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# ON CERTAIN HESSENBERG MATRICES RELATED WITH LINEAR RECURRENCES

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**Abstract.** In this paper, we present various results for permanents and determinants of some Hessenberg matrices. Also, some special cases for permanents are given.

**Keywords**: Hessenberg matrices, permanents, determinants.

## 1. Introduction

Matrix methods are useful tools deriving some properties of linear recurrences. Some authors obtained many connections between certain sequences and permanents of Hessenberg matrices in the literature [1]-[4],[6],[10]-[12].

The permanent of an $n-$ square matrix $\mathbf{A_n} = [a_{ij}]$ is defined by

$$per\mathbf{A_n} = \sum_{\sigma \in D_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations $\sigma$ of the symmetric group $D_n$.

In [9], Minc defined the super-diagonal matrix and showed that the permanent of the matrix equals the order k-Fibonacci number.

In [5], Kılıç derived recurrence relations and generating matrices for the sums of usual tribonacci numbers and 4n subscripted tribonacci sequences $\{T_{4n}\}$, and their sums. Also, the relationships between these sequences and permanents of certain matrices are obtained.

In [6], Kılıç and Taşcı found the relationships between the sums of the Fibonacci and Lucas numbers and 1-factors of bipartite graphs.

In [7], Kılıç and Taşcı defined the $n \times n$ tridiagonal Toeplitz $(0, -1, 1)$-matrix $\mathbf{M_n} = [m_{i,j}]$ with $m_{i,i} = -1$ for $1 \le i \le n, m_{i,i+1} = m_{i+1,i} = 1$ for $1 \le i \le n-1$

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and 0 otherwise, and the $n \times n$ tridiagonal Toeplitz $(0,-1,1)$-matrix $\mathbf{L_n} = [l_{i,j}]$ with $l_{i,i} = -1$ for $2 \leq i \leq n, l_{i,i+1} = l_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $l_{1,1} = -\frac{1}{2}$ and 0 otherwise. They showed $per\mathbf{M_n} = F_{-(n+1)}$ and $per\mathbf{L_n} = \frac{L_{-n}}{2}$, where $F_n$ and $L_n$ is the $n$th Fibonacci and Lucas numbers, respectively.

In [8], Li showed new Fibonacci-Hessenberg matrices and gave another proof of the well-known results relative to the Pell and Perrin numbers.

In [3], Kalman showed that the $(n+k)$-th term of a sequence is defined recursively as a linear combination of the preceding $k$ terms:

$$u_{n+k} = c_0 u_n + c_1 u_{n+1} + ... + c_{k-1} u_{n+k-1} \tag{1.1}$$

in which the initial terms $u_0 = ... = u_{k-2} = 0, u_{k-1} = 1$ and $c_0, c_1, ..., c_{k-1}$ are constants.

In [10], considering the generalized Fibonacci-Narayana sequence $\{G_n(a,c,r)\}$, Ramírez derived some relations between this sequence and a permanent of one type of the upper Hessenberg matrix. For example,

$$per\begin{bmatrix} a & c & c & \cdots & c & & & 0 \\ 1 & a & 0 & 0 & \cdots & c & & \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & 1 & a & 0 & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & a & 0 & 0 \\ & & & & & 1 & a & 0 \\ & & & & & & 1 & a \end{bmatrix} = G_{n+r-1}(a,c,r),$$

where the generalized Fibonacci-Narayana sequence $\{G_n(a,c,r)\}_{n\in\mathbb{N}}$ is defined as follows:

$$G_n(a,c,r) = aG_{n-1}(a,c,r) + cG_{n-r}(a,c,r),\ 2 \leq r \leq n,$$

with the initial conditions $G_0(a,c,r) = 0$, $G_i(a,c,r) = 1$, for $i = 1, 2, ..., r-1$.

