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SOME NOTES CONCERNING TACHIBANA AND VISHNEVSKII OPERATORS IN THE TANGENT BUNDLE

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Abstract. The main aim of the present paper is to study Tachibana and Vishnevskii operators for the Lorentzian almost r-para-contact structure in the tangent bundle.
Keywords: Tangent bundle, Vertical lift, Complete lift, Lie derivative, Tachibana operator, Vishnevskii operator.

1. Introduction

The study of differential geometry of the tangent bundle is a very fruitful domain of differential geometry because the theory provides many new problems in modern differential geometry. The study of differential geometry of the tangent bundle started promptly in 1960s by Davis, Sasaki, Yano and Davis, Tachibana and many others. Yano and Ishihara have studied vertical, complete and horizontal lifts of tensors and connection. The first author studied lifts of a hypersurface with connections to tangent bundles and a Kähler manifold in 2014 [7] and 2016 [8]. Also, different structures on tanent bundles have been studied by several authors such as Das and the first author (2005) [4], Tekkoyun(2006) [6], the first author(2017) [15] and many others.

I. Sato [17] introduced the notion of almost contact structure on differential geometry. An almost paracontact Riemannian manifold and an almost product Riemannian manifold were studied by Adati [19] while the almost r-contact structure was introduced by Vanzura [10]. In [13], Motsumoto initiated the study of Lorentzian paracontact manifolds. The Lorentzian almost r-paracontact structure in the tangent bundle was studied by Khan and Jun [14].

The paper is organized as follows: In Section 2, we recall some basic definitions of vertical, complete and horizontal lifts and the Lie derivative. Section 3 deals with Tachibana and Vishnevskii operators associated with the Lorentzian almost r-para-contact structure in the tangent bundle.

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2. Preliminaries

Let M be an n-dimensional differentiable manifold and let $T(M) = \bigcup_{p\in M} T_p(M)$ be its tangent bundle. Then $T(M)$ is also a differentiable manifold [1]. Let $X = \sum_{i=1}^{n} x^i(\frac{\partial}{\partial x^i})$ and $\eta = \sum_{i=1}^{n} \eta^i dx^i$ be the expressions in local coordinates for the vector field X and the 1-form η in M. Let (x^i, y^i) be local coordinates of point in $T(M)$ induced naturally from the coordinate chart $U(x^i)$ in M.

2.1. Vertical lifts

If f is a function in M, we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \longrightarrow M$ and $f : M \longrightarrow R$, so that

$$f^V = f o \pi \tag{2.1}$$

Thus, if a point $\tilde{p} \epsilon \pi^{-1}(U)$ has induced the coordinates (x^h, y^h) then

$$f^V(\tilde{p}) = f^V(x, y) = f o \pi(\tilde{p}) = f(p) = f(x) \tag{2.2}$$

Thus the value of $f^V(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f . Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\tau(M)$ into the tensor algebra $\tau(T(M))$ with respect to constant coefficients by the conditions

$$(P \otimes Q)^V = P^V \otimes Q^V, (P + R)^V = P^V + R^V \tag{2.3}$$

P, Q and R being arbitrary elements of $\tau(M)$ [3].

Furthermore, the vertical lifts of tensor fields obey the general properties [1, 2]:

$$\begin{array}{ll} \text{(a)} & (f.g)^V = f^V g^V, (f + g)^V = f^V + g^V \\ \text{(b)} & (X + Y)^V = X^V + Y^V, (f.X)^V = f^V X^V, X^V f^V = 0, [X^V, Y^V] = 0 \\ \text{(c)} & (f.\eta)^V = f^V \eta^V, \eta^V(X^V) = 0, X^V(Y^V) = 0, \end{array}$$

$\forall f, g \in \tau_0^0(M), X, Y \in \tau_0^1(M), \phi \in \tau_1^1(M)$.

2.2. Complete lifts

If f is a function in M, we write f^C for the function in $T(M)$ defined by

$$f^C = i(df)$$

and call f^C the complete lift of the function f. The complete lift f^C of a function f has the local expression

$$f^C = y^i \partial_i f = \partial f$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \tau_0^1(M)$. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C$$

f being an arbitrary function in M and call X^C the complete lift of X in $T(M)$.

