

**BILATERAL AND BILINEAR GENERATING FUNCTIONS
FOR THE MODIFIED GENERALIZED SYLVESTER
POLYNOMIALS**

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Abstract. The present study deals with some new properties for the modified generalized Sylvester polynomials. The results obtained here include various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the modified generalized Sylvester polynomials and the generalized Lauricella functions. Finally, we get several interesting results of this theorem.

Keywords: Sylvester polynomial, generating function, Lauricella function.

1. Introduction

Generalized functions occupy pride of place in literature on special functions. Their importance, which is mounting everyday, stems from the fact that they generalize the well-known one variable special functions, namely, Hermite polynomials, Laguerre polynomials, Legendre polynomials, Gegenbauer polynomials, Jacobi polynomials, Rice polynomials, Generalized Sylvester polynomials, etc. All these polynomials are closely associated with problems of applied nature. For example, Gegenbauer polynomials are deeply connected with axially symmetric potentials in dimensions and contain the Legendre and Chebyshev polynomials as special cases. The hypergeometric functions of which the Jacobi polynomials is a special case are important in many cases of mathematics analysis and its applications.

We define the modified generalized Sylvester polynomials $f_n(x; a, b)$ as follows (see [10]):

$$(1.1) \quad f_n(x; a, b) = \frac{(bx)^n}{n!} {}_2F_0 \left[-n, ax; -; (-bx)^{-1} \right].$$

where $b \neq 0$ is an arbitrary constant.

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When $a = 1$ and $b = 1$ then (1.1) becomes

$$(1.2) \quad f_n(x; 1, 1) = \phi_n(x).$$

We call the polynomials $f_n(x; a, b)$ modified generalized Sylvester polynomials in view of the relations (1.2). For $a = 1$ and b by (1.1) becomes A.K. Agarwal and H.L. Manocha [8] generalization of Sylvester polynomials.

The following generating relations hold for (1.1) (see, [10]):

$$(1.3) \quad \sum_{n=0}^{\infty} f_n(x; a, b)t^n = (1-t)^{-ax} e^{bxt} \\ (|t| < 1)$$

and

$$(1.4) \quad \sum_{n=0}^{\infty} (\lambda)_n f_n(x; a, b)t^n = (1-bxt)^{-\lambda} {}_2F_0 \left[\lambda, ax; -; \left(\frac{t}{1-bxt} \right) \right],$$

where ${}_2F_0$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of p numerator and q denominator parameters ($p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$${}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n z^n}{(\beta_1)_n \dots (\beta_q)_n n!} \\ = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$$

and $(\lambda)_\nu$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 = 1 \text{ and } (\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C})$$

in terms of the familiar Gamma function.

Lemma 1.1. *The following generating function holds true [2]:*

$$(1.5) \quad \sum_{n=0}^{\infty} \binom{n+k}{n} f_{n+k}(x; a, b)t^n = (1-t)^{-ax-k} e^{bxt} f_k(x; a, b(1-t)).$$

Proof. If we write $t+u$ instead of t in (1.3), we get

$$\sum_{n=0}^{\infty} f_n(x; a, b)(t+u)^n = (1-t-u)^{-ax} e^{bx(t+u)} \\ \sum_{n=0}^{\infty} f_n(x; a, b) \sum_{k=0}^n \binom{n}{k} t^{n-k} u^k = (1-t)^{-ax} \left(1 - \frac{u}{1-t}\right)^{-ax} e^{bxt} e^{bxu}.$$

Replacing n by $n + k$ in the last relation, we may write that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n+k}{n} f_{n+k}(x; a, b) t^n u^k = (1-t)^{-ax} e^{bxt} \sum_{k=0}^{\infty} (1-t)^{-k} f_k(x; a, b(1-t)) u^k$$

From the coefficients of u^k on both sides of the equality, one can get the desired result. \square

Lemma 1.2. *The following addition formula holds for the modified generalized Sylvester polynomials $f_n(x; a, b)$:*

$$(1.6) \quad f_n(x_1 + x_2; a, b) = \sum_{m=0}^n f_{n-m}(x_1; a, b) f_m(x_2; a, b).$$

Proof. Replacing x by $x_1 + x_2$ in (1.3), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} f_n(x_1 + x_2; a, b) t^n &= (1-t)^{-ax_1 - ax_2} e^{b(x_1+x_2)t} \\ &= (1-t)^{-ax_1} e^{bx_1t} (1-t)^{-ax_2} e^{bx_2t} \\ &= \sum_{n=0}^{\infty} f_n(x_1; a, b) t^n \sum_{m=0}^{\infty} f_m(x_2; a, b) t^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_n(x_1; a, b) f_m(x_2; a, b) t^{n+m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n f_{n-m}(x_1; a, b) f_m(x_2; a, b) t^n. \end{aligned}$$

From the coefficients of t^n on both sides of the last equality, one can get the desired result. \square

The main objective of this paper is to study different properties of the modified generalized Sylvester polynomials. Various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are given. In addition, we derive a theorem giving certain families of bilateral generating functions for the modified generalized Sylvester polynomials and the generalized Lauricella functions.

