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PROJECTIVE CHANGE BETWEEN RANDERS METRIC AND EXPONENTIAL (α, β)-METRIC

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Abstract. In this paper, we find conditions to characterize the projective change between two (α, β)-metrics, such as exponential (α, β)-metric, $L = \alpha e^{\frac{\beta}{\alpha}}$ and Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$ on a manifold with dim $n > 2$, where α and $\overline{\alpha}$ are two Riemannian metrics, β and $\overline{\beta}$ are two non-zero 1-forms. We also discuss special curvature properties of two classes of (α, β)-metrics.

Keywords: Finsler space, (α, β)-metric, projective change, Randers metric, Berwlad, Riemannian metric.

1. Introduction

M. Matsumoto [10] introduced the concept of (α, β)-metric on a differentiable manifold with local coordinates x^i, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n. M. Hashiguchi and Y. Ichijyo [6] studied some special (α, β)-metrics and obtained interesting results. In the projective Finsler geometry, there is a remarkable theorem called Rapcsak [14] theorem, which plays an important role in the projective geometry of Finsler spaces. In fact, this theorem gives the necessary and sufficient condition for a Finsler space to be projective to another Finsler space.

The projective change between two Finsler spaces has been studied by many authors ([2], [5], [8], [11], [12], [16]). In 1994, S. Bacso and M. Matsumoto [2] studied the projective change between Finsler spaces with (α, β)-metric. In 2008, H. S. Park and Y. Lee [11] studied the projective changes between a Finsler space with (α, β)-metric and the associated Riemannian metric. The authors Z. Shen and Civi Yildirim [16] studied a class of projectively flat metrics with a constant flag curvature. In 2009, Ningwei Cui and Yi-Bing Shen [5] studied projective change between two classes of (α, β)-metrics. Also the author N. Cui [4] studied the S-curvature of some (α, β)-metrics. In this paper, we find conditions to characterize

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the projective change between two (α,β)-metrics, such as the exponential (α,β)-metric, $L=\alpha e^{\frac{\beta}{\alpha}}$ and Randers metric $\overline{L}=\overline{\alpha}+\overline{\beta}$ on a manifold with dim $n>2$, where α and $\overline{\alpha}$ are two Riemannian metrics, β and $\overline{\beta}$ are two non-zero 1-forms. In addition, we discuss special curvature properties of two classes of (α,β)-metrics.

2. Preliminaries

The terminology and notation are referred to ([15], [9], [1]). Let M^n be a real smooth manifold of dimension n and let $F^n=(M^n,L)$ be a Finsler space on the differentiable manifold M^n endowed with the fundamental function $L(x,y)$. We use the following notation:

$$\begin{cases} g_{ij}=\frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \\ C_{ijk}=\frac{1}{2}\dot{\partial}_k g_{ij}, \\ h_{ij}=g_{ij}-l_i l_j, \\ \gamma^i_{jk}=\frac{1}{2}g^{ir}(\partial_j g_{rk}+\partial_k g_{rj}-\partial_r g_{jk}), \\ G^i=\frac{1}{2}\gamma^i_{jk}y^j y^k,\ G^i_j=\dot{\partial}_j G^i, \\ G^i_{jk}=\dot{\partial}_k G^i_j,\ G^i_{jkl}=\dot{\partial}_l G^i_{jk}, \end{cases} \tag{2.1}$$

where $\dot{\partial}_i \equiv \frac{\partial}{\partial y^i}$.

Definition 2.1. A change $L\to\overline{L}$ of a Finsler metric on the same underlying manifold M is called projective change if any geodesic in (M,L) remains to be geodesic in $(M,\overline{L})$ and vice versa.

A Finsler metric is projectively related to another metric if they have the same geodesic as point sets. In Riemannian geometry, two Riemannian metrics α and $\overline{\alpha}$ are projectively related if and only if their spray coefficients have the relation [5]

$$G^i_\alpha=G^i_{\overline{\alpha}}+\lambda_{x^k}y^k y^i, \tag{2.2}$$

where $\lambda=\lambda(x)$ is a scalar function on the based manifold.

