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TOTALLY REAL SUBMANIFOLDS OF $(LCS)_n$-MANIFOLDS

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Abstract. The present paper deals with the study of totally real submanifolds and C-totally real submanifolds of $(LCS)_n$-manifolds with respect to the Levi-Civita connection and the quarter symmetric metric connection. It is proved that the scalar curvatures of C-totally real submanifolds of $(LCS)_n$-manifold with respect to both the said connections are the same.

Keywords: $(LCS)_n$-manifold, totally real submanifold, quarter symmetric metric connection.

1. Introduction

As a generalization of LP-Sasakian manifold, Shaikh [13] recently introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example. Such manifolds have many applications in the general theory of relativity and cosmology ([15], [16]).

The notion of semisymmetric linear connection on a smooth manifold was introduced by Friedmann and Schouten [4]. Then Hayden [6] introduced the idea of metric connection with torsion on a Riemannian manifold. Thereafter Yano [19] studied the semisymmetric metric connection on a Riemannian manifold systematically. As a generalization of the semisymmetric connection, Golab [5] introduced the idea of quarter symmetric linear connection on smooth manifolds. A linear connection $\bar{\nabla}$ in an n-dimensional smooth manifold $\tilde{M}$ is said to be a quarter symmetric connection [5] if its torsion tensor T is of the form

$$(1.1)\qquad \begin{aligned} T(X,Y) &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y, \end{aligned}$$

where η is a 1-form and ϕ is a tensor of type $(1,1)$. In particular, if $\phi X = X$ then the quarter symmetric connection reduces to a semisymmetric connection. Further,

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if the quarter symmetric connection $\bar{\nabla}$ satisfies the condition $(\bar{\nabla}_X g)(Y,Z) = 0$, for all $X,\ Y,\ Z \in \chi(\tilde{M})$, then $\bar{\nabla}$ is said to be a quarter symmetric metric connection.

Due to important applications in applied mathematics and theoretical physics, the geometry of submanifolds has become a subject of growing interest. Analogous to almost Hermitian manifolds, the invariant and anti-invariant submanifolds [2] are dependent on the behaviour of almost contact metric structure ϕ. A submanifold M of a $(LCS)_n$-manifold manifold $\tilde{M}$ is said to be anti-invariant (or totally real) if for any $X \in T(M)$, $\phi X \in T^{\perp}M$ i.e., $\phi(TM) \subset T^{\perp}M$ at every point of M. A totally real submanifold M of $\tilde{M}$ is a C-totally real submanifold if ξ is normal to M [18]. Consequently, C-totally real submanifolds are anti-invariant. Recently Hui et al. ([1], [7], [8], [9], [17]) studied submanifolds of $(LCS)_n$-manifolds. The present paper deals with the study of totally real submanifolds and C-totally real submanifolds of $(LCS)_n$-manifolds with respect to the Levi-Civita connection and the quarter symmetric metric connection. It is shown that the scalar curvature of a C-totally real submanifold of $(LCS)_n$-manifold with respect to the Levi-Civita connection and the quarter symmetric metric connection is the same. However, in the case of totally real submanifolds of $(LCS)_n$-manifolds with respect to the Levi-Civita connection and the quarter symmetric metric connection, they are different. An inequality for the square length of the shape operator in the case of a totally real submanifold of $(LCS)_n$-manifold is derived. The equality case is also considered.

2. Preliminaries

Let $\tilde{M}$ be an n-dimensional Lorentzian manifold [12] admitting a unit time-like concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X,\xi) = \eta(X), \tag{2.2}$$

satisfies [20]

$$(\tilde{\nabla}_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0, \tag{2.3}$$

$$\tilde{\nabla}_X \xi = \alpha\{X + \eta(X)\xi\}, \quad \alpha \neq 0, \tag{2.4}$$

for $X,\ Y \in \chi(\tilde{M})$, where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function that satisfies

$$\tilde{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. Let us take

$$\phi X = \frac{1}{\alpha}\tilde{\nabla}_X \xi, \tag{2.6}$$

then from (2.4) and (2.6), we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

$$g(\phi X, Y) = g(X, \phi Y), \tag{2.8}$$

from which it follows that ϕ is a symmetric (1,1) tensor called the structure tensor of the manifold. Thus the Lorentzian manifold $\tilde{M}$ together with the unit time-like concircular vector field ξ, its associated 1-form η and a (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [13]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [11]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold ([13], [14]):