2. Bilinear and Bilateral Generating Functions

This section presents several families of bilinear and bilateral generating functions for the modified generalized Sylvester polynomials $f_n(x; a, b)$ given by (1.1) without using Lie algebraic techniques but with the help of a similar method as considered in [4], [5], [6].

We begin by stating the following theorem.

Theorem 2.1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, \dots, y_r)$ of r complex variables y_1, \dots, y_r ($r \in \mathbb{N}$) and of complex order μ, ψ , let*

$$\Lambda_{\mu, \psi}(y_1, \dots, y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, \dots, y_r) \zeta^k \quad (a_k \neq 0)$$

and

$$\Theta_{n,p}^{\mu, \psi}(x; a, b; y_1, \dots, y_r; \xi) := \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x; a, b) \Omega_{\mu+\psi k}(y_1, \dots, y_r) \xi^k.$$

Then, for $p \in \mathbb{N}$, we have

$$(2.1) \quad \sum_{n=0}^{\infty} \Theta_{n,p}^{\mu, \psi}\left(x; a, b; y_1, \dots, y_r; \frac{\eta}{t^p}\right) t^n = (1-t)^{-ax} e^{bxt} \Lambda_{\mu, \psi}(y_1, \dots, y_r; \eta)$$

provided that each member of (2.1) exists.

Proof. For convenience, let S denote the first member of the assertion (2.1) of Theorem 2.1. Then,

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x; a, b) \Omega_{\mu+\psi k}(y_1, \dots, y_r) \eta^k t^{n-pk}.$$

Replacing n by $n + pk$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k f_n(x; a, b) \Omega_{\mu+\psi k}(y_1, \dots, y_r) \eta^k t^n \\ &= \sum_{n=0}^{\infty} f_n(x; a, b) t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, \dots, y_r) \eta^k \\ &= (1-t)^{-ax} e^{bxt} \Lambda_{\mu, \psi}(y_1, \dots, y_r; \eta) \end{aligned}$$

which completes the proof. \square

By using a similar idea, we also get the next result immediately.

Theorem 2.2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, \dots, y_r)$ of r complex variables y_1, \dots, y_r ($r \in \mathbb{N}$) and of complex order μ, ψ , let*

$$\Lambda_{\mu, \psi}^{n,p}(x_1 + x_2; a, b; y_1, \dots, y_r; t) := \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x_1 + x_2; a, b) \Omega_{\mu+\psi k}(y_1, \dots, y_r) t^k,$$

where $a_k \neq 0$, $n, p \in \mathbb{N}$ and the notation $[n/p]$ means the greatest integer less than or equal to n/p .

Then, for $p \in \mathbb{N}$, we have

$$(2.2) \quad \sum_{k=0}^n \sum_{l=0}^{[k/p]} a_l f_{n-k}(x_1; a, b) f_{k-pl}(x_2; a, b) \Omega_{\mu+\psi l}(y_1, \dots, y_r) t^l = \Lambda_{\mu, \psi}^{n,p}(x_1 + x_2; a, b; y_1, \dots, y_r; t)$$

provided that each member of (2.2) exists.

Proof. For convenience, let T denote the first member of the assertion (2.2) of Theorem 2.2. Then, upon substituting for the polynomials $f_n(x_1 + x_2; a, b)$ from the (1.6) into the left-hand side of (2.2), we obtain

$$\begin{aligned} T &= \sum_{l=0}^{[n/p]} \sum_{k=0}^{n-pl} a_l f_{n-k-pl}(x_1; a, b) f_k(x_2; a, b) \Omega_{\mu+\psi l}(y_1, \dots, y_r) t^l \\ &= \sum_{l=0}^{[n/p]} a_l \left(\sum_{k=0}^{n-pl} f_{n-k-pl}(x_1; a, b) f_k(x_2; a, b) \right) \Omega_{\mu+\psi l}(y_1, \dots, y_r) t^l \\ &= \sum_{l=0}^{[n/p]} a_l f_{n-pl}(x_1 + x_2; a, b) \Omega_{\mu+\psi l}(y_1, \dots, y_r) t^l \\ &= \Lambda_{\mu, \psi}^{n,p}(x_1 + x_2; a, b; y_1, \dots, y_r; t). \end{aligned}$$