Two Finsler metric F and $\overline{F}$ are projectively related if and only if their spray coefficients have the relation [5]

$$G^i=\overline{G}^i+P(y)y^i, \tag{2.3}$$

where $P(y)$ is a scalar function and homogeneous of degree one in y^i.

Definition 2.2. A Finsler metric is called a projectively flat metric if it is projectively related to a locally Minkowskian metric.

For a given Finsler metric $L=L(x,y)$, the geodesic of L is given by

$$\frac{d^2x^i}{dt^2}+2G^i(x,\frac{dx}{dt})=0, \tag{2.4}$$

where $G^i = G^i(x, y)$ are called geodesic coefficients, which are given by

$$G^i = \frac{g^{il}}{4}\Big\{[L^2]_{x^m y^l} y^m - [L^2]_{x^l}\Big\}. \tag{2.5}$$

Let $\phi = \phi(s)$, $|s| < b_0$, be a positive C^∞ satisfying the following

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \ (|s| \le b < b_0). \tag{2.6}$$

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric, $\beta = b_i y^i$ is a 1-form satisfying $\|\beta_x\|_\alpha < b_0$ for all $x \in M$, then $L = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, is called an (regular) (α, β)-metric. In this case, the fundamental form of the metric tensor induced by L is positive definite. Let $\nabla\beta = b_{i|j} dx^i \otimes dx^j$ be the covariant derivative of β with respect to α.
Denote

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}) \ \ s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

β is closed if and only if $s_{ij} = 0$ [17]. Let $s_j = b^i s_{ij}$, $s^i_j = a^{il} s_{lj}$, $s_0 = s_i y^i$, $s^i_0 = s^i_j y^j$ and $r_{00} = r_{ij} y^i y^j$.

The relation between the geodesic coefficient G^i of L and the geodesic coefficient G^i_α of α is given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{r_{00} - 2Q\alpha s_0\}\{\psi b^i + \Theta\alpha^{-1} y^i\}, \tag{2.7}$$

where

$$\begin{aligned} \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \\ Q &= \frac{\phi'}{\phi - s\phi'}, \\ \psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

Definition 2.3. [5] Let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right), \tag{2.8}$$

where G^i is the spray coefficient of L. The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called a Douglas metric if the Douglas tensor vanishes.

We know that the Douglas tensor is a projective invariant [13]. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes (2.8). This shows that the Douglas tensor is a non-Riemannian quantity. In what follows, we use quantities with a bar to denote the corresponding quantities of metric $\overline{L}$. We compute the Douglas tensor of a general (α,β)-metric. Let

$$\widehat{G}^i = G^i_\alpha + \alpha Q s^i_0 + \psi\{r_{00} - 2Q\alpha s_0\}b^i. \tag{2.9}$$

Using (2.9) in (2.7), we have

$$G^i = \widehat{G}^i + \Theta\{r_{00} - 2Q\alpha s_0\}\alpha^{-1}y^i. \tag{2.10}$$

Clearly, G^i and $\widehat{G}^i$ are projective equivalents according to (2.3). They have the same Douglas tensor. Let

$$T^i = \alpha Q s^i_0 + \psi\{r_{00} - 2Q\alpha s_0\}b^i. \tag{2.11}$$

Then $\widehat{G}^i = G^i_\alpha + T^i$, thus

$$\begin{aligned} D^i_{jkl} &= \widehat{D}^i_{jkl} \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m}y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right). \end{aligned} \tag{2.12}$$

