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CERTAIN INEQUALITIES USING SAIGO FRACTIONAL INTEGRAL OPERATOR

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Abstract. The main objective of this paper is to establish some new results on fractional integral inequalities by considering the extended Chebyshev functional in the case of synchronous function. The result is concerned with some inequalities using one fractional parameter and two parameters.
Keywords: Chebyshev functional, Saigo fractional integral operator and fractional integral inequality.

1. Introduction

During the past few years, many authors have established some well-known inequalities and their applications using Riemann-Liouville fractional derivative and integral (see [1, 2, 6-10]). Recently in [11-18] the authors obtained the inequalities using Saigo fractional integral operator. In [4] the authors proved the following inequalities using Hadamard fractional integral for the extended Chebyshev functional.

Theorem 1.1. *Let f and g be two synchronous functions on $[0,\infty)$, and $r, p, q : [0,\infty) \to [0,\infty)$. Then for all $t > 0$, $\alpha > 0$, we have*

$$\begin{gathered}
2\,{}_HD^{-\alpha}_{1,t}r(t)\left[{}_HD^{-\alpha}_{1,t}p(t)\,{}_HD^{-\alpha}_{1,t}(qfg)(t) + {}_HD^{-\alpha}_{1,t}q(t)\,{}_HD^{-\alpha}_{1,t}(pfg)(t)\right] + \\
2\,{}_HD^{-\alpha}_{1,t}p(t)\,{}_HD^{-\alpha}_{1,t}q(t)\,{}_HD^{-\alpha}_{1,t}(rfg)(t) \geq \\
{}_HD^{-\alpha}_{1,t}r(t)\left[{}_HD^{-\alpha}_{1,t}(pf)(t)\,{}_HD^{-\alpha}_{1,t}(qg)(t) + {}_HD^{-\alpha}_{1,t}(qf)(t)\,{}_HD^{-\alpha}_{1,t}(pg)(t)\right] + \\
{}_HD^{-\alpha}_{1,t}p(t)\left[{}_HD^{-\alpha}_{1,t}(rf)(t)\,{}_HD^{-\alpha}_{1,t}(qg)(t) + {}_HD^{-\alpha}_{1,t}(qf)(t)\,{}_HD^{-\alpha}_{1,t}(rg)(t)\right] + \\
{}_HD^{-\alpha}_{1,t}q(t)\left[{}_HD^{-\alpha}_{1,t}(rf)(t)\,{}_HD^{-\alpha}_{1,t}(pg)(t) + {}_HD^{-\alpha}_{1,t}(pf)(t)\,{}_HD^{-\alpha}_{1,t}(rg)(t)\right]
\end{gathered}$$

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And

Theorem 1.2. *Let f and g be two synchronous functions on $[0,\infty)$, and $r,p,q:[0,\infty)\to[0,\infty)$. Then for all $t>0$, $\alpha>0$, we have:*

$$\begin{gathered}
{}_HD_{1,t}^{-\alpha}r(t)[{}_HD_{1,t}^{-\alpha}q(t){}_HD_{1,t}^{-\beta}(pfg)(t)+2{}_HD_{1,t}^{-\alpha}p(t){}_HD_{1,t}^{-\beta}(qfg)(t)\\
+{}_HD_{1,t}^{-\beta}q(t){}_HD_{1,t}^{-\alpha}(pfg)(t)]\\
+\left[{}_HD_{1,t}^{-\alpha}p(t){}_HD_{1,t}^{-\beta}q(t)+{}_HD_{1,t}^{-\beta}p(t){}_HD_{1,t}^{-\alpha}q(t)\right]{}_HD_{1,t}^{-\alpha}(rfg)(t)\geq\\
{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}(pf)(t){}_HD_{1,t}^{-\beta}(qg)(t)+{}_HD_{1,t}^{-\beta}(qf)(t){}_HD_{1,t}^{-\alpha}(pg)(t)\right]+\\
{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t){}_HD_{1,t}^{-\beta}(qg)(t)+{}_HD_{1,t}^{-\beta}(qf)(t){}_HD_{1,t}^{-\alpha}(rg)(t)\right]+\\
{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t){}_HD_{1,t}^{-\beta}(pg)(t)+{}_HD_{1,t}^{-\beta}(pf)(t){}_HD_{1,t}^{-\alpha}(rg)(t)\right].
\end{gathered}$$

In [3, 5] the authors established some other fractional integral inequality using Hadamard fractional integral.