□

Theorem 2.3. Corresponding to an identically non-vanishing function $\Omega_{\mu}(y_1, \dots, y_r)$ of r complex variables y_1, \dots, y_r ($r \in \mathbb{N}$) and of complex order μ , let

$$\Lambda_{\mu,p,q}(x; a, b; y_1, \dots, y_r; t) := \sum_{n=0}^{\infty} a_n f_{m+qn}(x; a, b) \Omega_{\mu+pn}(y_1, \dots, y_r) t^n$$

where $a_n \neq 0$ and

$$\theta_{n,p,q}(y_1, \dots, y_r; z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k \Omega_{\mu+pk}(y_1, \dots, y_r) z^k.$$

Then, for $p, q \in \mathbb{N}$; we have

$$(2.3) \quad \sum_{n=0}^{\infty} f_{m+n}(x; a, b) \theta_{n,p,q}(y_1, \dots, y_r; z) t^n = (1-t)^{-ax-m} e^{bxt} \Lambda_{\mu,p,q} \left(x; a, b(1-t); y_1, \dots, y_r; z \left(\frac{t}{1-t} \right)^q \right)$$

provided that each member of (2.3) exists.

Proof. For convenience, let T denote the first member of the assertion (2.3) of Theorem 2.3. Then,

$$T = \sum_{n=0}^{\infty} f_{m+n}(x; c) \sum_{k=0}^{[n/q]} \binom{m+n}{n- qk} a_k \Omega_{\mu+pk}(y_1, \dots, y_r) z^k t^n.$$

Replacing n by $n + qk$ and then using (1.5), we may write that

$$\begin{aligned} T &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{m+n+qk}{n} f_{m+n+qk}(x; c) a_k \Omega_{\mu+pk}(y_1, \dots, y_r) z^k t^{n+qk} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \binom{m+n+qk}{n} f_{m+n+qk}(x; a, b) t^n \right) a_k \Omega_{\mu+pk}(y_1, \dots, y_r) (zt^q)^k \\ &= \sum_{k=0}^{\infty} a_k (1-t)^{-ax-m- qk} e^{bxt} f_{m+qk}(x; a, b(1-t)) \Omega_{\mu+pk}(y_1, \dots, y_r) (zt^q)^k \\ &= (1-t)^{-ax-m} e^{bxt} \sum_{k=0}^{\infty} a_k (1-t)^{-qk} f_{m+qk}(x; a, b(1-t)) \Omega_{\mu+pk}(y_1, \dots, y_r) (zt^q)^k \\ &= (1-t)^{-ax-m} e^{bxt} \Lambda_{\mu,p,q} \left(x; a, b(1-t); y_1, \dots, y_r; z \left(\frac{t}{1-t} \right)^q \right) \end{aligned}$$

which completes the proof. \square

3. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(y_1, \dots, y_r)$, $k \in \mathbb{N}_0$, $r \in \mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(y_1, \dots, y_r) = \Phi_{\mu+\psi k}^{(\alpha)}(y_1, \dots, y_r)$$

in Theorem 2.1, where the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1, \dots, x_r)$ [4], generated by

$$(3.1) \quad (1-x_1 t)^{-\alpha} e^{(x_2+\dots+x_r)t} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x_1, \dots, x_r) t^n. \\ \left(\alpha \in \mathbb{C}; |t| < \left\{ |x_1|^{-1} \right\} \right).$$

We are thus led to the following result which provides a class of bilateral generating functions for the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(y_1, \dots, y_r)$ and the modified generalized Sylvester polynomials.

Corollary 3.1. *If*

$$\Lambda_{\mu,\psi}(y_1, \dots, y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Phi_{\mu+\psi k}^{(\alpha)}(y_1, \dots, y_r) \zeta^k \quad (a_k \neq 0, \mu, \psi \in \mathbb{C}),$$

then, we have

$$(3.2) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x; a, b) \Phi_{\mu+\psi k}^{(\alpha)}(y_1, \dots, y_r) \frac{\zeta^k}{t^{pk}} t^n = (1-t)^{-ax} e^{bxt} \Lambda_{\mu, \psi}(y_1, \dots, y_r; \zeta)$$

provided that each member of (3.2) exists.