The complete lift X^C of X with components x^h in M has components

$$X^C : \begin{bmatrix} x^h \\ \partial x^h \end{bmatrix}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\eta \in \tau_0^1(M)$ Then a 1-form η^C in $T(M)$ defined by

$$\eta^C(X^C) = (\eta(X))^C$$

X being an arbitrary vector field in M. We call η^C the complete lift of η.

The complete lifts to a unique algebra isomorphism of the tensor algebra $\tau(M)$ into the tensor algebra $\tau(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C, (P + R)^C = P^C + R^C$$

P, Q and R being arbitrary elements of $\tau(M)$.

Moreover, the complete lifts of tensor fields obey the general properties [1, 2]:

(a) $$(fX)^C = f^C X^V + f^V X^C = (Xf)^C, X^C f^V = (Xf)^V, X^V f^C = (Xf)^V,$$

(b) $$\phi^V XC = (\phi X)^V, \phi^C X^V = (\phi X)^V, (\phi X)^C = \phi^C X^C;$$

(c) $$\eta^V X^C = (\eta(X))^C, \eta^C X^V = (\eta(X))^V$$

(d) $$[X^V, Y^C] = [X, Y]^C, I^C = I, I^V I^C = X^V, [X^C, Y^C] = [X, Y]^C$$

$\forall f, g \in \tau_0^0(M), X, Y \in \tau_0^1(M), \phi \in \tau_1^1(M).$

2.3. Horizontal lifts

Let (x^h, y^h) be a local coordinate system in an open set $\pi^{-1}(U) \subset T(M)$ where U is an arbitrary coordinate neighborhood in M. Suppose that a tensor field S in M by

$$S = S^{i,\ldots,h}_{l,k,\ldots,j} \frac{\partial}{\partial x^i} \otimes \ldots\ldots \otimes \frac{\partial}{\partial x^h} \otimes dx^l \otimes dx^k \otimes \ldots\ldots \otimes dx^j$$

and a tensor field $\gamma_x S$ in $\pi^{-1}(U)$ by

$$\gamma_x S = \left(X^l S^{i,\dots,h}_{l,k,\dots,j}\right) \frac{\partial}{\partial y^i} \otimes \dots\dots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \dots\dots \otimes dx^j$$

and a tensor field γS in $\pi^{-1}(U)$ by

$$\gamma S = \left(y^l S^{i,\dots,h}_{l,k,\dots,j}\right) \frac{\partial}{\partial y^i} \otimes \dots\dots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \dots\dots \otimes dx^j.$$

The tensor fields $\gamma_x S$ and γS defined in each $\pi^{-1}(U)$ determine respectively global tensor fields in $T(M)$.

Let ∇ be an affine connection in M. If f is a function in M then the gradient of f dented by ∇f in M.

Apply the operation γ to γf and get $\gamma(\nabla f)$.

Put

$$\nabla_\gamma f = \gamma(\nabla f).$$

The horizontal lift f^H of $f \in \tau^0_0(M)$ to the tangent bundle $T(M)$ by

$$(f)^H = f^C - \nabla_\gamma f \tag{2.4}$$

Let $X \in \tau^1_0(M)$. Then the horizontal lift X^H of X defined by

$$X^H = X^C - \nabla_\gamma X \tag{2.5}$$

in $T(M)$, where

$$\nabla_\gamma X = \gamma(\nabla X)$$

The horizontal lift X^H of X has the components

$$\begin{bmatrix} x^h \\ -\Gamma^h_i x^i \end{bmatrix} \tag{2.6}$$

with respect to the induced coordinates in $T(M)$, where $\Gamma^h_i = y^j \Gamma^h_{ji}$.

Suppose that $\eta \in \tau^0_1(M)$. Then the 1-form η^C in $T(M)$ defined by the horizontal lift S^H of the tensor field S of an arbitrary type in M to $T(M)$ is defined by

$$S^H = S^C - \nabla_\gamma S \tag{2.7}$$

for all $P, Q \in \tau(M)$. We have

$$\nabla_\gamma (P \otimes Q) = (\nabla_\gamma P) \otimes Q^V + P^V \otimes (\nabla_\gamma Q)$$

or

$$(P \otimes Q)^H = P^H \otimes Q^V + P^V \otimes Q^H. \tag{2.8}$$

In addition, the horizontal lifts of tensor fields obey the general properties [1, 2]:

$$\begin{array}{ll} \text{(a)} & X^H f^V = (Xf)^V, F^V X^H = (FX)^V, F^C X^H = (FX)^H + (\nabla_\gamma F)X^H \\ \text{(b)} & \eta^V(X^H) = (\eta(X))^H, \eta^C(X^H) = (\eta(X))^C - \gamma(\eta \circ (\nabla X), ; \\ \text{(c)} & \eta^H(X^C) = \eta^H(\nabla_\gamma X), \eta^H(X^H) = 0 \end{array}$$