To simplify (2.12), we use the following identities

$$\alpha_{y^k} = \alpha^{-1}y_k, \;\; s_{y^k} = \alpha^{-2}(b_k\alpha - sy_k),$$

where $y_i = a_{il}y^l$, $\alpha_{y^k} = \frac{\partial\alpha}{\partial y^k}$. Then

$$\begin{aligned} [\alpha Q s^m_0]_{y^m} &= \alpha^{-1}y_m Q s^m_0 + \alpha^{-2}Q'[b_m\alpha^2 - \beta y_m]s^m_0 \\ &= Q's_0 \end{aligned}$$

and

$$\begin{aligned} \psi(r_{00} - 2Q\alpha s_0)b^m]_{y^m} &= \psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &+ 2\psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0], \end{aligned}$$

where $r_j = b^i r_{ij}$ and $r_0 = r_i y^i$. Thus from (2.11), we get

$$\begin{aligned} T^m_{y^m} &= Q's_0 + \psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &+ 2\psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \end{aligned} \tag{2.13}$$

We assume that the (α,β)-metrics L and $\overline{L}$ have the same Douglas tensor, i.e., $D^i_{jkl} = \widehat{D}^i_{jkl}$. Thus from (2.8) and (2.12), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \overline{T}^i - \frac{1}{n+1}(T^m_{y^m} - \overline{T}^m_{y^m})y^i\right) = 0.$$

Then there exists a class of scalar functions $H^i_{jk} = H^i_{jk}(x)$, such that

$$H^i_{00} = T^i - \overline{T}^i - \frac{1}{n+1}(T^m_{y^m} - \overline{T}^m_{y^m})y^i, \tag{2.14}$$

where $H^i_{00} = H^i_{jk}y^j y^k$.

Theorem 2.1. *[4] For the special form of (α, β)-metric, $L = \alpha + \epsilon\beta + k\left(\frac{\beta^2}{\alpha}\right)$, where ϵ, k are non-zero constant, the following are equivalent:*

- *L has an isotropic S-curvature, i.e., $S = (n+1)c(x)L$ for some scalar function $c(x)$ on M.*
- *L has an isotropic mean Berwald curvature.*
- *β is a killing one form of constant length with respect to α. This is equivalent to $r_{00} = s_0 = 0$.*
- *L has a vanished S-curvature, i.e., $S = 0$.*
- *L is a weak Berwald metric, i.e., $E = 0$.*

3. Projective Change between Randers Metric and Exponential (α, β)-metric

In this section, we find the projective relation between two (α, β)-metrics on the same underlying manifold M of dimension $n > 2$. For (α, β)-metric $L = \alpha e^{\frac{\beta}{\alpha}}$, one can prove by (2.6) that L is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha < 1$ for any $x \in M$. The geodesic coefficient are given by (2.7) with

$$\begin{cases} \Theta = \frac{1-2s}{2(1+b^2-s-s^2)}, \\ Q = \frac{1}{1-s}, \\ \psi = \frac{1}{2(1+b^2-s-s^2)}. \end{cases} \tag{3.1}$$

Using (3.1) in (2.7), we get

$$\begin{aligned} G^i &= G^i_\alpha + \frac{\alpha^2}{\alpha - \beta}s^i_0 + \frac{1}{2(\alpha^2 - \beta^2 + \alpha^2 b^2 - \alpha\beta)}\left[r_{00} - \frac{2\alpha^2}{\alpha - \beta}s_0\right] \\ &\times \quad [\alpha^2 b^i + (\alpha - 2\beta)y^i]. \end{aligned} \tag{3.2}$$

For the Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$, one can also prove by (2.6) that $\overline{L}$ is a regular Finsler metric if and only if $\|\beta_x\|_\alpha < 1$ for any $x \in M$. The geodesic coefficients are given by (2.7) with

$$\overline{\Theta} = \frac{1}{2(1+s)}, \quad \overline{Q} = 1, \quad \overline{\psi} = 0. \tag{3.3}$$

First, we prove the following lemma:

Lemma 3.1. *Let $L = \alpha e^{\frac{\beta}{\alpha}}$ and $\overline{L} = \overline{\alpha} + \overline{\beta}$ be two (α, β)-metrics on a manifold M with dimension $n > 2$. Then they have the same Douglas tensor if and only if both metrics are Douglas metrics.*