Remark 3.1. Using the generating relation (3.1) for the multivariable polynomials $\Phi_n^{(\alpha)}(x_1, \dots, x_r)$ and getting $a_k = 1, \mu = 0, \psi = 1$ in Corollary 3.1, we find that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} f_{n-pk}(x; a, b) \Phi_k^{(\alpha)}(y_1, \dots, y_r) \zeta^k t^{n-pk} = (1-t)^{-ax} e^{bxt} (1-y_1\zeta)^{-\alpha} e^{(y_2+\dots+y_r)\zeta}.$$

$$(|\zeta| < \{|y_1|^{-1}\}, \quad |t| < 1)$$

If we set $r = 1, y_1 = x_3$ and

$$\Omega_{\mu+\psi k}(x_3) = f_{\mu+\psi k}(x_3; a, b)$$

in Theorem 2.2, we have the following bilinear generating functions for the modified generalized Sylvester polynomials.

Corollary 3.2. *If*

$$\Lambda_{\mu, \psi}^{n,p}(x_1 + x_2; a, b; x_3; a, b; t) \quad : \quad = \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x_1 + x_2; a, b) f_{\mu+\psi k}(x_3; a, b) t^k$$

$$(a_k \neq 0, \mu, \psi \in \mathbb{C})$$

then, we have

$$(3.3) \quad \sum_{k=0}^n \sum_{l=0}^{[k/p]} a_l f_{n-k}(x_1; a, b) f_{k-pl}(x_2; a, b) f_{\mu+\psi l}(x_3; a, b) t^l = \Lambda_{\mu, \psi}^{n,p}(x_1 + x_2; a, b; x_3; a, b; t)$$

provided that each member of (3.3) exists.

Remark 3.2. Taking $a_l = 1, \mu = 0, \psi = 1, p = 1, t = 1$ in Corollary 3.2, we have

$$\sum_{k=0}^n \sum_{l=0}^k f_{n-l}(x_1 + x_2; a, b) f_l(x_3; a, b) = f_n(x_1 + x_2 + x_3; a, b).$$

If we set $s = r$ and

$$\Omega_{\mu+\psi k}(y_1, \dots, y_r) = u_{\mu+\psi k}^{(\alpha_1, \dots, \alpha_r)}(y_1, \dots, y_r)$$

in Theorem 2.3, where the Erkus-Srivastava polynomials $u_n^{(\alpha_1, \dots, \alpha_r)}(y_1, \dots, y_r)$ is generated by [7],

$$\prod_{j=1}^r \{(1 - x_j t^{m_j})^{-\alpha_j}\} = \sum_{n=0}^{\infty} u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) t^n$$

$$(\alpha_j \in \mathbb{C} \ (j = 1, \dots, r) ; |t| < \min \{|x_1|^{-1/m_1}, \dots, |x_r|^{-1/m_r}\})$$

we get a family of the bilateral generating functions for the Erkus-Srivastava polynomials and the modified generalized Sylvester polynomials as follows:

Corollary 3.3. *If*

$$\Lambda_{\mu,p,q}(x; a, b; y_1, \dots, y_r; t) : = \sum_{n=0}^{\infty} a_n f_{m+qn}(x; a, b) u_{\mu+pn}^{(\alpha_1, \dots, \alpha_r)}(y_1, \dots, y_r) t^n$$

($a_n \neq 0, m \in \mathbb{N}_0, \mu, \psi \in \mathbb{C}$)

and

$$\theta_{n,p,q}(y_1, \dots, y_r; z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n- qk} a_k u_{\mu+pk}^{(\alpha_1, \dots, \alpha_r)}(y_1, \dots, y_r) z^k$$

where $n, p \in \mathbb{N}$, then we have

$$(3.4) \quad \sum_{n=0}^{\infty} f_{m+n}(x; a, b) \theta_{n,p,q}(y_1, \dots, y_r; z) t^n$$

$$= (1 - t)^{-ax-m} e^{bxt} \Lambda_{\mu,p,q} \left(x; a, b(1 - t); y_1, \dots, y_r; z \left(\frac{t}{1 - t} \right)^q \right)$$

provided that each member of (3.4) exists.

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable functions $\Omega_{\mu+\psi k}(y_1, \dots, y_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 2.1, 2.2, 2.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the modified generalized Sylvester polynomials given explicitly by (1.1).

4. Miscellaneous Properties

In this section we give some properties for the modified generalized Sylvester polynomials $f_n(x; a, b)$ given by (1.1).

Theorem 4.1. *The modified generalized Sylvester polynomials $f_n(x; a, b)$ have the following integral representation:*

$$f_n(x; a, b) = \frac{1}{n! \Gamma(ax)} \int_0^{\infty} e^{-u} u^{ax-1} (bx + u)^n du$$

where, $\operatorname{Re}(ax) > 0$.