$\forall f, g \in \tau_0^0(M), X, Y \in \tau_0^1(M), \eta \in \tau_1^0(M), F \in \tau_1^1(M).$

Let$X$ be a vector field in an n-dimensional differentiable manifold M. The differential transformation L_X is called the Lie derivative with respect to X if

$$\begin{array}{ll} \text{(a)} & L_X f = Xf, \forall f \in \tau_0^0(M) \\ \text{(b)} & L_X Y = [X, Y]. \end{array}$$

The Lie derivative $L_X F$ of a tensor field F of type (1, 1) with respect to a vector field X is defined by [1]

$$(L_X F) = [X, FY] - F[X, Y] \tag{2.9}$$

where $[,]$ is the Lie bracket.

Let M be an n-dimensional differentiable manifold. Differential transformation of algebra $T(M)$ defined by

$$D = \nabla_X : T(M) \to T(M), X \in \tau_0^1(M), \tag{2.10}$$

is called covariant derivation with respect to a vector field X if

$$\begin{array}{ll} \text{(a)} & \nabla_{fX+gY} t = f\nabla_X t + g\nabla_Y t, \\ \text{(b)} & \nabla_X f = Xf, \forall f, g \in \tau_0^0(M), \forall X, Y \in \tau_0^1(M), \forall t \in \tau(M). \end{array}$$

and a transformation defined by

$$\nabla : \tau_0^1(M) \times \tau_0^1(M) \to \tau_0^1(M) \tag{2.11}$$

is called affine connection [1].

Proposition 2.1. *For any $X, Y \in \tau_0^1(M)$[1]*

$$\begin{array}{ll} \textit{(a)} & [X^V, Y^H] = [X, Y]^V - (\nabla_X Y)^V = -(\hat{\nabla}_X Y)^V \\ \textit{(b)} & [X^C, Y^H] = [X, Y]^H - \gamma(L_X Y), \\ \textit{(c)} & [X^H, Y^V] = [X, Y]^V + (\nabla_Y X)^V, \\ \textit{(d)} & [X^C, Y^H] = [X, Y]^H - \gamma\hat{R}(X, Y) \end{array}$$

where $\hat{\nabla}$ is an affine connection in M defined by

$$\hat{\nabla}_X Y = \nabla_Y X + [X, Y]$$

and $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$.

Proposition 2.2. *For any $X, Y \in \tau_0^1(M), f \in \tau_0^0(M)$ and ∇^H is the horizontal lift of the affine connection ∇ to $T(M)$ [1]*

$$\begin{array}{ll} (a) & \nabla^H_{X^V} Y^V = 0, \\ (b) & \nabla^H_{X^V} Y^H = 0, \\ (c) & \nabla^H_{X^H} Y^V = (\nabla_X Y)^V, \\ (d) & \nabla^H_{X^H} Y^H = (\nabla_X Y)^H. \end{array}$$

3. Tachibana and Vishnevskii operators associated with the Lorentzian almost r-para-contact structure in the tangent bundle

Let M be a differentiable manifold of C^∞ class and $T(M)$ denotes the tangent bundle of M. Suppose that there are a tensor field ϕ of type (1, 1), a vector field ξ_p and a 1-form $\eta_p, p = 1, 2, ...r$ satisfying [5, 6, 11]

$$\begin{array}{lll} & \text{(a)} & \phi^2 = I - \sum_{p=1}^r \xi_p \otimes \eta_p \\ & \text{(b)} & \phi \xi_p = 0 \\ & \text{(c)} & \eta_p \circ \phi = 0 \\ (3.1) & \text{(d)} & \eta_p(\xi_q) = \delta_{pq} \end{array}$$

where $p = 1, 2, ...r$ and δ_{pq} denote the Kronecker delta. Thus the manifold M satisfying conditions (3.1) will be said to possess a Lorentzian almost r-para-contact structure [13, 14].