Proof. First, we prove the sufficient condition. Let L and $\overline{L}$ be Douglas metrics and the corresponding Douglas tensor D^i_{jkl} and $\widehat{D}^i_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\widehat{D}^i_{jkl} = 0$, that is, both metrics have the same Douglas tensor. Next, we prove the necessary condition. If L and $\overline{L}$ have the same Douglas tensor, then (2.14) holds.
Using (2.13), (3.1) and (3.3) in (2.14), we have

$$H^i_{00} = \frac{A^i\alpha^7 + B^i\alpha^6 + C^i\alpha^5 + D^i\alpha^4 + E^i\alpha^3 + F^i\alpha^2}{K\alpha^6 + U\alpha^5 + M\alpha^4 + N\alpha^3 + V\alpha^2 + R} - \overline{\alpha}\,\overline{s}^i_0, \tag{3.4}$$

where

$$\begin{aligned}
A^i &= (1+b^2)[2s^i_0(1+b^2) - 2s_0],\\
B^i &= (1+b^2)[r_{00}b^i - 2\beta(3+b^2)s^i_0 + 2\beta s_0 b^i\\
&- 2\lambda s s_0(1+s)y^i - 2\lambda r_0 y^i] - 2\lambda s_0 y^i,\\
C^i &= \beta(3+2b^2)(2\lambda r_0 y^i - r_{00}b^i) - 2\lambda\beta s s_0(2+b^2)y^i\\
&+ 4\lambda\beta s_0(1+b^2)y^i + 2\beta^2(1-2b^2)s^i_0 - \lambda b^2 r_{00} y^i,\\
D^i &= 2\beta^3(3+2b^2)s^i_0 + r_{00}\beta^2(2+b^2)b^i\\
&+ 2\lambda\beta(\beta s_0 + 2\beta s^2 s_0 - \beta b^2 r_0 - 2\beta r_0 - s^2 r_{00})y^i,\\
E^i &= \beta^2 r_{00}[\beta b^i + \lambda(3b^2 - 4s^2 - 2\beta b^2)y^i]\\
&+ 2\lambda\beta^3(s s_0 - r_0 - 2s_0)y^i - 2\beta^4 s^i_0,\\
F^i &= 2\lambda\beta^3(\beta r_0 - \beta s_0 + s^2 r_{00})y^i\\
&- 2\beta^5 s^i_0 - \beta^4 r_{00} b^i,\\
\lambda &= \frac{1}{n+1}
\end{aligned}$$

and

$$\begin{aligned}
K &= 2(1+b^2)^2,\ \ U = 4\beta(b^4 - 3b^2 - 2),\ \ M = 2\beta^2(b^2+2)^2,\\
N &= 4\beta^3(1+b^2),\ \ V = -4\beta^4(2+b^2),\ \ R = 2\beta^6.
\end{aligned}$$

Then (3.4) is equivalent to

$$\begin{aligned}
&A^i\alpha^7 + B^i\alpha^6 + C^i\alpha^5 + D^i\alpha^4 + E^i\alpha^3 + F^i\alpha^2\\
&= (K\alpha^6 + U\alpha^5 + M\alpha^4 + N\alpha^3 + V\alpha^2 + R)(H^i_{00} + \overline{\alpha}\,\overline{s}^i_0).
\end{aligned} \tag{3.5}$$

Replacing y^i in (3.5) by $-y^i$, we have

$$\begin{aligned}
&- A^i\alpha^7 + B^i\alpha^6 - C^i\alpha^5 + D^i\alpha^4 - E^i\alpha^3 + F^i\alpha^2\\
&= (K\alpha^6 - U\alpha^5 + M\alpha^4 - N\alpha^3 + V\alpha^2 + R)(H^i_{00} - \overline{\alpha}\,\overline{s}^i_0).
\end{aligned} \tag{3.6}$$

Subtracting (3.6) from (3.5), we get

$$\begin{aligned} A^i\alpha^7 + C^i\alpha^5 + E^i\alpha^3 &= (U\alpha^5 + N\alpha^3)H^i_{00} \\ &+ (K\alpha^6 + M\alpha^4 + V\alpha^2 + R)\overline{\alpha}\ \overline{s}^i_0. \end{aligned} \tag{3.7}$$