Proof. If we use the identity

$$a^{-v} = \frac{1}{\Gamma(v)} \int_0^{\infty} e^{-at} t^{v-1} dt, \quad (\operatorname{Re}(v) > 0)$$

on the left-hand side of the generating function (1.3), we have

$$\begin{aligned} \sum_{n=0}^{\infty} f_n(x; a, b) t^n &= \frac{1}{\Gamma(ax)} \int_0^{\infty} e^{-(1-t)u} u^{ax-1} e^{bxt} du \\ &= \frac{1}{\Gamma(ax)} \int_0^{\infty} e^{-u} u^{ax-1} e^{(bx+u)t} du \\ &= \frac{1}{\Gamma(ax)} \int_0^{\infty} e^{-u} u^{ax-1} \sum_{n=0}^{\infty} (bx+u)^n \frac{t^n}{n!} du \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n! \Gamma(ax)} \int_0^{\infty} e^{-u} u^{ax-1} (bx+u)^n du \right) t^n \end{aligned}$$

From the coefficients of t^n on both sides of the last equality, one can get the desired result. \square

We now discuss some miscellaneous recurrence relations of the modified generalized Sylvester polynomials. By differentiating each member of the generating function relation (1.3) with respect to x and using

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n-k),$$

we have

$$\frac{d}{dx} f_n(x; a, b) = b f_{n-1}(x; a, b) + a \sum_{m=0}^{n-1} \frac{1}{(m+1)} f_{n-m-1}(x; a, b).$$

Besides, by differentiating each member of the generating function relation (1.3) with respect to t , we have the following recurrence relation for these polynomials:

$$(n+1) f_{n+1}(x; a, b) = x \left(b f_n(x; a, b) + a \sum_{m=0}^n f_{n-m}(x; a, b) \right).$$

5. The Generalized Lauricella Functions

In the present section, we derive various families of bilateral generating functions for the modified generalized Sylvester polynomials and the generalized Lauricella (or the Srivastava-Daoust) functions. The four Appell functions of two variables, denoted by F_1, F_2, F_3 and F_4 were generalized by Lauricella functions of n variables which are denoted by $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ [2] and

$$F_A^{(2)} = F_2, \quad F_B^{(2)} = F_3, \quad F_C^{(2)} = F_4, \quad F_D^{(2)} = F_1.$$

A further generalization of the familiar Kampé de Fériet hypergeometric function in two variables is due to Srivastava and Daoust who defined the generalized Lauricella (or the Srivastava-Daoust) function as follows [3]:

$$F_{C:D^{(1)};\dots;D^{(n)}}^{A:B^{(1)};\dots;B^{(n)}} \left(\begin{matrix} [(a) : \theta^{(1)}, \dots, \theta^{(n)}] : [(b^{(1)}) : \phi^{(1)}] ; \dots ; [(b^{(n)}) : \phi^{(n)}] ; \\ [(c) : \psi^{(1)}, \dots, \psi^{(n)}] : [(d^{(1)}) : \delta^{(1)}] ; \dots ; [(d^{(n)}) : \delta^{(n)}] ; \end{matrix} \right. \left. z_1, \dots, z_n \right) = \sum_{m_1, \dots, m_n=0}^{\infty} \Omega(m_1, \dots, m_n) \frac{z_1^{m_1}}{m_1!} \dots \frac{z_n^{m_n}}{m_n!},$$

where, for convenience,

$$\Omega(m_1, \dots, m_n) := \frac{\prod_{j=1}^A (a_j)_{m_1 \theta_j^{(1)} + \dots + m_n \theta_j^{(n)}} \prod_{j=1}^{B^{(1)}} (b_j^{(1)})_{m_1 \phi_j^{(1)}} \dots \prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n \phi_j^{(n)}}}{\prod_{j=1}^C (c_j)_{m_1 \psi_j^{(1)} + \dots + m_n \psi_j^{(n)}} \prod_{j=1}^{D^{(1)}} (d_j^{(1)})_{m_1 \delta_j^{(1)}} \dots \prod_{j=1}^{D^{(n)}} (d_j^{(n)})_{m_n \delta_j^{(n)}}$$

the coefficients

$$\theta_j^{(k)} \quad (j = 1, \dots, A; \quad k = 1, \dots, n) \quad \text{and} \quad \phi_j^{(k)} \quad (j = 1, \dots, B^{(k)}; \quad k = 1, \dots, n),$$

$$\psi_j^{(k)} \quad (j = 1, \dots, C; \quad k = 1, \dots, n) \quad \text{and} \quad \delta_j^{(k)} \quad (j = 1, \dots, D^{(k)}; \quad k = 1, \dots, n)$$

are real constants and $\left(b_{B^{(k)}}^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)} \quad (j = 1, \dots, B^{(k)}; \quad k = 1, \dots, n)$$

with similar interpretations for other sets of parameters [1]. Here, as usual, $(\lambda)_v$ denotes the Pochhammer symbol.