Let us suppose that the base space M admits the Lorentzian almost r-para-contact structure. Then there exists a tensor field ϕ of type (1, 1), $r(C^\infty)$ vector fields $\xi_1, \xi_2, \ldots \xi_p$, and $r(C^\infty)$ 1-forms $\eta_1, \eta_2, \ldots \eta_p$, such that equation (3.1) are satisfied. Taking the complete lifts of the equation (3.1) we obtain the following:

$$\begin{array}{lll} & \text{(a)} & (\phi^H)^2 = I + \sum_{p=1}^r \{\xi_p^V \otimes \eta_p^H - \xi_p^H \otimes \eta_p^V\} \\ & \text{(b)} & \phi^H \xi_p^V = 0, \phi^H \xi_p^C = 0 \\ & \text{(c)} & \eta_p^V \circ \phi^H = 0, \eta_p^H \circ \phi^V = 0, \eta_p^H \circ \phi^H = 0, \eta_p^V \circ \phi^V = 0 \\ (3.2) & \text{(d)} & \eta_p^H(\xi_p^H) = \eta_p^V(\xi_p^V) = 0, \eta_p^H(\xi_p^V) = \eta_p^V(\xi_p^H) = 1 \end{array}$$

Let us define the element $\tilde{J}$ of $J_0^1 T(M)$ by

$$\tilde{J} = \phi^H + \sum_{p=1}^{r} \left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H \right) \tag{3.3}$$

then in the view of the equation (3.2), it is easily shown that

$$\tilde{J}^2 X^V = X^V, \tilde{J}^2 X^H = X^H$$

which means that $\tilde{J}$ is an almost product structure in $T(M)$ [3, 14]. Now in view of the equation (3.3), we have

$$\begin{aligned} &\text{(a)} \qquad \tilde{J}X^H = (\phi X)^H + \textstyle\sum_{p=1}^{r} \left\{ (\eta_p(X))^V \xi_p^V \right\} \\ (3.4) \qquad &\text{(b)} \qquad \tilde{J}X^V = (\phi X)^V - \textstyle\sum_{p=1}^{r} \left\{ (\eta_p(X))^V \xi_p^H \right\} \end{aligned}$$

for all $X \in \tau_0^1(M)$.

3.1. Tachibana Operator

Let $\phi \in \tau_1^1(M)$ and $\tau(M) = \sum_{r,s=0}^{\infty} \tau_r^s(M)$ be a tensor algebra over R. A map $\Phi_\phi|_{r+s>0}$ is called the Tachibana operator or Φ_ϕ operator on M if [9]

$$\begin{aligned} &\text{(a)} && \Phi_\phi \textit{ is linear with respect to constant coefficient },\\ &\text{(b)} && \Phi_\phi : \tau^*(M) \to \tau_{s+1}^r(M) \textit{ for all } r \textit{ and } s\\ &\text{(c)} && \Phi_\phi(K \otimes^C L) = (\Phi_\phi K) \otimes L + K \otimes \Phi_\phi L \textit{ for all } K, L \in \tau^*(M),\\ &\text{(d)} && \Phi_{\phi X} Y = -(L_Y \phi) X forall X, Y \in \tau_0^1(M)\\ & && \textit{where } L_Y \textit{ is Lie derivation with respect to } Y,\\ &\text{(e)} && (\Phi_{\phi\eta})Y = (d(\tau_Y \eta(\Phi X) - (d(\tau_Y(\eta \circ \Phi)X + \eta((L_Y \phi)X)\\ (3.5) & && = (\Phi X(\tau_Y \eta))(\Phi X) - X(\tau_{\phi X} \eta) + \eta((L_Y \phi)X) \end{aligned}$$

for all $\eta \in \tau_1^0(M)$ and $X, Y \in \tau_0^1$, where $\tau_Y \eta = \eta(X) = \eta \otimes^C Y, \tau_r^{*s}(M)$ the module of the pure tensor field of type (r, s) on M with respect to the affinor field φ.