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.9}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.10}$$

$$\tilde{R}(X, Y)Z = \phi\tilde{R}(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \tag{2.11}$$

for all $X,\ Y,\ Z \in \chi(\tilde{M})$. Using (2.8) in (2.11), we get

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= \tilde{R}(X, Y, Z, \phi W) + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) \\ &\quad - g(X, Z)\eta(Y)\}\eta(W). \end{aligned} \tag{2.12}$$

Let M be a submanifold of dimension m of a $(LCS)_n$-manifold $\tilde{M}$ ($m < n$) with induced metric g. Also, let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.13}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.14}$$

for all $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are the second fundamental form and the shape operator (corresponding to the normal vector field V), respectively, for the immersion of M into $\tilde{M}$ and they are related by [21]

$$g(h(X, Y), V) = g(A_V X, Y) \tag{2.15}$$

for any $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. The equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W)+g(h(X,Z),h(Y,W))-g(h(X,W),h(Y,Z)) \tag{2.16}$$

for any vectors $X,\ Y,\ Z,\ W$ tangent to M.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space $\tilde{M}$ such that refracting to M^m, $\{e_1, e_2, \cdots, e_m\}$ is the orthonormal basis to the tangent space T_xM with respect to the induced connection.
We write

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad i, j \in \{1, 2, \cdots, m\} \text{ and } r \in \{m + 1, \cdots, n\}.$$

Then the square length of h is

$$||h||^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j))$$

and the mean curvature H of M associated to ∇ is $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$.
The quarter symmetric metric connection $\bar{\tilde{\nabla}}$ and the Riemannian connection $\tilde{\nabla}$ on a $(LCS)_n$-manifold $\tilde{M}$ are related by [10]

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.17}$$

If $\bar{\tilde{R}}$ and $\tilde{R}$ are the curvature tensors of an $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection $\bar{\tilde{\nabla}}$ and the Riemannian connection $\tilde{\nabla}$, then

$$\begin{aligned} \bar{\tilde{R}}(X,Y,Z,W) &= \tilde{R}(X,Y,Z,W) + (2\alpha - 1)[g(\phi X, Z)g(\phi Y, W) \\ &\quad -g(\phi Y, Z)g(\phi X, W)] + \alpha[\eta(Y)g(X, W) \\ &\quad -\eta(X)g(Y, W)]\eta(Z) + \alpha[g(Y, Z)\eta(X) \\ &\quad -g(X, Z)\eta(Y)]\eta(W) \end{aligned} \tag{2.18}$$

for all $X,\ Y,\ Z,\ W \in \chi(\tilde{M})$.
We now recall the following [3]:

Let L be a k-plane section of T_xM and X be a unit vector in L. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_k\}$ of L such that $e_1 = X$. Then the Ricci curvature Ric_L of L at X is defined by [3]

$$Ric_L(X) = K_{12} + K_{13} + \cdots + K_{1k}, \tag{2.19}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. Such a curvature is called a k-Ricci curvature.

The scalar curvature τ of the k-plane section L is given by [3]

$$\tau(L) = \sum_{1\le i<j\le k} K_{ij}. \tag{2.20}$$

For each integer k, $2 \leq k \leq n$, the invariant Θ_k on M is defined by [3]

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L.X} Ric_L(X), \quad x \in M, \tag{2.21}$$

where L runs over all k-plane sections in T_xM and X runs over all unit vectors in L.

The relative null space for M at a point $x \in M$ is defined by [3]

$$\mathcal{N}_x = \{X \in T_xM | h(X,Y) = 0, Y \in T_xM\}. \tag{2.22}$$

3. Theorem-like Environments

This section deals with the study of totally real submanifolds of $(LCS)_n$-manifolds with respect to the Levi-Civita and quarter symmetric metric connection. We prove the following:

Theorem 3.1. *Let M be a totally real submanifold of dimension m $(m < n)$ of a $(LCS)_n$-manifold $\tilde{M}$. Then*

$$m^2||H||^2 = 2\tau + ||h||^2 + (m-1)(\alpha^2 - \rho), \tag{3.1}$$

where τ is the scalar curvature of M.

Proof. Let M be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Now from (2.12) and (2.16), we get

$$\begin{aligned} R(X,Y,Z,W) \quad = \quad & \tilde{R}(X,Y,Z,\phi W) + (\alpha^2 - \rho)\{g(Y,Z)\eta(X) \\ & -g(X,Z)\eta(Y)\}\eta(W) + g(h(X,W),h(Y,Z)) \\ & -g(h(X,Z),h(Y,W)) \end{aligned} \tag{3.2}$$

for any $X,\ Y,\ Z,\ W \in \Gamma(TM)$.