From (3.7), we have

$$\begin{aligned} &\alpha^2[A^i\alpha^5 + C^i\alpha^3 + E^i\alpha - (U\alpha^3 + N\alpha)H^i_{00} \\ &-\overline{\alpha}\ \overline{s}^i_0(K\alpha^4 + M\alpha^2 + V)] = R\overline{\alpha}\ \overline{s}^i_0. \end{aligned} \tag{3.8}$$

From (3.8), $R\,\overline{\alpha}\ \overline{s}^i_0$ has the factor α^2, i.e., the term $R\,\overline{\alpha}\ \overline{s}^i_0 = 2\beta^6\,\overline{\alpha}\ \overline{s}^i_0$ has the factor α^2. We can study two cases for Riemannian metric.

Case (i): If $\overline{\alpha} \neq \mu(x)\alpha$, then $R\,\overline{s}^i_0 = 2\beta^6\overline{s}^i_0$ has the factor α^2. Note that β^2 has no factor α^2. Then the only possibility is that $\beta\overline{s}^i_0$ has the factor α^2. Then for each i there exists a scalar function $\eta^i = \eta^i(x)$ such that $\beta\overline{s}^i_0 = \eta^i\alpha^2$ which is equivalent to $b_j\overline{s}^i_k + b_k\overline{s}^i_j = 2\eta^i\alpha_{jk}$. When $n > 2$ and we assume that $\eta^i \neq 0$, then

$$\begin{aligned} 2 &\geq rank(b_j\overline{s}^i_k) + rank(b_k\overline{s}^i_j) \\ &> rank(b_j\overline{s}^i_k + b_k\overline{s}^i_j) \\ &= rank(2\eta^i\alpha_{jk}) > 2, \end{aligned}$$

which is impossible unless $\eta^i = 0$. Then $\beta\overline{s}^i_0 = 0$. Since $\beta \neq 0$, we have $\overline{s}^i_0 = 0$, which says that $\overline{\beta}$ is closed.

Case (ii): If $\overline{\alpha} = \mu(x)\alpha$, then (3.7), becomes

$$\begin{aligned} R\mu(x)\overline{s}^i_0 &= \alpha^2[A^i\alpha^4 + C^i\alpha^2 + E^i - (U\alpha^2 + N)H^i_{00} \\ &- \mu(x)\overline{s}^i_0(K\alpha^4 + M\alpha^2 + V)]. \end{aligned} \tag{3.9}$$

From (3.9), we can see that $\mu(x)R\overline{s}^i_0$ has the factor α^2. i.e., $\mu(x)R\overline{s}^i_0 = 2\mu(x)\overline{s}^i_0\beta^6$ has the factor α^2. Note that $\mu(x) \neq 0$ for all $x \in M$ and β^2 has no factor α^2. The only possibility is that $\beta\overline{s}^i_0$ has the factor α^2. As the similar reason in case (i), we have $\overline{s}^i_0 = 0$, when $n > 2$, which says that $\overline{\beta}$ is closed.

M. Hashiguchi [7] proved that the Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$ is a Douglas metric if and only if $\overline{\beta}$ is closed. Thus $\overline{L}$ is a Douglas metric. Since L is projectively related to $\overline{L}$, then both L and $\overline{L}$ are Douglas metrics. □

Theorem 3.1. *The Finsler metric $L = \alpha e^{\frac{\beta}{\alpha}}$ is projectively related to $\overline{L} = \overline{\alpha} + \overline{\beta}$ if and only if the following conditions are satisfied*

$$\begin{cases} G^i_\alpha = G^i_{\overline{\alpha}} + \theta y^i - \tau\xi\alpha^2 b^i, \\ b_{i|j} = \tau[(1 + 2b^2)a_{ij} - 3b_ib_j], \\ d\overline{\beta} = 0, \end{cases} \tag{3.10}$$

where $b^i = a^{ij}b_j$, $b = \|\beta\|_\alpha$, $b_{i|j}$ denotes the coefficient of the covariant derivatives of β with respect to α, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on a manifold M with dimension $n > 2$.