In [12], Trojovský defined tridiagonal matrices $\mathbf{B_n^\delta} = \left[b_{ij}^\delta\right]$ in the form

$$\begin{cases} 1 & \text{if } i = j \text{ or } i = j-1, \\ (-1)^{j+\delta} & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta \in \{0,1\}$ and showed

$$\det \mathbf{B_n^\delta} = \begin{cases} F_{(n+4-6\delta)/2} & \text{if } n \equiv 0 \pmod 2, \\ F_{(n+1)/2} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

## 2. Some results

In this section, we define the sequence $\{R_n(a,b,c,d)\}$ and determine some relationships between the terms of this sequence and permanents of certain upper Hessenberg matrices. A sequence $\{R_n(a,b,c,d)\}$ is defined by for $3 \leq d \leq n$,

$$R_n(a,b,c,d) = aR_{n-1}(a,b,c,d) + bR_{n-2}(a,b,c,d) + cR_{n-d}(a,b,c,d), \tag{2.1}$$

in which $R_0(a,b,c,d) = 0, R_1(a,b,c,d) = R_2(a,b,c,d) = ... = R_{d-2}(a,b,c,d) = 1$ and $R_{d-1}(a,b,c,d) = a$. The sequence $\{R_n(a,b,c,d)\}$ is a generalization of the tribonacci sequence. When $a = b = c = 1$ and $d = 3$, $R_n(1,1,1,3) = T_n$ (the $n$th tribonacci number). If $c = 0$ and $d = 3$, the generalized Fibonacci sequence $\{U_n(a,b)\}$ is obtained. If $a = b = 1, c = 0$ and $d = 3$, the Fibonacci sequence $\{F_n\}$ is obtained and if $a = c = 1$, $b = 0$ and $d = 3$, the Narayana sequence is obtained.

The generating function $R(z)$ of $R_n(a,b,c,d)$ is given by

$$R(z) = \frac{(a-1+bz)\,z^{d-1} - bz^3 - az^2 + z}{(1-z)\,(1-az-bz^2-cz^d)}.$$

Now we give relationships between terms of the sequence $\{R_n(a,b,c,d)\}$ and the permanents of certain matrices.

For $n \geq 1$, define a $n \times n$ matrix $\mathbf{H_n}(a,b,c,d,k,t) = [h_{i,j}]$ with $h_{i+1,i} = 1$ for $1 \leq i \leq n-2$, $h_{i,i} = a$ for $1 \leq i \leq n-1$, $h_{i,i+1} = b$ for $1 \leq i \leq n-1$, $h_{1,i} = c$ for $3 \leq i \leq d$, $h_{i,d+i-1} = c$ for $2 \leq i \leq n-d+1, h_{n,n-1} = k, h_{n,n} = t$, and 0 otherwise, i.e.,

$$\mathbf{H_n}(a,b,c,d,k,t) = \begin{bmatrix} a & b & c & \cdots & c & 0 & ... & 0 \\ 1 & a & b & 0 & \cdots & c & ... & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ & & 1 & a & b & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & 1 & a & b \\ 0 & & & & & & k & t \end{bmatrix}. \tag{2.2}$$

Then we give the following Theorem.

**Theorem 2.1.** *Let $\mathbf{H_n}(a,b,c,d,k,t)$ be the matrix defined in (2.2). Then, for $n \geqslant 1$ and $d \geq 3$,*

$$per\mathbf{H_n}(a,b,c,d,k,t) = kR_{n+d-2}(a,b,c,d) - (ka-t)\,R_{n+d-3}(a,b,c,d), \tag{2.3}$$

*where the real numbers $k$ and $t$.*

*Proof.* (Induction on $n$) If $n = 1$, then we have

$$per\mathbf{H_1}(a,b,c,d,k,t) = t = kR_{d-1}(a,b,c,d) - (ka-t)\,R_{d-2}(a,b,c,d).$$

Suppose that the equation holds for $n-1$. Then we show that the equation holds for $n$. Expanding the $per\mathbf{H_n}$ with respect to the last column $d$ times, we write

$$\begin{aligned} & per\mathbf{H_n}(a,b,c,d,k,t) \\ = \; & aper\mathbf{H_{n-1}}(a,b,c,d,k,t)+bper\mathbf{H_{n-2}}(a,b,c,d,k,t)+cper\mathbf{H_{n-d}}(a,b,c,d,k,t). \end{aligned}$$

