(1-t)b,(1-k)c+kd) \\ \leq\ & h(t)h(1-k)f(a,c)+h(t)h(k)f(a,d) \\ & +h(1-t)h(1-k)f(b,c)+h(1-t)h(k)f(b,d) \end{aligned}$$

$$\begin{aligned} & f((1-t)a+tb,(1-k)c+kd) \\ \leq\ & h(1-t)h(1-k)f(a,c)+h(1-t)h(k)f(a,d) \\ & +h(t)h(1-k)f(b,c)+h(t)h(k)f(b,d). \end{aligned}$$

By adding these inequalities we have

(2.3)

$$\begin{aligned} & f(ta+(1-t)b, kc+(1-k)d)+f((1-t)a+tb, kc+(1-k)d) \\ & +f(ta+(1-t)b,(1-k)c+kd)+f((1-t)a+tb,(1-k)c+kd) \\ \leq\ & \left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right] \\ & \times[h(t)h(k)+h(t)h(1-k)+h(1-t)h(k)+h(1-t)h(1-k)]. \end{aligned}$$

Then, multiplying both sides of (2.3) by $t^{\alpha-1}k^{\beta-1}$ and integrating with respect to (t,k) over $[0,1]\times[0,1]$, we get

$$\begin{aligned} & \int_0^1\int_0^1 t^{\alpha-1}k^{\beta-1} \\ & \times\left[f(ta+(1-t)b, kc+(1-k)d)+f((1-t)a+tb, kc+(1-k)d)\right. \\ & \left.+f(ta+(1-t)b,(1-k)c+kd)+f((1-t)a+tb,(1-k)c+kd)\right]dkdt \\ \leq\ & \left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right] \\ & \times\left\{\int_0^1\int_0^1 t^{\alpha-1}k^{\beta-1}\right. \\ & \left.\times[h(t)h(k)+h(t)h(1-k)+h(1-t)h(k)+h(1-t)h(1-k)]dkdt\right\}. \end{aligned}$$

Here, using the change of the variable we have

$$
\begin{aligned}
& \left[h\left(\frac{1}{2}\right)\right]^2 \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}} \\
& \times\left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) + J^{\alpha,\beta}_{b^-,d^-} f(a,c)\right] \\
\leq\ & \left[h\left(\frac{1}{2}\right)\right]^2 \alpha\beta\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] \\
& \times\left[\int_0^1\int_0^1 t^{\alpha-1}k^{\beta-1}\right. \\
& \left.\times\left[h(t)h(k)+h(t)h(1-k)+h(1-t)h(k)+h(1-t)h(1-k)\right]dkdt\right].
\end{aligned}
$$

The proof is completed. □

Remark 2.1. If we take $h(\alpha)=\alpha$ in Theorem 2.1, then the inequality (2.1) becomes the inequality (1.2) of Theorem 1.1.

Corollary 2.1. *If we take $h(\alpha)=\alpha^s$ in Theorem 2.1, we have the following inequality:*

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \left(\frac{1}{2}\right)^{2s} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}} \\
& \times\left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) + J^{\alpha,\beta}_{b^-,d^-} f(a,c)\right] \\
\leq\ & \left(\frac{1}{2}\right)^{2s} \alpha\beta\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] \\
& \times\left(\frac{1}{\alpha+s}+B(\alpha,s+1)\right)\left(\frac{1}{\beta+s}+B(\beta,s+1)\right)
\end{aligned}
$$

where B is the Beta function,

$$
B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt.
$$

Theorem 2.2. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$. If*

$\left|\frac{\partial^2 f}{\partial t \partial k}\right|$ *is a h-convex function on the co-ordinates on* Δ, *then one has the inequalities:*

(2.4)

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right. \\
&\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\Bigg| \\
\leq\ &\frac{(b-a)(d-c)}{4} \\
&\times\left\{\left|\frac{\partial^2 f}{\partial t\partial k}(a,c)\right|\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(k^{\beta}+(1-k)^{\beta}\right)h(t)h(k)\,dkdt\right. \\
&+\left|\frac{\partial^2 f}{\partial t\partial k}(b,c)\right|\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(k^{\beta}+(1-k)^{\beta}\right)h(1-t)h(k)\,dkdt \\
&+\left|\frac{\partial^2 f}{\partial t\partial k}(a,d)\right|\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(k^{\beta}+(1-k)^{\beta}\right)h(t)h(1-k)\,dkdt \\
&\left.+\left|\frac{\partial^2 f}{\partial t\partial k}(b,d)\right|\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(k^{\beta}+(1-k)^{\beta}\right)h(1-t)h(1-k)\,dkdt\right\}
\end{aligned}$$