Since M is a totally real submanifold i.e., anti-invariant, so

$$\tilde{R}(X,Y,Z,\phi W) = g(\tilde{R}(X,Y)Z,\phi W) = 0$$

as $\tilde{R}(X,Y)Z$ is tangent to M and ϕW is normal to M and hence (3.2) yields

$$\begin{aligned} R(X,Y,Z,W) \quad = \quad & (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(W) \\ & +g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \end{aligned} \tag{3.3}$$

for any $X,\ Y,\ Z,\ W \in \Gamma(TM)$. Putting $X = W = e_i$ and $Y = Z = e_j$ in (3.3) and taking summation over $1 \leq i < j \leq m$, we get

$$\begin{aligned} \sum_{1\leq i<j\leq m} R(e_i,e_j,e_j,e_i) \quad = \quad & (\alpha^2 - \rho) \sum_{1\leq i<j\leq m} [g(e_j,e_j)\eta(e_i)\eta(e_i) - g(e_i,e_j)\eta(e_j)\eta(_j)] \\ & + \sum_{1\leq i<j\leq m} g(h(e_i,e_i),h(e_j,e_j)) \\ & - \sum_{1\leq i<j\leq m} g(h(e_i,e_j),h(e_j,e_i)) \end{aligned}$$

i.e.,

$$2\tau = -(m-1)(\alpha^2 - \rho) + m^2||H||^2 - ||h||^2, \tag{3.4}$$

which implies (3.1). □

Corollary 3.1. *Let M be a C-totally real submanifold of dimension m $(m < n)$ of a $(LCS)_n$-manifold $\tilde{M}$. Then*

$$m^2||H||^2 = 2\tau + ||h||^2.$$

Proof. In a C-totally real submanifold, since $\xi \in \Gamma(T^{\perp}M)$ so, $\eta(X) = 0$ for all $X \in \Gamma(TM)$. Then (3.3) yields

$$R(X, Y, Z, W) = g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

from which, similarly to the above, we can prove that $m^2||H||^2 = 2\tau + ||h||^2$. □

Now let M be a submanifold of dimension m $(m < n)$ of a $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection $\bar{\bar{\nabla}}$ and $\bar{\nabla}$ be the induced connection of M associated to the quarter symmetric metric connection. Also let $\bar{h}$ be the second fundamental form of M with respect to $\bar{\nabla}$. Then the Gauss formula can be written as

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X, Y) \tag{3.5}$$

and hence by virtue of (2.13) and (2.17), we get

$$\bar{\nabla}_X Y + \bar{h}(X, Y) = \nabla_X Y + h(X, Y) + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{3.6}$$

If M is a totally real submanifold of $\tilde{M}$ then $\phi X \in T^{\perp}M$ for any $X \in TM$ and hence $g(\phi X, Y) = 0$ for $X,\ Y \in TM$. So, equating the normal part from (3.6), we get

$$\bar{h}(X, Y) = h(X, Y) + \eta(Y)\phi X. \tag{3.7}$$

Further, if M is C-totally real submanifold of $\tilde{M}$ then $\xi \in T^{\perp}M$ and hence $\eta(Y) = 0$ for all $Y \in TM$. So, (3.7) yields

$$\bar{h}(X, Y) = h(X, Y). \tag{3.8}$$

Let U be a unit tangent vector at $x \in \tilde{M}$ and $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space $\tilde{M}$ such that $e_1 = U$ refracting to M^m, $\{e_1, e_2, \cdots, e_m\}$ is the orthonormal basis to the tangent space T_xM with respect to the induced quarter symmetric metric connection. Then we have the following:

Theorem 3.2. *Let M be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection, then*

$$m^2||H||^2 = 2\bar{\tau} + ||h||^2 + (2m-1)\alpha + m\alpha\eta^2(U), \tag{3.9}$$

where $\bar{\tau}$ is the scalar curvature of M with respect to the induced connection associated to the quarter symmetric metric connection.