By our assumption, we have

$$\begin{aligned} per\mathbf{H_n}(a,b,c,d,k,t) &= a\left(kR_{n+d-3}(a,b,c,d)-(ka-t)\,R_{n+d-4}(a,b,c,d)\right) \\ &\quad +b\left(kR_{n+d-4}(a,b,c,d)-(ka-t)\,R_{n+d-5}(a,b,c,d)\right) \\ &\quad +c\left(kR_{n-2}(a,b,c,d)-(ka-t)\,R_{n-3}(a,b,c,d)\right) \\ &= kR_{n+d-2}(a,b,c,d)-(ka-t)\,R_{n+d-3}(a,b,c,d). \end{aligned}$$

Thus, the proof is complete. □

When $t=a$ and $k=1$ in (2.3), we have $per\mathbf{H_n}(a,b,c,d,1,a)=R_{n+d-2}(a,b,c,d)$.

For $n\geq 1$, define a $n\times n$ matrix $\mathbf{E_n}(a,b,c,d,k,t)=[e_{i,j}]$ with $e_{i+1,i}=-1$ for $1\leq i\leq n-2$, $e_{i,i}=a$ for $1\leq i\leq n-1$, $e_{i,i+1}=b$ for $1\leq i\leq n-1$, $e_{1,i}=c$ for $3\leq i\leq d$, $e_{i,d+i-1}=c$ for $2\leq i\leq n-d+1$, $e_{n,n-1}=-k$, $e_{n,n}=t$, and 0 otherwise, i.e.,

$$\mathbf{E_n}(a,b,c,d,k,t)=\begin{bmatrix} a & b & c & \cdots & c & 0 & \ldots & 0 \\ -1 & a & b & 0 & \cdots & c & \ldots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ & & -1 & a & b & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & -1 & a & b \\ 0 & & & & & & -k & t \end{bmatrix}.$$

It is clearly showed from [2] that

$$\det\mathbf{E_n}(a,b,c,d,k,t)=per\mathbf{H_n}(a,b,c,d,k,t).$$

Now, we take the $n\times n$ matrix $\mathbf{H_n}(a,b,c,3,k,t)$ by the following form:

$$\mathbf{H_n}(a,b,c,3,k,t)=\begin{bmatrix} a & b & c & & & & & 0 \\ 1 & a & b & c & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & 1 & a & b & c & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots & c \\ & & & & & 1 & a & b \\ 0 & & & & & & k & t \end{bmatrix}.$$

Then, we have

$$perH_n(a,b,c,3,k,t) = kR_{n+1}(a,b,c,3) - (ka-t)\,R_n(a,b,c,3). \tag{2.4}$$

For example, from [9], for $a=b=c=t=k=1$ in (2.4), we have that

$$perH_n(1,1,1,3,1,1) = T_{n+1} = perF(n,3),$$

where $T_n$ is the $n$th tribonacci number.

For $n>1$; we define an $n\times n$ matrix $\mathbf{W_n}(a,b,c,3,k,t)$ as in the compact form, by the definition of $\mathbf{H_n}(a,b,c,3,k,t)$;

$$\mathbf{W_n}(a,b,c,3,k,t) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ 0 & & & \\ \vdots & & \mathbf{H_{n-1}}(a,b,c,3,k,t) & \\ 0 & & & \end{bmatrix}. \tag{2.5}$$

Now, we have the following theorem:

**Theorem 2.2.** *Let $\mathbf{W_n}(a,b,c,3,k,t)$ be the matrix defined in (2.5). Then, for $n>2$*

$$per\mathbf{W_n}(a,b,c,3,k,t) = k\sum_{i=1}^{n} R_i(a,b,c,3) - (ka-t)\sum_{i=1}^{n} R_{i-1}(a,b,c,3).$$

*Proof.* (Induction on $n$) If $n=3$, we write

$$\begin{aligned} & per\mathbf{W_3}(a,b,c,3,k,t) \\ = \; & k+t+at+bk = k\sum_{i=1}^{3} R_i(a,b,c,3) - (ka-t)\sum_{i=1}^{3} R_{i-1}(a,b,c,3). \end{aligned}$$