Proof. First, we prove the necessary condition. Since the Douglas tensor is invariant under projective changes between two Finsler metrics, if L is projectively related to $\overline{L}$, then they have the same Douglas tensor. According to Lemma 3.1, we obtain that both L and $\overline{L}$ are Douglas metrics.

We know that the Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$ is a Douglas metric if and only if $\overline{\beta}$ is closed, i.e., $d\overline{\beta} = 0$.
The Finsler metric $L = \alpha e^{\frac{\beta}{\alpha}}$ is a Douglas metric if and only if

$$b_{i|j} = \tau[(1 + 2b^2) - 3b_i b_j], \tag{3.11}$$

for some scalar function $\tau = \tau(x)$ [3], where $b_{i|j}$ denotes the coefficient of the covariant derivatives of $\beta = b_i y^i$ with respect to α. In this case, β is closed. Since β is closed, $s_{ij} = 0 \Rightarrow b_{i|j} = b_{j|i}$. Thus $s^i_0 = 0$ and $s_0 = 0$.
By using (3.11), we have $r_{00} = \tau[(1 + 2b^2)\alpha^2 - 3\beta^2]$. Substituting all these in (3.2), we get

$$G^i = G^i_\alpha + \tau\frac{[(1 + 2b^2)(\alpha^3 - 2\alpha^2\beta) - 3\alpha\beta^2 + 6\beta^3]}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)} y^i + \tau\xi\alpha^2 b^i, \tag{3.12}$$

where $\xi = \frac{\tau[(1+2b^2)\alpha^2 - 3\beta^2]b^i}{2(\alpha^2-\beta^2+b^2\alpha^2-\alpha\beta)}$.
Since L is projectively related to $\overline{L}$, this is a Randers change between L and $\overline{\alpha}$. Noticing that $\overline{\beta}$ is closed, then L is projectively related to $\overline{\alpha}$. Thus, there is a scalar function $P = P(y)$ on $TM - \{0\}$ such that

$$G^i = G^i_{\overline{\alpha}} + P y^i. \tag{3.13}$$

From (3.12) and (3.13), we have

$$\left[P + \frac{3\alpha\beta^2 - 6\beta^3 - (1 + 2b^2)(\alpha^3 - 3\alpha^2\beta)}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}\right] y^i = G^i_\alpha - G^i_{\overline{\alpha}} + \tau\xi\alpha^2 b^i. \tag{3.14}$$

Note that the RHS of the above equation is a quadratic form. Then there must be one form $\theta = \theta_i y^i$ on M, such that

$$P + \frac{3\alpha\beta^2 - 6\beta^3 - (1 + 2b^2)()\alpha^3 - 3\alpha^2\beta}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)} = \theta.$$

Thus (3.14) becomes

$$G^i_\alpha = G^i_{\overline{\alpha}} + \theta y^i - \tau\xi\alpha^2 b^i. \tag{3.15}$$

Equations (3.11) and (3.12) together with (3.15) complete the proof of the necessity. Since $\overline{\beta}$ is closed, it suffices to prove that L is projectively related to $\overline{\alpha}$. From (3.12) and (3.15), we have

$$G^i = G^i_{\overline{\alpha}} + \left[\theta + \frac{\tau[(1 + 2b^2)(\alpha^3 - 3\alpha^2\beta) - 3\alpha\beta^2 + 6\beta^3]}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}\right] y^i,$$

that is, L is projectively related to $\overline{\alpha}$ □

From the above theorem, we get the following corollaries.

Corollary 3.1. *The Finsler metric $L=\alpha e^{\frac{\beta}{\alpha}}$ is projectively related to $\overline{L}=\overline{\alpha}+\overline{\beta}$ if and only if they are Douglas metrics and the spray coefficients of α and $\overline{\alpha}$ have the following relation*

$$G^i_{\alpha}=G^i_{\overline{\alpha}}+\theta y^i-\tau\xi\alpha^2 b^i,$$

where $b^i=a^{ij}b_j$, $\tau=\tau(x)$ is a scalar function and $\theta=\theta_i y^i$ is one form on a manifold M with dimension $n\geq 2$.