where

$$\begin{aligned}
A\ =\ &\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J^{\beta}_{c^+}f(a,d)+J^{\beta}_{c^+}f(b,d)+J^{\beta}_{d^-}f(a,c)+J^{\beta}_{d^-}f(b,c)\right] \\
&+\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b,c)+J^{\alpha}_{a^+}f(b,d)+J^{\alpha}_{b^-}f(a,c)+J^{\alpha}_{b^-}f(a,d)\right].
\end{aligned}$$

Proof. From Lemma 1.1, we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.$$
$$\left.\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left\{\int_0^1\int_0^1 t^{\alpha}k^{\beta}\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|dkdt\right.$$
$$+\int_0^1\int_0^1(1-t)^{\alpha}k^{\beta}\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|dkdt$$
$$+\int_0^1\int_0^1 t^{\alpha}(1-k)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|dkdt$$
$$\left.+\int_0^1\int_0^1(1-t)^{\alpha}(1-k)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|dkdt\right\}.$$

Since $\left|\frac{\partial^2 f}{\partial t\partial k}\right|$ is h-convex function on the co-ordinates on Δ, then one has:

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.$$
$$\left.\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\right|$$
$$\leq \frac{(b-a)(d-c)}{4}$$
$$\times\left\{\left|\frac{\partial^2 f}{\partial t\partial k}(a,c)\right|\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(k^{\beta}+(1-k)^{\beta}\right)h(t)h(k)\,dkdt\right.$$
$$+\left|\frac{\partial^2 f}{\partial t\partial k}(b,c)\right|\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(k^{\beta}+(1-k)^{\beta}\right)h(1-t)h(k)\,dkdt$$
$$+\left|\frac{\partial^2 f}{\partial t\partial k}(a,d)\right|\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(k^{\beta}+(1-k)^{\beta}\right)h(t)h(1-k)\,dkdt$$
$$\left.+\left|\frac{\partial^2 f}{\partial t\partial k}(b,d)\right|\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(k^{\beta}+(1-k)^{\beta}\right)h(1-t)h(1-k)\,dkdt\right\}.$$

The proof is completed. □

Remark 2.2. If we take $h(\alpha) = \alpha$ in Theorem 2.2, then the inequality (2.4) becomes the inequality (1.3) of Theorem 1.2.

Corollary 2.2. *If we take $h(\alpha) = \alpha^s$ in Theorem 2.2, we have*

$$\begin{aligned}
&\Bigg| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \\
&+ \left\{ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \right. \\
&\left. \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) + J^{\alpha,\beta}_{b^-,d^-} f(a,c) \right] \right\} - A \Bigg| \\
\leq\ & \frac{(b-a)(d-c)}{4} \\
&\times \left[\left| \frac{\partial^2 f}{\partial t \partial k}(a,c) \right| + \left| \frac{\partial^2 f}{\partial t \partial k}(b,c) \right| + \left| \frac{\partial^2 f}{\partial t \partial k}(a,d) \right| + \left| \frac{\partial^2 f}{\partial t \partial k}(b,d) \right| \right] \\
&\times \left(\frac{1}{\alpha+s+1} + B(s+1,\alpha+1) \right) \left(\frac{1}{\beta+s+1} + B(s+1,\beta+1) \right)
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{\Gamma(\beta+1)}{4(d-c)^{\beta}} \left[J^{\beta}_{c^+} f(a,d) + J^{\beta}_{c^+} f(b,d) + J^{\beta}_{d^-} f(a,c) + J^{\beta}_{d^-} f(b,c) \right] \\
&\quad + \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f(b,c) + J^{\alpha}_{a^+} f(b,d) + J^{\alpha}_{b^-} f(a,c) + J^{\alpha}_{b^-} f(a,d) \right]
\end{aligned}$$

and B is the Beta function,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Theorem 2.3. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$. If $\left| \frac{\partial^2 f}{\partial t \partial k} \right|^q$, $q > 1$, is an h-convex function on the co-ordinates on Δ, then one has the inequalities:*

(2.5)