Suppose that the equation holds for $n$. Then we show that the equation holds for $n+1$. From the definitions of matrices $\mathbf{H_n}(a,b,c,3,k,t)$ and $\mathbf{W_n}(a,b,c,3,k,t)$, expanding the $per\mathbf{W_{n+1}}(a,b,c,3,k,t)$ with respect to the first column gives us

$$per\mathbf{W_{n+1}}(a,b,c,3,k,t) = per\mathbf{H_n}(a,b,c,3,k,t) + per\mathbf{W_n}(a,b,c,3,k,t).$$

By our assumption and (2.4), we have

$$\begin{aligned} & per\mathbf{W_{n+1}}(a,b,c,3,k,t) \\ = \; & kR_{n+1}(a,b,c,3) - (ka-t)\,R_n(a,b,c,3) \\ & +k\sum_{i=1}^{n} R_i(a,b,c,3) - (ka-t)\sum_{i=1}^{n} R_{i-1}(a,b,c,3) \\ = \; & k\sum_{i=1}^{n+1} R_i(a,b,c,3) - (ka-t)\sum_{i=1}^{n+1} R_{i-1}(a,b,c,3). \end{aligned}$$

Thus the proof is obtained. $\square$

When $a = b = c = 1$ in (2.1), the sequence $\{R_n\}$, special case of the sequence $\{R_n(a, b, c, d)\}$, is defined by the recurrence

$$R_n = R_{n-1} + R_{n-2} + R_{n-d},\ 3 \leq d \leq n \tag{2.6}$$

in which $R_0 = 0$, $R_1 = R_2 = ... = R_{d-2} = R_{d-1} = 1$ and especially from (1.1), the sequence $\{S_n\}$ is defined by

$$S_n = S_{n-1} + S_{n-2} + S_{n-d},\ 3 \leq d \leq n \tag{2.7}$$

in which $S_0 = S_1 = S_2 = ... = S_{d-3} = 0$ and $S_{d-2} = S_{d-1} = 1$. For $d = 3$ in (2.6) and (2.7), the sequences $\{R_n\}$ and $\{S_n\}$ coincide as the tribonacci sequence.

For $n > 1$, define an $n \times n$ matrix $\mathbf{Z_n} = [z_{i,j}]$ with $z_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $z_{i,i} = 1$ for $1 \leq i \leq n$, $z_{1,2} = 0$, $z_{i,i+1} = 1$ for $2 \leq i \leq n-1$, $z_{1,i} = 1$ for $3 \leq i \leq d$, $z_{i,d+i-1} = 1$ for $2 \leq i \leq n-d+1$ and 0 otherwise, i.e.,

$$\mathbf{Z_n} = \begin{bmatrix} 1 & 0 & 1 & \cdots & 1 & 0 & ... & 0 \\ 1 & 1 & 1 & 0 & \cdots & 1 & ... & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ & & 1 & 1 & 1 & 0 & \cdots & 1 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & 1 & 1 & 1 \\ 0 & & & & & & 1 & 1 \end{bmatrix}. \tag{2.8}$$

Then we give the following Theorem.

**Theorem 2.3.** *Let $\mathbf{Z_n}$ be the matrix defined in (2.8). Then, for $n \geq 5$ and $d \geq 3$,*

$$per\mathbf{Z_n} = R_{n+d-3} + R_{n+d-4} - R_{n+d-5} + S_{n-5}.$$

*Proof.* We prove this by induction on $n$. For $n = 5$, we write

$$per\mathbf{Z_5} = 2R_{d+1} + 1 = R_{d+2} + R_{d+1} - R_d + S_0.$$

The claim is true for $n = 5$. Assume that the claim is true for $n - 1$. Thus we show that the claim is true for $n$. Expanding the $perZ_n$ according to the last column $d$ times, we have

$$per\mathbf{Z_n} = per\mathbf{Z_{n-1}} + per\mathbf{Z_{n-2}} + per\mathbf{Z_{n-d}}.$$