Proof. In the case of an $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection, the relation (2.16) becomes

$$\begin{aligned} \bar{\tilde{R}}(X,Y,Z,W) &= \bar{R}(X,Y,Z,W) + g(\bar{h}(X,Z),\bar{h}(Y,W)) \\ &\quad - g(\bar{h}(X,W),\bar{h}(Y,Z)). \end{aligned} \tag{3.10}$$

In view of (2.7) and (2.8), (3.10) yields

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= \tilde{R}(X,Y,Z,\phi W) + (\alpha^2-\rho)\{g(Y,Z)\eta(X) \\ &\quad - g(X,Z)\eta(Y)\}\eta(W) + (2\alpha-1)[g(\phi X,Z)g(\phi Y,W) \\ &\quad - g(\phi Y,Z)g(\phi X,W)] + \alpha[\eta(Y)g(X,W) \\ &\quad - \eta(X)g(Y,W)]\eta(Z) + \alpha[g(Y,Z)\eta(X) \\ &\quad - g(X,Z)\eta(Y)]\eta(W) \\ &\quad + g(\bar{h}(X,W),\bar{h}(Y,Z)) - g(\bar{h}(X,Z),\bar{h}(Y,W)). \end{aligned} \tag{3.11}$$

Since M is totally real, therefore $g(\phi X, Y) = 0$ for all $X,\ Y \in TM$ and (3.7) holds. Thus (3.11) becomes

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= (\alpha^2-\rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(W) \\ &\quad + \alpha[\eta(Y)g(X,W) - \eta(X)g(Y,W)]\eta(Z) \\ &\quad + \alpha[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(W) \\ &\quad + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \\ &\quad - \eta(Z)g(h(X,W),\phi Y) - \eta(W)g(\phi X,h(Y,Z)) \\ &\quad + \eta(Z)g(\phi X,h(Y,W)) + \eta(W)g(h(X,Z),\phi Y). \end{aligned} \tag{3.12}$$

Putting $X = W = e_i$ and $Y = Z = e_j$ in (3.12) and taking summation over $1 \le i < j \le m$, we get

$$\begin{aligned} 2\bar{\tau} &= -(m-1)(\alpha^2-\rho) - \alpha(1+\eta^2(U))m - \alpha(m-1) \\ &\quad + m^2||H||^2 - ||h||^2, \end{aligned} \tag{3.13}$$

from which (3.9) follows. □

Corollary 3.2. *Let M be a C-totally real submanifold of an $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection. Then*

$$m^2||H||^2 = 2\bar{\tau} + ||h||^2. \tag{3.14}$$

Proof. If M is a C-totally real submanifold then $\eta(Y) = 0$ for all $Y \in TM$ and hence (3.12) implies that

$$\bar{R}(X,Y,Z,W) = g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \tag{3.15}$$

from which, similarly to the above, (3.14) follows. □

From Corollary 3.1 and Corollary 3.2 we get $\tau = \bar{\tau}$ i.e., the scalar curvatures of a C-totally real submanifold of a $(LCS)_n$-manifold with respect to the induced Levi-Civita connection and the induced quarter symmetric metric connection are identical. Thus we can state the following:

Theorem 3.3. *Let M be a C-totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Then the scalar curvatures of M with respect to the induced Levi-Civita connection and induced quarter symmetric metric connection are the same.*

Next, we prove the following:

Theorem 3.4. *Let M be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Then*
(i) for each unit vector $X \in T_xM$,

$$4Ric(X) \le m^2||H||^2 + 2(\alpha^2 - \rho)(m-2) + 4(m-2)(\alpha^2 - \rho)\eta^2(X); \tag{3.16}$$

(ii) in the case of H(x)=0, a unit tangent vector X at x satisfies the equality case of (3.16) if and only if X lies in the relative null space $\mathcal{N}_x$ at x.
(iii) the equality case of (3.16) holds identically for all unit tangent vectors at x if and only if either x is a totally geodesic point or $m = 2$ and x is a totally umbilical point.

Proof. Let $X \in T_xM$ be a unit tangent vector at x. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_m, e_{m+1}, \cdots, e_n\}$ such that $\{e_1, \cdots, e_m\}$ are tangent to M at x and $e_1 = X$. Then from (3.1), we have

$$\begin{aligned}
m^2\|H\|^2 &= 2\tau + \sum_{r=m+1}^{n}\{(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{mm}^r)^2)\} \\
&\quad -2\sum_{r=m+1}^{n}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r + (m-1)(\alpha^2-\rho) \\
&= 2\tau + \frac{1}{2}\sum_{r=m+1}^{n}\{(h_{11}^r + \cdots + h_{mm}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{mm}^r)^2\} \qquad (3.17)\\
&\quad +2\sum_{r=m+1}^{n}\sum_{i<j}(h_{ij}^r)^2 - 2\sum_{r=m+1}^{n}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r + (m-1)(\alpha^2-\rho).
\end{aligned}$$