For a suitably bounded non-vanishing multiple sequence $\{\Omega(m_1, m_2, \dots, m_s)\}_{m_1, \dots, m_s \in \mathbb{N}_0}$ of real or complex parameters, let $\varphi_n(u_1; u_2, \dots, u_s)$ of s (real or complex) variables $u_1; u_2, \dots, u_s$ defined by [1]

$$\begin{aligned} \varphi_n(u_1; u_2, \dots, u_s) & : = \sum_{m_1=0}^n \sum_{m_2, \dots, m_s=0}^{\infty} \frac{(-n)_{m_1} ((b))_{m_1 \phi}}{((d))_{m_1 \delta}} \\ (5.1) \quad & \times \Omega(f(m_1, \dots, m_s), m_2, \dots, m_s) \frac{u_1^{m_1}}{m_1!} \dots \frac{u_s^{m_s}}{m_s!} \end{aligned}$$

where, for convenience,

$$((b))_{m_1 \phi} = \prod_{j=1}^B (b_j)_{m_1 \phi_j} \quad \text{and} \quad ((d))_{m_1 \delta} = \prod_{j=1}^D (d_j)_{m_1 \delta_j}.$$

Theorem 5.1. *The following bilateral generating function holds true:*

$$\begin{aligned} & \sum_{n=0}^{\infty} f_n(x; a, b) \varphi_n(u_1; u_2, \dots, u_s) t^n \\ = & (1-t)^{-ax} e^{bxt} \sum_{m_1, k, m_2, \dots, m_s=0}^{\infty} \frac{((b))_{(m_1+k)\phi} (ax)_k}{((d))_{(m_1+k)\delta}} \\ & \times \Omega(f((m_1+k), \dots, m_s), m_2, \dots, m_s) \frac{(-u_1 bxt)^{m_1}}{m_1!} \frac{\left(\frac{u_1 t}{t-1}\right)^k}{k!} \frac{u_2^{m_2}}{m_2!} \dots \frac{u_s^{m_s}}{m_s!}, \end{aligned}$$

where $\varphi_n(u_1; u_2, \dots, u_s)$ is given by (5.1).

Proof. By using the relationship (1.5), it is easily observed that

$$\begin{aligned} & \sum_{n=0}^{\infty} f_n(x; a, b) \varphi_n(u_1; u_2, \dots, u_s) t^n \\ = & \sum_{n=0}^{\infty} f_n(x; a, b) \sum_{m_1=0}^n \sum_{m_2, \dots, m_s=0}^{\infty} \frac{(-n)_{m_1} ((b))_{m_1 \phi}}{((d))_{m_1 \delta}} \\ & \times \Omega(f(m_1, \dots, m_s), m_2, \dots, m_s) \frac{u_1^{m_1}}{m_1!} \dots \frac{u_s^{m_s}}{m_s!} t^n \\ = & \sum_{m_1, m_2, \dots, m_s=0}^{\infty} \frac{((b))_{m_1 \phi}}{((d))_{m_1 \delta}} \Omega(f(m_1, \dots, m_s), m_2, \dots, m_s) \end{aligned}$$

$$\begin{aligned}
 & \times (-u_1 t)^{m_1} \frac{u_2^{m_2}}{m_2!} \dots \frac{u_s^{m_s}}{m_s!} (1-t)^{-ax-m_1} e^{bxt} f_{m_1}(x; a, b(1-t)) \\
 = & (1-t)^{-ax} e^{bxt} \sum_{m_1, m_2, \dots, m_s=0}^{\infty} \frac{((b))_{m_1} \phi}{((d))_{m_1} \delta} \Omega(f(m_1, \dots, m_s), m_2, \dots, m_s) \\
 & \times \left(-\frac{u_1 t}{1-t}\right)^{m_1} \frac{u_2^{m_2}}{m_2!} \dots \frac{u_s^{m_s}}{m_s!} \frac{(bx(1-t))^{m_1}}{m_1!} \sum_{k=0}^{m_1} (-m_1)_k (ax)_k \frac{(-bx(1-t))^{-k}}{k!} \\
 = & (1-t)^{-ax} e^{bxt} \\
 & \times \sum_{m_1, k, m_2, \dots, m_s=0}^{\infty} \frac{((b))_{(m_1+k)} \phi}{((d))_{(m_1+k)} \delta} \Omega(f((m_1+k), m_2, \dots, m_s), m_2, \dots, m_s) (ax)_k \\
 & \frac{(-u_1 bxt)^{m_1}}{m_1!} \frac{\left(\frac{u_1 t}{1-t}\right)^k}{k!} \frac{u_2^{m_2}}{m_2!} \dots \frac{u_s^{m_s}}{m_s!}.
 \end{aligned}$$

□

By appropriately choosing the multiple sequence $\Omega(m_1, m_2, \dots, m_s)$ in Theorem 5.1, we obtain several interesting results as follows which give bilateral generating functions for the generalized Sylvester polynomials and the generalized Lauricella (or the Srivastava-Daoust) functions.