Further, we assume that the Randers metric $\overline{L}=\overline{\alpha}+\overline{\beta}$ is locally Minkowskian, where $\overline{\alpha}$ is a Euclidean metric and $\overline{\beta}=\overline{b}_i y^i$ is one form with $\overline{b}_i$ =constant. Then (3.10) can be written as

$$\begin{cases} G^i_{\alpha}=\theta y^i-\tau\xi\alpha^2 b^i,\\ b_{i|j}=\tau[(1+2b^2)a_{ij}-3b_ib_j]. \end{cases}\tag{3.16}$$

Thus, we state

Corollary 3.2. *The Finsler metric $\alpha e^{\frac{\beta}{\alpha}}$ is projectively related to $\overline{L}=\overline{\alpha}+\overline{\beta}$ if and only if L is projectively flat, that is, L is projectively flat if and only if* (3.16) *holds.*

4. Special Curvature Properties of two (α,β)-metrics

We know that the Berwald curvature tensor of a Finsler metric L is defined by [9]

$$G=G^i_{jkl}dx^j\otimes\partial_i\otimes dx^k\otimes dx^l,\tag{4.1}$$

where $G^i_{jkl}=[G^i]_{y^jy^ky^l}$ and G^i are the spray coefficients of L. The mean Berwald curvature tensor is defined by

$$E=E_{ij}dx^i\otimes dx^j,\tag{4.2}$$

where $E_{ij}=\frac{1}{2}G^m_{mij}$.
A Finsler space is said to be of the isotropic mean Berwald curvature if

$$E_{ij}=\frac{n+1}{2}c(x)L_{y^iy^j},\tag{4.3}$$

where $c(x)$ is scalar function on M.

In this section, we assume that (α,β)-metric $L=\alpha e^{\frac{\beta}{\alpha}}$ has some special curvature properties.

Theorem 4.1. *The Finsler metric $L=\alpha e^{\frac{\beta}{\alpha}}$ having an isotropic S-curvature or isotropic mean Berwald curvature is projectively related to $\overline{L}=\overline{\alpha}+\overline{\beta}$ if and only if the following conditions hold:*

- *α is projectively related to $\overline{\alpha}$,*
- *β is parallel with respect to α, i.e., $b_{i|j} = 0$,*
- *$\overline{\beta}$ is closed, i.e., $d\overline{\beta} = 0$,*

where $b_{i|j}$ denotes the coefficient of the covariant derivative of β with respect to α.

Proof. The sufficiency is obvious from Theorem (3.2). For the necessary condition, from Theorem 3.1, if L is projectively related to $\overline{L}$, then

$$b_{i|j} = \tau[(1 + 2b^2)a_{ij} - 3b_i b_j],$$

where $\tau = \tau(x)$ is scalar function. Transvecting the above equation with y^i and y^j, we have

$$r_{00} = \tau[(1 + 2b^2)\alpha^2 - 3\beta^2]. \tag{4.4}$$

From Theorem 2.4, if L has an isotropic S-curvature or an equivalently isotropic mean Berwald curvature, then $r_{00} = 0$. If $\tau \neq 0$, then (4.4) gives

$$(1 + 2b^2)\alpha^2 - 3\beta^2 = 0, \tag{4.5}$$

which is equivalent to

$$(1 + 2b^2)a_{ij} - 3b_i b_j = 0. \tag{4.6}$$

Transvecting (4.6) with a^{il}, we get

$$(1 + 2b^2)\delta^l_j - 3b^l b_j = 0. \tag{4.7}$$

Contracting l and j in (4.7), we have $n + (2n - 3)b^2 = 0$, which is impossible. Thus $\tau = 0$. Substituting in Theorem 3.2, we complete the proof. □

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