In literature few results have been obtained on some fractional integral inequalities using Saigo fractional integral operator in [14, 15]. Motivated by [3, 4, 14], our purpose in this paper is to establish some new results using Saigo fractional integral inequalities for the extended Chebyshev functional. The paper is organized as follows. In Section 2, we provide basic definitions and propositions related to Saigo fractional derivatives and integrals. In Section 3, we give the main results.

2. Preliminaries

We give some necessary definitions. For more details, see [14].

Definition 2.1. A real-valued function $f(x),(x>1)$, is said to be in space C_μ $(\mu\in R)$, if there exists a real number $p>\mu$ such that $f(x)=x^p\phi(x)$; where $\phi(x)\in C(0,\infty)$.

Definition 2.2. Let $\alpha>0$,$\beta,\eta\in R$, then the Saigo fractional integral $I_{0,x}^{\alpha,\beta,\eta}[f(x)]$ of order α for a real-valued continuous function $f(x)$ is defined by ([14]),

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)]=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})f(t)dt. \tag{2.1}$$

where, the function ${}_2F_1(-)$ in the right-hand side of (2.1) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \tag{2.2}$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)...(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0 = 1.$$

For $f(x) = x^{\mu}$ in (2.1) we have the known result [14] as:

$$I_{0,x}^{\alpha,\beta,\eta}[x^{\mu}] = \frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)}x^{\mu-\beta}. \tag{2.3}$$

$(\alpha > 0, min(\mu, \mu-\beta+\eta) > -1, x > 0)$.

3. Main Result

We now prove the following lemma.

Lemma 3.1. *Let f and g be two integrable and synchronous functions on $[0,\infty)$. and $u, v : [0,\infty) \to [0,\infty)$. Then for all $x > 0$, $\alpha > max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, we have,*

$$\begin{aligned} I_{0,x}^{\alpha,\beta,\eta}[u(x)]I_{0,x}^{\alpha,\beta,\eta}[vfg(x)] + I_{0,x}^{\alpha,\beta,\eta}[v(x)]I_{0,x}^{\alpha,\beta,\eta}[ufg(x)] \geq \\ I_{0,x}^{\alpha,\beta,\eta}[uf(x)]I_{0,x}^{\alpha,\beta,\eta}[vg(x)] + I_{0,x}^{\alpha,\beta,\eta}[vf(x)]I_{0,x}^{\alpha,\beta,\eta}[ug(x)]. \end{aligned} \tag{3.1}$$

Proof: Since f and g are synchronous functions on $[0,\infty)$ for all $\tau \geq 0$, $\rho \geq 0$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0. \tag{3.2}$$

From (3.2),

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{3.3}$$

Consider

$$\begin{aligned} G(x,\tau) &= \tfrac{x^{-\alpha-\beta}(x-\tau)^{\alpha-1}}{\Gamma(\alpha)} \, {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\tfrac{\tau}{x}), \ (\tau \in (0,x); x > 0) \\ &= \tfrac{1}{\Gamma(\alpha)} \tfrac{(x-\tau)^{\alpha-1}}{x^{\alpha+\beta}} + \tfrac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)} \tfrac{(x-\tau)^{\alpha}}{x^{\alpha+\beta+1}} \\ &+ \tfrac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)}{\Gamma(\alpha+2)} \tfrac{(x-\tau)^{\alpha+1}}{x^{\alpha+\beta+2}} + ... \end{aligned} \tag{3.4}$$

Clearly, the function $G(x,\tau)$ remains positive because for all $\tau \in (0,x)$, $(x > 0)$ since each term of the (3.4) is positive. Multiplying both sides of (3.3) by $G(x,\tau)u(\tau)$, then integrating the resulting identity with respect to τ from 0 to x, we get

$$\begin{aligned} &\tfrac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \, {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\tfrac{\tau}{x}) u(\tau) f(\tau) g(\tau) d\tau \\ &+ \tfrac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \, {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\tfrac{\tau}{x}) u(\tau) f(\rho) g(\rho) d\tau \\ &\geq \tfrac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \, {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\tfrac{\tau}{x}) u(\tau) f(\tau) g(\rho) d\tau \\ &+ \tfrac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \, {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\tfrac{\tau}{x}) u(\tau) f(\rho) g(\tau) d\tau, \end{aligned} \tag{3.5}$$

consequently,

$$I_{0,x}^{\alpha,\beta,\eta}[ufg(x)] + f(\rho)g(\rho)I_{0,x}^{\alpha,\beta,\eta}[u(x)] \geq g(\rho)I_{0,x}^{\alpha,\beta,\eta}[uf(x)] + f(\rho)I_{0,x}^{\alpha,\beta,\eta}[ug(x)]. \tag{3.6}$$