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right. \\
&\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\Bigg| \\
\leq\ & \frac{(b-a)(d-c)}{\left[(\alpha p+1)(\beta p+1)\right]^{\frac{1}{p}}} \\
&\times\left(\left|\frac{\partial^2 f}{\partial k\partial t}(a,c)\right|^q\int_0^1\int_0^1 h(t)h(k)\,dkdt\right. \\
&+\left|\frac{\partial^2 f}{\partial k\partial t}(a,d)\right|^q\int_0^1\int_0^1 h(t)h(1-k)\,dkdt \\
&+\left|\frac{\partial^2 f}{\partial k\partial t}(b,c)\right|^q\int_0^1\int_0^1 h(1-t)h(k)\,dkdt \\
&\left.+\left|\frac{\partial^2 f}{\partial k\partial t}(b,d)\right|^q\int_0^1\int_0^1 h(1-t)h(1-k)\,dkdt\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
A =\ & \frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J^{\beta}_{c^+}f(a,d)+J^{\beta}_{c^+}f(b,d)+J^{\beta}_{d^-}f(a,c)+J^{\beta}_{d^-}f(b,c)\right] \\
&+\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b,c)+J^{\alpha}_{a^+}f(b,d)+J^{\alpha}_{b^-}f(a,c)+J^{\alpha}_{b^-}f(a,d)\right]
\end{aligned}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1.1, we have

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right. \\
&\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\Bigg|
\end{aligned}$$

$$\leq \frac{(b-a)(d-c)}{4}\left\{\int_0^1\int_0^1 t^{\alpha}k^{\beta}\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|dkdt\right.$$
$$+\int_0^1\int_0^1 (1-t)^{\alpha}k^{\beta}\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|dkdt$$
$$+\int_0^1\int_0^1 t^{\alpha}(1-k)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|dkdt$$
$$\left.+\int_0^1\int_0^1 (1-t)^{\alpha}(1-k)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|dkdt\right\}.$$

By using the well known Hölder's inequality for double integrals, we get

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.$$
$$\left.\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left\{\left(\int_0^1\int_0^1 t^{p\alpha}k^{p\beta}dkdt\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1 (1-t)^{p\alpha}k^{p\beta}dkdt\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_0^1\int_0^1 t^{p\alpha}(1-k)^{p\beta}dkdt\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1 (1-t)^{p\alpha}(1-k)^{p\beta}dkdt\right)^{\frac{1}{p}}\right\}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial k}(ta+(1-t)b,kc+(1-k)d)\right|^q dkdt\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 f}{\partial t\partial k}\right|^q$ is h-convex function on the co-ordinates on Δ, then one has:

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.$$
$$\left.\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\right|$$

$$\begin{aligned}
&\leq \frac{(b-a)(d-c)}{[(\alpha p+1)(\beta p+1)]^{\frac{1}{p}}} \\
&\times\left(\left|\frac{\partial^2 f}{\partial k\partial t}(a,c)\right|^q \int_0^1\int_0^1 h(t)h(k)\,dkdt\right. \\
&+\left|\frac{\partial^2 f}{\partial k\partial t}(a,d)\right|^q \int_0^1\int_0^1 h(t)h(1-k)\,dkdt \\
&+\left|\frac{\partial^2 f}{\partial k\partial t}(b,c)\right|^q \int_0^1\int_0^1 h(1-t)h(k)\,dkdt \\
&\left.+\left|\frac{\partial^2 f}{\partial k\partial t}(b,d)\right|^q \int_0^1\int_0^1 h(1-t)h(1-k)\,dkdt\right)^{\frac{1}{q}}
\end{aligned}$$

and the proof is completed. □

Remark 2.3. If we take $h(\alpha)=\alpha$ in Theorem 2.3, then the inequality (2.5) becomes the inequality (1.4) of Theorem 1.3.

Corollary 2.3. *If we take $h(\alpha)=\alpha^s$ in Theorem 2.3, we have*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&+\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\right. \\
&\left.\left.\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right\}-A\right| \\
&\leq \frac{(b-a)(d-c)}{[(\alpha p+1)(\beta p+1)]^{\frac{1}{p}}} \\
&\times\left(\frac{1}{(s+1)^2}\left[\left|\frac{\partial^2 f}{\partial k\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial k\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial k\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial k\partial t}(b,d)\right|^q\right]\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right] \\
&+\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right].
\end{aligned}$$

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