From the equation of Gauss, we find

$$K_{ij} = \sum_{r=m+1}^{n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + (\alpha^2 - \rho)\eta^2(e_i),$$

and consequently

$$(3.18) \sum_{2 \leq i < j \leq m} K_{ij} = \sum_{r=m+1}^{n} \sum_{2 \leq i < j \leq m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + (\alpha^2 - \rho)[m - 2 + \eta^2(X)].$$

Using (3.18) in (3.17), we get

$$(3.19) \quad m^2||H||^2 \geq 2\tau + \frac{m^2}{2}||H||^2 + 2\sum_{r=m+1}^{n}\sum_{j=2}^{m}(h_{1j}^r)^2 - 2\sum_{2 \leq i < j \leq m} K_{ij} - (m-3)(\alpha^2 - \rho) - 2(m-2)(\alpha^2 - \rho)\eta^2(X).$$

Therefore,

$$\frac{1}{2}m^2||H||^2 \geq 2Ric(X) - (m-3)(\alpha^2 - \rho) - 2(m-2)(\alpha^2 - \rho)\eta^2(X),$$

from which we get (3.16).

Let us assume that $H(x) = 0$. Then the equality holds in (3.16) if and only if

$$h_{11}^r = h_{22}^r = \cdots = h_{1m}^r = 0 \text{ and } \quad h_{11}^r = h_{22}^r + \cdots + h_{mm}^r, \quad r \in \{m+1, \cdots, n\}.$$

Then $h_{1j}^r = 0$ for every $j \in \{1, \cdots m\}, r \in \{m+1 \cdots n\}$, i.e., $X \in \mathcal{N}_x$.

(iii) The equality case of (3.16) holds for every unit tangent vector at x if and only if

$$h_{ij}^r = 0, i \neq j \text{ and } h_{11}^r + h_{22}^r + \cdots + h_{mm}^r - 2h_{ii}^r = 0.$$

We distinguish two cases:
(a) $m \neq 2$, then x is a totally geodesic point;
(b) $m = 2$, it follows that x is a totally umbilical point.
The converse is trivial. □

Next we obtain the following:

Theorem 3.5. *Let M be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Then*

$$(3.20) \qquad ||H||^2 \geq \frac{2\tau}{m(m-1)} + \frac{1}{m}(\alpha^2 - \rho).$$

Proof. We choose an orthonormal basis $\{e_1, \cdots e_m, e_{m+1}, \cdots, e_n\}$ at x such that e_{m+1} is parallel to the mean curvature vector $H(x)$, and $e_1, \cdots, e_m$ diagonalise the shape operator A_{m+1}. Then the shape operator takes the form

$$A_{m+1} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}, \tag{3.21}$$

$$A_r = (h^r_{ij}), \quad i,j = 1, \cdots, m; r = m+2, \cdots, n, \qquad traceA_r = \sum_{i=1}^{m} h^r_{ii} = 0$$

and from (3.1), we get

$$m^2||H||^2 = 2\tau + \sum_{i=1}^{m} a_i^2 + \sum_{r=m+2}^{n} \sum_{i,j=1}^{m} (h^r_{ij})^2 + (m-1)(\alpha^2 - \rho). \tag{3.22}$$

On the other hand, since

$$0 \le \sum_{i<j} (a_i - a_j)^2 = (m-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j, \tag{3.23}$$

we obtain

$$m^2||H||^2 = \left(\sum_{i=1}^{m} a_i\right)^2 + 2\sum_i a_i^2 - 2\sum_{i<j} a_i a_j \le m\sum_{i=1}^{m} a_i^2, \tag{3.24}$$

which implies that

$$\sum_i a_i^2 \ge m\|H\|^2. \tag{3.25}$$

In view of (3.25), (3.22) yields

$$m^2\|H\|^2 \ge 2\tau + m\|H\|^2 + (m-1)(\alpha^2 - \rho), \tag{3.26}$$

which implies (3.20). □

Theorem 3.6. *Let M be a totally real submanifold of an $(LCS)_n$-manifold $\tilde{M}$. Then for any integer k, $2 \le k \le m$ and for any point $x \in M$*

$$\|H\|^2(x) \ge \Theta_k(x) + \frac{1}{m}(\alpha^2 - \rho). \tag{3.27}$$

Proof. Let $\{e_1, e_2, \cdots, e_m\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1,\cdots,i_k}$ the k-plane section spanned by $e_{i_1}, \cdots, e_{i_k}$. Then, we have [3]

$$\tau(x) \ge \frac{m(m-1)}{2}\Theta_k(x). \tag{3.28}$$

Using (3.28) in (3.20), (3.27) follows. □

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