Multiplying both sides of (3.6) by $G(x,\rho)v(\rho)$, ($\rho \in (0,x)$, $x > 0$), where $G(x,\rho)$ is defined in view of (3.4). Now integrating the resulting identity with respect to ρ from 0 to x, we have

$$\begin{aligned} & I_{0,x}^{\alpha,\beta,\eta}[ufg(x)]\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\tfrac{\rho}{x})v(\rho)d\rho \\ & +I_{0,x}^{\alpha,\beta,\eta}[u(x)]\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\tfrac{\rho}{x})v(\rho)f(\rho)g(\rho)d\rho \\ & \geq I_{0,x}^{\alpha,\beta,\eta}[uf(x)]\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\tfrac{\rho}{x})v(\rho)g(\rho)d\rho \\ & +I_{0,x}^{\alpha,\beta,\eta}[ug(x)]\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\tfrac{\rho}{x})v(\rho)f(\rho)d\rho. \end{aligned} \tag{3.7}$$

This completes the proof of Inequality 3.1.

Now, we give our main result.

Theorem 3.1. *Let f and g be two integrable and synchronous functions on $[0,\infty)$, and $r,p,q : [0,\infty) \to [0,\infty)$. Then for all $x > 0$, $\alpha > max\{0,-\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, we have,*

$$\begin{aligned} & 2I_{0,x}^{\alpha,\beta,\eta}[r(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\alpha,\beta,\eta}[qfg(x)] + I_{0,x}^{\alpha,\beta,\eta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[pfg(x)]\right] + \\ & 2I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\alpha,\beta,\eta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[rfg(x)] \geq \\ & I_{0,x}^{\alpha,\beta,\eta}[r(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[pf(x)]I_{0,x}^{\alpha,\beta,\eta}[qg(x)] + I_{0,x}^{\alpha,\beta,\eta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[pg(x)]\right] + \\ & I_{0,x}^{\alpha,\beta,\eta}[p(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\alpha,\beta,\eta}[qg(x)] + I_{0,x}^{\alpha,\beta,\eta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)]\right] + \\ & I_{0,x}^{\alpha,\beta,\eta}[q(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\alpha,\beta,\eta}[pg(x)] + I_{0,x}^{\alpha,\beta,\eta}[pf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)]\right]. \end{aligned} \tag{3.8}$$

Proof: To prove the theorem, put $u = p,\ v = q$, and using lemma 3.1, we get

$$\begin{aligned} & I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\alpha,\beta,\eta}[qfg(x)] + I_{0,x}^{\alpha,\beta,\eta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[pfg(x)] \geq \\ & I_{0,x}^{\alpha,\beta,\eta}[pf(x)]I_{0,x}^{\alpha,\beta,\eta}[qg(x)] + I_{0,x}^{\alpha,\beta,\eta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[pg(x)]. \end{aligned} \tag{3.9}$$

Now, multiplying both sides by (3.9) $I_{0,x}^{\alpha,\beta,\eta}[r(x)]$, we have

$$\begin{aligned} & I_{0,x}^{\alpha,\beta,\eta}[r(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\alpha,\beta,\eta}[qfg(x)] + I_{0,x}^{\alpha,\beta,\eta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[pfg(x)]\right] \geq \\ & I_{0,x}^{\alpha,\beta,\eta}[r(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[pf(x)]I_{0,x}^{\alpha,\beta,\eta}[qg(x)] + I_{0,x}^{\alpha,\beta,\eta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[pg(x)]\right], \end{aligned} \tag{3.10}$$

again, put $u = r, v = q$, and using lemma 3.1, we get

$$\begin{aligned} & I_{0,x}^{\alpha,\beta,\eta}[r(x)]I_{0,x}^{\alpha,\beta,\eta}[qfg(x)] + I_{0,x}^{\alpha,\beta,\eta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[rfg(x)] \geq \\ & I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\alpha,\beta,\eta}[qg(x)] + I_{0,x}^{\alpha,\beta,\eta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)], \end{aligned} \tag{3.11}$$

multiplying both sides of (3.11) by $I_{0,x}^{\alpha,\beta,\eta}[p(x)]$, we have

$$(3.12)\qquad I_{0,x}^{\alpha,\beta,\eta}[p(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[r(x)]I_{0,x}^{\alpha,\beta,\eta}[qfg(x)]+I_{0,x}^{\alpha,\beta,\eta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[rfg(x)]\right]\geq I_{0,x}^{\alpha,\beta,\eta}[p(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\alpha,\beta,\eta}[qg(x)]+I_{0,x}^{\alpha,\beta,\eta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)]\right].$$