Theorem 3.1. *For the Tachibana operator on M, L_X the operator Lie derivation with respect to $X, \tilde{J} \in \tau_1^1(T(M))$ defined by $\tilde{J} = \phi^H + \sum_{p=1}^{r} \left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H \right)$ and $\eta(Y) = 0$, we have*

$$\begin{aligned}
&(a) && \Phi_{\tilde{J}Y^V}X^H = -((\hat{\nabla}_X\phi)Y)^V + \textstyle\sum_{p=1}^r((\hat{\nabla}_X\eta_p)Y)^V\xi_p^H\\
&(b) && \Phi_{\tilde{J}Y^H}X^H = -((L_X\phi)Y)^H + \gamma\hat{R}(X,\phi Y) - \textstyle\sum_{p=1}^r((L_X\eta_p)Y)^V\xi_p^V - \tilde{J}\gamma\hat{R}(X,Y)\\
&(c) && \Phi_{\tilde{J}Y^V}X^V = 0\\
&(d) && \Phi_{\tilde{J}Y^H}X^V = -((L_X\phi)Y)^V + ((\nabla_X\phi)Y)^V + \textstyle\sum_{p=1}^r((L_X\eta_p)Y)^V\xi_p^H\\
&(3.6) && \qquad - \textstyle\sum_{p=1}^r((\nabla_X\eta_p)Y)^V\xi_p^H
\end{aligned}$$

where $X, Y \in \tau_0^1(M)$, a tensor field $\phi \in \tau_1^1(M)$, a vector field ξ and a 1-form $\eta \in \tau_1^0(M)$.

Proof.

$$\begin{aligned}
\text{(a)}\Phi_{\tilde{J}Y^V}X^H &= -(L_{X^H}\tilde{J})Y^V = -(L_{X^H}\tilde{J}Y^V - \tilde{J}L_{X^H}Y^V), \quad \textit{since} \ \ L_XY = [X,Y]\\
&= -[X^H, \tilde{J}Y^V] + \left(\phi^H + \sum_{p=1}^r\left(\xi_p^V\otimes\eta_p^V - \xi_p^H\otimes\eta_p^H\right)\right)[X^H, Y^V]\\
&= -[X^H.(\phi Y)^V] + \phi^H([X,Y]^V + (\nabla_XY)^V) + \sum_{p=1}^r\eta_p^V([X,Y]^V + (\nabla_YX)^V)\xi_p^V\\
&\quad - \sum_{p=1}^r\eta_p^H([X,Y]^V + (\nabla_YX)^V)\xi_p^H\\
&= -[X^H,(\phi Y)^V](\nabla_{\phi Y}X)^V + \phi^H([X,Y]^V + (\nabla_YX)^V) + \sum_{p=1}^r\eta_p^V([X,Y]^V\\
&\quad + (\nabla_YX)^V)\xi_p^V - \sum_{p=1}^r\eta_p^H([X,Y]^V + (\nabla_YX)^V)\xi_p^H\\
&= -((\hat{\nabla}_X\phi)Y)^V - (\phi\hat{\nabla}_XY)^V - \sum_{p=1}^r((L_X\eta_p)Y)^V\xi_p^H + \sum_{p=1}^r((\hat{\nabla}_X\eta_p)Y)^V\xi_p^H\\
&\quad - \sum_{p=1}^r(\eta_p(L_XY))^V\xi_p^H\\
&\qquad \textit{as} \ \ \eta(L_XY) = -(L_X\eta_p)Y\\
&= -((\hat{\nabla}_X\phi)Y)^V + \sum_{p=1}^r((\hat{\nabla}_X\eta_p)Y)^V\xi_p^H.
\end{aligned}$$

(3.7)

$$\text{(b)}\Phi_{\tilde{J}Y^H}X^H = -(L_{X^H}\tilde{J})Y^H = -(L_{X^H}\tilde{J}Y^H - \tilde{J}L_{X^H}Y^H) \quad \textit{since} \ \ L_XY = [X,Y]$$

$$= -[X^H, \tilde{J}Y^H] + \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)[X^H, Y^H]$$

$$= -[X^H, (\phi Y)^H] + \phi^H[X^H, Y^H] + \sum_{p=1}^{r}\eta_p^V[X^H, Y^H]\xi_p^V - \sum_{p=1}^{r}\eta_p^H[X^H, Y^H]\xi_p^H$$

$$since [X^H, Y^H] = [X, Y]^H - \gamma\hat{R}(X, Y),$$

$$= -((L_X\phi)Y)^H + \gamma\hat{R}(X, \phi Y) - \phi^H\gamma\hat{R}(X, Y)) - \sum_{p=1}^{r}((L_X\eta_p)Y)^V\xi_p^V$$

$$-\sum_{p=1}^{r}((\eta_p^V\gamma\hat{R}(X, Y)\xi_p^V + \sum_{p=1}^{r}((\eta_p^H\gamma\hat{R}(X, Y)\xi_p^H$$