I. By letting

$$\begin{aligned}
 & \Omega(f(m_1, \dots, m_s), m_2, \dots, m_s) \\
 = & \frac{\prod_{j=1}^A (a_1 j)_{m_1} \theta_j^{(1)} + \dots + m_s \theta_j^{(s)}}{\prod_{j=1}^E (c_j)_{m_1} \psi_j^{(1)} + \dots + m_s \psi_j^{(s)}} \frac{\prod_{j=1}^{B^{(2)}} (b_j^{(2)})_{m_2} \phi_j^{(2)}}{\prod_{j=1}^{D^{(2)}} (d_j^{(2)})_{m_2} \delta_j^{(2)}} \dots \frac{\prod_{j=1}^{B^{(s)}} (b_j^{(s)})_{m_s} \phi_j^{(s)}}{\prod_{j=1}^{D^{(s)}} (d_j^{(s)})_{m_s} \delta_j^{(s)}}
 \end{aligned}$$

in Theorem 5.1, we obtain the following result.

Corollary 5.1. *The following bilateral generating function holds true:*

$$\begin{aligned}
 & \sum_{n=0}^{\infty} f_n(x; a, b) F_{E:D;D^{(2)};\dots;D^{(s)}}^{A:B+1;B^{(2)};\dots;B^{(s)}} \\
 & \left(\begin{array}{l} [(a_1) : \theta^{(1)}, \dots, \theta^{(s)}] : [-n : 1], \quad [(b) : \phi]; \quad [(b^{(2)}) : \phi^{(2)}]; \quad \dots; \quad [(b^{(s)}) : \phi^{(s)}]; \\ [(c) : \psi^{(1)}, \dots, \psi^{(s)}] : \quad \quad \quad [(d) : \delta]; \quad [(d^{(2)}) : \delta^{(2)}]; \quad \dots; \quad [(d^{(s)}) : \delta^{(s)}]; \end{array} \right) t^n \\
 & \frac{u_1, u_2, \dots, u_s}{} \\
 = & (1-t)^{-ax} e^{bxt} F_{E+D:0;0;D^{(2)};\dots;D^{(s)}}^{A+B:0;1;B^{(2)};\dots;B^{(s)}} \\
 & \left(\begin{array}{l} [(e) : \varphi^{(1)}, \dots, \varphi^{(s+1)}] : \quad -, \quad [ax : 1]; \quad [(b^{(2)}) : \phi^{(2)}]; \quad \dots; \quad [(b^{(s)}) : \phi^{(s)}]; \\ [(f) : \xi^{(1)}, \dots, \xi^{(s+1)}] : \quad - \quad -; \quad [(d^{(2)}) : \delta^{(2)}]; \quad \dots; \quad [(d^{(s)}) : \delta^{(s)}]; \end{array} \right)
 \end{aligned}$$

$$\left. (-u_1 bxt), \left(\frac{u_1 t}{t-1}\right), u_2, \dots, u_s \right)$$

where the coefficients e_j , f_j , $\varphi_j^{(s)}$ and $\xi_j^{(s)}$ are given by

$$e_j = \begin{cases} a_{1j}, & (1 \leq j \leq A) \\ b_{j-A}, & (A < j \leq A + B) \end{cases}$$

$$f_j = \begin{cases} c_j, & (1 \leq j \leq E) \\ d_{j-E}, & (E < j \leq E + D) \end{cases}$$

$$\varphi_j^{(r)} = \begin{cases} \theta_j^{(1)} & (1 \leq j \leq A; 1 \leq r \leq 2) \\ \theta_j^{(r-1)} & (1 \leq j \leq A; 2 < r \leq s + 1) \\ \phi_{j-A} & (A < j \leq A + B; 1 \leq r \leq 2) \\ 0 & (A < j \leq A + B; 2 < r \leq s + 1) \end{cases}$$

and

$$\xi_j^{(r)} = \begin{cases} \psi_j^{(1)} & (1 \leq j \leq E; 1 \leq r \leq 2) \\ \psi_j^{(r-1)} & (1 \leq j \leq E; 2 < r \leq s + 1) \\ \delta_{j-E} & (E < j \leq E + D; 1 \leq r \leq 2) \\ 0 & (E < j \leq E + D; 2 < r \leq s + 1) \end{cases}$$

respectively.