With the same arguments as in equations (3.11) and (3.12), we can write

$$(3.13)\qquad I_{0,x}^{\alpha,\beta,\eta}[q(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[r(x)]I_{0,x}^{\alpha,\beta,\eta}[pfg(x)]+I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\alpha,\beta,\eta}[rfg)(x)]\right]\geq I_{0,x}^{\alpha,\beta,\eta}[q(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\alpha,\beta,\eta}[pg(x)]+I_{0,x}^{\alpha,\beta,\eta}[pf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)]\right].$$

Adding the inequalities (3.10), (3.12) and (3.13), we get the required inequality (3.8).

Lemma 3.2. *Let f and g be two integrable and synchronous functions on* $[0,\infty)$, *and* $u,v:[0,\infty[\to[0,\infty)$. *Then for all* $x>0$, $\alpha>max\{0,-\beta\}$, $\psi>max\{0,-\phi\}$ $\beta<1$, $\beta-1<\eta<0$, $\phi<1$, $\phi-1<\zeta<0$, *we have*

$$(3.14)\qquad I_{0,x}^{\alpha,\beta,\eta}[u(x)]I_{0,x}^{\psi,\phi,\zeta}[vfg(x)]+I_{0,x}^{\psi,\phi,\zeta}[v(x)]I_{0,x}^{\alpha,\beta,\eta}[ufg(x)]\geq I_{0,x}^{\alpha,\beta,\eta}[uf(x)]I_{0,x}^{\psi,\phi,\zeta}[vg(x)]+I_{0,x}^{\psi,\phi,\zeta}[vf(x)]I_{0,x}^{\alpha,\beta,\eta}[ug(x)].$$

Proof: Now multiplying both sides of (3.6) by $v(\rho)\frac{x^{-\psi-\phi}}{\Gamma(\psi)}(x-\rho)^{\psi-1}\,{}_2F_1(\psi+\phi,-\zeta;\psi;1-\frac{\rho}{x})$ $(\rho\in(0,x), x>0)$, which (in view of the argument mentioned above in the proof of theorem 3.1) remain positive. Then integrating the resulting identity with respect to ρ from 0 to x, we have

$$(3.15)\qquad I_{0,x}^{\alpha,\beta,\eta}[ufg(x)]\frac{x^{-\psi-\phi}}{\Gamma(\psi)}\int_0^x(x-\rho)^{\psi-1}\,{}_2F_1(\psi+\phi,-\zeta;\psi;1-\tfrac{\rho}{x})v(\rho)d\rho$$
$$+I_{0,x}^{\alpha,\beta,\eta}[u(x)]\frac{x^{-\psi-\phi}}{\Gamma(\psi)}\int_0^x(x-\rho)^{\psi-1}\,{}_2F_1(\psi+\phi,-\zeta;\psi;1-\tfrac{\rho}{x})v(\rho)f(\rho)g(\rho)d\rho$$
$$\geq I_{0,x}^{\alpha,\beta,\eta}[uf(x)]\frac{x^{-\psi-\phi}}{\Gamma(\psi)}\int_0^x(x-\rho)^{\psi-1}\,{}_2F_1(\psi+\phi,-\zeta;\psi;1-\tfrac{\rho}{x})v(\rho)g(\rho)d\rho$$
$$+I_{0,x}^{\alpha,\beta,\eta}[ug(x)]\frac{x^{-\psi-\phi}}{\Gamma(\psi)}\int_0^x(x-\rho)^{\psi-1}\,{}_2F_1(\psi+\phi,-\zeta;\psi;1-\tfrac{\rho}{x})v(\rho)f(\rho)d\rho.$$

This completes the proof of Inequality (3.14).

Theorem 3.2. *Let f and g be two integrable and synchronous functions on* $[0,\infty)$, *and* $r,p,q:[0,\infty)\to[0,\infty)$.
Then for all $x>0$, $\alpha>max\{0,-\beta\}$, $\psi>max\{0,-\phi\}\beta<1$, $\beta-1<\eta<0$, $\phi<1$, $\phi-1<\zeta<0$, *we have*

$$(3.16)\qquad I_{0,x}^{\alpha,\beta,\eta}[r(x)][I_{0,x}^{\alpha,\beta,\eta}[q(x)]I_{0,x}^{\psi,\phi,\zeta}[pfg(x)]+2I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\psi,\phi,\zeta}(qfg)(x)+I_{0,x}^{\psi,\phi,\zeta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[pfg(x)]]+\left[I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\psi,\phi,\zeta}[q(x)]+I_{0,x}^{\psi,\phi,\zeta}[p(x)]I_{0,x}^{\alpha,\beta,\eta}[q(x)]\right]I_{0,x}^{\alpha,\beta,\eta}[rfg(x)]\geq$$