By our assumption, we have

$$\begin{aligned} per\mathbf{Z_n} &= R_{n+d-4} + R_{n+d-5} - R_{n+d-6} + S_{n-6} \\ &\quad + R_{n+d-5} + R_{n+d-6} - R_{n+d-7} + S_{n-7} \\ &\quad + R_{n-3} + R_{n-4} - R_{n-5} + S_{n-d-5}. \end{aligned}$$

From the sequences $\{R_n\}$ and $\{S_n\}$, we write

$$per\mathbf{Z_n} = R_{n+d-3} + R_{n+d-4} - R_{n+d-5} + S_{n-5}.$$

So the proof is complete. □

For $n \geq 1$, define the $n \times n$ matrix $\mathbf{V_n^\delta} = [v_{i,j}]$ with $v_{i+1,i} = 1$ for $1 \leq i \leq n-2$, $v_{i,i} = 1$ for $1 \leq i \leq n-1$, $v_{1,i+1} = v_{i,i+1} = (-1)^{i-\delta}$for $1 \leq i \leq n$ and 0 otherwise, where $\delta \in \{0,1\}$. i.e.,

$$\mathbf{V_n^\delta} = \begin{bmatrix} 1 & (-1)^{1-\delta} & (-1)^{2-\delta} & \cdots & & (-1)^{n-2-\delta} & (-1)^{n-1-\delta} \\ 1 & 1 & (-1)^{2-\delta} & 0 & \cdots & & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots \\ & & 1 & 1 & & 0 & \cdots \\ & & & \ddots & \ddots & (-1)^{n-2-\delta} & \ddots \\ & & & & \ddots & 1 & (-1)^{n-1-\delta} \\ 0 & & & & & 1 & 1 \end{bmatrix}. \tag{2.9}$$

**Theorem 2.4.** *Let $\mathbf{V_n^\delta}$ be the matrix defined in (2.9). Then, for $n \geq 1$,*

$$\det \mathbf{V_n^\delta} = \begin{cases} (-1)^\delta \left(F_{(n+5+3(-1)^\delta)/2} - 2 - (-1)^\delta\right) & \text{if } n \equiv 0 \pmod 2, \\ (-1)^\delta \left(F_{(n+5)/2} - 2 + (-1)^\delta\right) & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

*where $F_n$ is the nth Fibonacci number.*

*Proof.* For $n = 1$, $\det \mathbf{V_1^0} = 1 = (F_3 - 1)$, $\det \mathbf{V_1^1} = 1 = -(F_3 - 3)$ and for $n = 2$, $\det \mathbf{V_2^1} = 0 = -(F_2 - 1)$, $\det \mathbf{V_2^0} = 2 = (F_5 - 3)$.

We show that the claim is true for $n-1$. Using expansion on the first column of $\det V_n^1$ , we get as follows

$$\det \mathbf{V_n^1} = \det \mathbf{B_{n-1}^0} - \det \mathbf{V_{n-1}^0}.$$

From (1.2) and the induction hypothesis, we have

$$\begin{aligned} \det \mathbf{V_n^1} &= \begin{cases} F_{n/2} - \left(F_{(n+4)/2} - 1\right) & \text{if } n \equiv 0 \pmod 2, \\ F_{(n+3)/2} - \left(F_{(n+7)/2} - 3\right) & \text{if } n \equiv 1 \pmod 2, \end{cases} \\ &= \begin{cases} -\left(F_{((n+2)/2} - 1\right) & \text{if } n \equiv 0 \pmod 2, \\ -\left(F_{((n+5)/2} - 3\right) & \text{if } n \equiv 1 \pmod 2. \end{cases} \end{aligned}$$

Similarly, using $\det \mathbf{V_n^0} = \det \mathbf{B_{n-1}^1} + \det \mathbf{B_{n-2}^0} + \det \mathbf{V_{n-2}^0}$, the desired result is given. We have the proof. □