II. Upon setting

$$\Omega(f(m_1, \dots, m_s), m_2, \dots, m_s) = \frac{(a_1)_{m_1+\dots+m_s} (b_2)_{m_2} \dots (b_s)_{m_s}}{(c_1)_{m_1} \dots (c_s)_{m_s}}$$

and

$$\phi = \delta = 0 \quad (\text{that is, } \phi_1 = \dots = \phi_B = \delta_1 = \dots = \delta_D = 0)$$

in Theorem 5.1, we obtain the following result.

Corollary 5.2. *The following bilateral generating function holds true:*

$$\begin{aligned} & \sum_{n=0}^{\infty} f_n(x; a, b) F_A^{(s)} [a_1, -n, b_2, \dots, b_s; c_1, \dots, c_s; u_1, u_2, \dots, u_s] t^n \\ = & (1-t)^{-ax} e^{bxt} F_{1:0;0;1;\dots;1}^{1:0;1;1;\dots;1} \\ & \left(\begin{array}{l} [(a_1) : 1, \dots, 1] : \quad -; \quad [ax : 1]; \quad [b_2 : 1]; \quad \dots; \quad [b_s : 1]; \\ [(c_1) : \psi^{(1)}, \dots, \psi^{(s+1)}] : \quad -; \quad -; \quad [c_2 : 1]; \quad \dots; \quad [c_s : 1]; \end{array} \right. \\ & \left. (-u_1 bxt), \left(\frac{u_1 t}{t-1}\right), u_2, \dots, u_s \right), \end{aligned}$$

where the coefficients $\psi^{(\eta)}$ are given by

$$\psi^{(\eta)} = \begin{cases} 1, & (1 \leq \eta \leq 2) \\ 0, & (2 < \eta \leq s + 1) \end{cases}$$

and $F_A^{(s)}$ is the first kind of Lauricella functions.

III. If we put

$$\Omega(f(m_1, \dots, m_s), m_2, \dots, m_s) = \frac{(a_1^{(1)})_{m_2} \dots (a_1^{(s-1)})_{m_s} (a_2^{(1)})_{m_2} \dots (a_2^{(s-1)})_{m_s}}{(c)_{m_1 + \dots + m_s}}$$

and

$$B = 1, \phi_1 = 1 \text{ and } \delta = 0$$

in Theorem 5.1, we obtain the following result.

Corollary 5.3. *The following bilateral generating function holds true:*

$$\begin{aligned} & \sum_{n=0}^{\infty} f_n(x; a, b) F_B^{(s)} \left[-n, a_1^{(1)}, \dots, a_1^{(s-1)}, b_1, a_2^{(1)}, \dots, a_2^{(s-1)}; c; u_1, u_2, \dots, u_s \right] t^n \\ &= (1-t)^{-ax} e^{bxt} F_{1:0;0;0;\dots;0}^{1:0;1;2;\dots;2} \\ & \left(\begin{array}{cccccc} [(b_1) : \theta^{(1)}, \dots, \theta^{(s+1)}] : & -; & [ax : 1]; & [a^{(1)} : 1]; & \dots; & [a^{(s-1)} : 1]; \\ [(c) : 1, \dots, 1] : & -; & -; & -; & \dots; & -; \end{array} \right. \\ & \left. (-u_1 bxt), \left(\frac{u_1 t}{t-1}\right), u_2, \dots, u_s \right), \end{aligned}$$

where the coefficients $\theta^{(\eta)}$ are given by

$$\theta^{(\eta)} = \begin{cases} 1, & (1 \leq \eta \leq 2) \\ 0, & (2 < \eta \leq k + 1) \end{cases}$$

and $F_B^{(s)}$ is the second kind of Lauricella functions.

IV. By letting

$$\Omega(f(m_1, \dots, m_s), m_2, \dots, m_s) = \frac{(a_1)_{m_1 + \dots + m_s} (b_2)_{m_2} \dots (b_s)_{m_s}}{(c)_{m_1 + \dots + m_s}}$$

and

$$\phi = \delta = 0,$$

in Theorem 5.1, we obtain the following result.

Corollary 5.4. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} f_n(x; a, b) F_D^{(s)} [a_1, -n, b_2, \dots, b_s; c; u_1, u_2, \dots, u_s] t^n$$

$$= (1-t)^{-ax} e^{bxt} F_D^{(s+1)} \left[a_1, 0, ax, b_2, \dots, b_s; c; (-u_1 bxt), \left(\frac{u_1 t}{t-1} \right), u_2, \dots, u_s \right]$$

and $F_D^{(s)}$ is the forth kind of Lauricella functions.

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