$$I_{0,x}^{\alpha,\beta,\eta}[r(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[pf(x)]I_{0,x}^{\psi,\phi,\zeta}[qg(x)]+I_{0,x}^{\psi,\phi,\zeta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[pg(x)]\right]+$$
$$I_{0,x}^{\alpha,\beta,\eta}[p(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\psi,\phi,\zeta}[qg(x)]+I_{0,x}^{\psi,\phi,\zeta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)]\right]+$$
$$I_{0,x}^{\alpha,\beta,\eta}[q(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\psi,\phi,\zeta}[pg(x)]+I_{0,x}^{\psi,\phi,\zeta}[pf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)]\right].$$

Proof: To prove the theorem, we put $u = p,\ v = q$ and using lemma 3.2 we get

$$I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\psi,\phi,\zeta}[qfg(x)]+I_{0,x}^{\psi,\phi,\zeta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[pfg(x)]\geq I_{0,x}^{\alpha,\beta,\eta}[pf(x)]I_{0,x}^{\psi,\phi,\zeta}[qg(x)]+I_{0,x}^{\psi,\phi,\zeta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[pg(x)]. \tag{3.17}$$

Now, multiplying both sides by (3.17) $I_{0,x}^{\alpha,\beta,\eta}[r(x)]$, we have

$$I_{0,x}^{\alpha,\beta,\eta}[r(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[p(x)]I_{0,x}^{\psi,\phi,\zeta}[qfg(x)]+I_{0,x}^{\psi,\phi,\zeta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[pfg(x)]\right]\geq I_{0,x}^{\alpha,\beta,\eta}[r(x)]\left[I_{0,x}^{\psi,\phi,\zeta}[pf(x)]I_{0,x}^{\psi,\phi,\zeta}[qg(x)]+I_{0,x}^{\psi,\phi,\zeta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[pg(x)]\right], \tag{3.18}$$

putting $u = r,\ v = q$, and using lemma 3.2, we get

$$I_{0,x}^{\alpha,\beta,\eta}[r(x)]I_{0,x}^{\psi,\phi,\zeta}[qfg(x)]+I_{0,x}^{\psi,\phi,\zeta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[rfg(x)]\geq I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\psi,\phi,\zeta}[qg(x)]+I_{0,x}^{\psi,\phi,\zeta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)], \tag{3.19}$$

multiplying both sides by (3.19) $I_{0,x}^{\alpha,\beta,\eta}[p(x)]$, we have

$$I_{0,x}^{\alpha,\beta,\eta}[p(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[r(x)]I_{0,x}^{\psi,\phi,\zeta}[qfg(x)]+I_{0,x}^{\psi,\phi,\zeta}[q(x)]I_{0,x}^{\alpha,\beta,\eta}[rfg(x)]\right]\geq I_{0,x}^{\alpha,\beta,\eta}[p(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\psi,\phi,\zeta}[qg(x)]+I_{0,x}^{\psi,\phi,\zeta}[qf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)]\right]. \tag{3.20}$$

With the same argument as in equations (3.19) and (3.20), we obtain

$$I_{0,x}^{\alpha,\beta,\eta}[q(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[r(x)]I_{0,x}^{\psi,\phi,\zeta}[pfg(x)]+I_{0,x}^{\psi,\phi,\zeta}[p(x)]I_{0,x}^{\alpha,\beta,\eta}[rfg(x)]\right]\geq I_{0,x}^{\alpha,\beta,\eta}[q(x)]\left[I_{0,x}^{\alpha,\beta,\eta}[rf(x)]I_{0,x}^{\psi,\phi,\zeta}[pg(x)]+I_{0,x}^{\psi,\phi,\zeta}[pf(x)]I_{0,x}^{\alpha,\beta,\eta}[rg(x)]\right]. \tag{3.21}$$

Adding the inequalities (3.18), (3.20) and (3.21), we get the inequality (3.16).

Remark 3.1. If f, g, r, p *and* q satisfy the following conditions,

1. The function f and g is asynchronous on $[0,\infty)$.
2. The function r,p,q are negative on $[0,\infty)$.
3. Two of the function r,p,q are positive and the third is negative on $[0,\infty)$.

then the inequality 3.8 and 3.16 are reversed.

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