**Theorem 2.5.** *For $n \geq 1$, we have*

$$per\mathbf{V_n^1} = \begin{cases} L_{(n+2)/2} - 1 & \text{if } n \equiv 0 \pmod 2, \\ L_{(n-1)/2} - 1 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

*and*

$$per\mathbf{V_n^0} = \begin{cases} F_{(n+2)/2} - 1 & \text{if } n \equiv 0 \pmod 2, \\ F_{(n+5)/2} - 1 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

*where $F_n$ is the nth Fibonacci number and $L_n$ is the nth Lucas number.*

*Proof.* Considering $per\mathbf{B_n^\delta} = \begin{cases} F_{(n-2+6\delta)/2} & \text{if } n \equiv 0 \pmod 2, \\ F_{(n+1)/2} & \text{if } n \equiv 1 \pmod 2, \end{cases}$ for $\delta \in \{0, 1\}$ and the equalities

$$per\mathbf{V_n^1} = per\mathbf{B_{n-1}^0} + per\mathbf{V_{n-1}^0} \text{ and } per\mathbf{V_n^0} = per\mathbf{B_{n-1}^1} - per\mathbf{B_{n-2}^0} + per\mathbf{V_{n-2}^0},$$

we have the proof from induction on $n$. □

## 3. Some special cases

In this section, we give some special cases of the above theorems:

- For $b = 1$, $c = 0$ and $d = 3$, the generalized Fibonacci sequence $\{U_n(a, 1)\}$,

$$\begin{aligned} & per\mathbf{H_n}(a, 1, 0, 3, k, t) \\ = \; & per \begin{bmatrix} a & 1 & 0 & \cdots & & \\ 1 & a & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & & & 0 \\ & & & 1 & a & 1 \\ & & & & k & t \end{bmatrix} = tU_n(a, 1) + kU_{n-1}(a, 1), \end{aligned}$$

$$\begin{aligned} & per\mathbf{W_n}(a, 1, 0, 3, k, t) \\ = \; & per \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & & & & \\ 0 & & H_{n-1} & & \\ \vdots & & & & \\ & & & & \end{bmatrix} \\ = \; & \frac{1}{a} \left( tU(_n(a, 1) + (k + t)U(_{n-1}(a, 1) + kU_{n-2}(a, 1) + k(a - 1) - t \right), \end{aligned}$$

and then,

$$per\mathbf{W_n}(a, 1, 0, 3, k, t) = per\mathbf{H_n}(a, 1, 0, 3, k, t) + per\mathbf{H_{n-1}}(a, 1, 0, 3, k, t) + k(a-1) + t.$$

- For $a = 1$, $b = 2$, $c = 0$ and $d = 3$ in (2.1), $\{J_n\}$ is the Jacobsthal sequence and for $n \geqslant 2$,

$$per\mathbf{H_n}(1,2,0,3,k,t) = per\begin{bmatrix} 1 & 2 & 0 & & & \\ 1 & 1 & 2 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots \\ & & & & & 0 \\ & & & 1 & 1 & 2 \\ & & & & k & t \end{bmatrix} = tJ_n + 2kJ_{n-1}.$$

- For $a = b = k = 1$, $c = 0$ and $d = 3$ in (2.1), $\{F_n\}$ is the Fibonacci sequence,

$$per\mathbf{H_n}(1,1,0,3,1,t) = per\begin{bmatrix} 1 & 1 & 0 & & & \\ 1 & 1 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots \\ & & & & & 0 \\ & & & 1 & 1 & 1 \\ & & & & 1 & t \end{bmatrix} = tF_n + F_{n-1}$$

and

$$per\mathbf{W_n}(1,1,0,3,1,t) = per\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & & & & \\ 0 & & H_{n-1} & & \\ \vdots & & & & \\ & & & & \end{bmatrix} = t(F_{n+1} - 1) + F_n.$$

- For $d = 3$ in (2.6) and (2.7), $\{T_n\}$ is the tribonacci sequence and

$$per\mathbf{Z_n} = per\begin{bmatrix} 1 & 0 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots \\ & & & & & 1 \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \end{bmatrix} = 2T_{n-1} + T_{n-3} + T_{n-5}.$$

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