$$= -((L_X\phi)Y)^H + \gamma\hat{R}(X, \phi Y) - \sum_{p=1}^{r}((L_X\eta_p)Y)^V\xi_p^V - \tilde{J}\gamma\hat{R}(X, Y). \tag{3.8}$$

$$(c)\Phi_{\tilde{J}Y^V}X^V = -(L_{X^V}\tilde{J})Y^V = -(L_{X^V}\tilde{J}Y^V - \tilde{J}L_{X^V}Y^V) \quad since \;\; L_XY = [X, Y]$$

$$= -[X^V, \tilde{J}Y^V] + \tilde{J}[X^V, Y^V], \qquad [X^V, Y^V] = 0$$

$$= -[X^V, \phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)Y^V]$$

$$as \;\; (\eta_p(Y)\xi_p)^H = 0$$

$$= -[X^V, (\phi Y)^V] + \sum_{p=1}^{r}[X^V, (\eta_p(Y)\xi_p)^H] = 0 \tag{3.9}$$

$$(d)\Phi_{\tilde{J}Y^H}X^V = -(L_{X^V}\tilde{J})Y^H = -L_{X^V}\tilde{J}Y^H + \tilde{J}L_{X^V}Y^H), \quad since \;\; L_XY = [X, Y]$$

$$= -[X^V, \tilde{J}Y^H] + \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)[X^V, Y^H]$$

$$= -[X, \phi Y]^V + (\nabla_X\phi Y)^V + \phi^H([X, Y]^V - (\nabla_XY)^V) + \sum_{p=1}^{r}\eta_p^V([X, Y]^V - (\nabla_XY)^V)\xi_p^V$$

$$-\sum_{p=1}^{r}\eta_p^H([X, Y]^V - (\nabla_XY)^V)\xi_p^H$$

$$since \quad \eta_pL_XY = L_X\eta_p(Y) - (L_X\eta_p)Y, \;\; \eta_p\nabla_XY = \nabla_X\eta_p(Y) - (\nabla_X\eta_p)Y$$

$$= -((L_X\phi)Y)^V + ((\nabla_X\phi)Y)^V + \sum_{p=1}^{r}((L_X\eta_p)Y)^V\xi_p^H - \sum_{p=1}^{r}((\nabla_X\eta_p)Y)^V\xi_p^H. \tag{3.10}$$

□

Corollary 3.1. *If we put* $Y = \xi_p$ *i.e.* $\eta_p^H(\xi_p^H) = \eta_p^V(\xi_p^V) = 0$, $\eta_p^H(\xi_p^V) = \eta_p^V(\xi_p^H) = 1$, *then we have*

$$\text{(a)} \qquad \Phi_{\tilde{J}\xi_p^V} X^H = \textstyle\sum_{p=1}^r (L_{\xi_p} X)^H - \gamma \hat{R}(X, \xi_P) - (\hat{\nabla}_X \phi)^V - (\hat{\nabla}_X \eta_p)\xi_p^V \xi_p^H$$

$$\text{(b)} \qquad \Phi_{\tilde{J}\xi_p^H} X^H = (\hat{\nabla}_X \xi_p)^V - ((L_X \phi)\xi_p)^H + \phi^H \gamma \hat{R}(X, \xi_p) - \textstyle\sum_{p=1}^r ((L_X \eta_p)\xi_p)^V \xi_p^V - \sum_{p=1}^r \eta_p^V \gamma \hat{R}(X, \xi_p)\xi_p^V + \sum_{p=1}^r \eta_p^H \gamma \hat{R}(X, \xi_p)\xi_p^H$$

$$\text{(c)} \qquad \Phi_{\tilde{J}\xi_p^V} X^V = (\hat{\nabla}_\xi)_p X^V$$

$$\text{(d)} \qquad \Phi_{\tilde{J}\xi_p^H} X^V = -((L_X \phi)\xi_p)^V + \textstyle\sum_{p=1}^r ((L_X \eta_p)\xi_p)^V \xi_p^H - \sum_{p=1}^r ((\nabla_X \eta_p)\xi_p)^V \xi_p^H.$$

3.2. Vishnevskii Operator

Let ∇ be a linear connection and ϕ be a tensor field of type (1,1) on M. If the condition (d) of the Tachibana operator is replaced by

$$\text{(D)} \qquad \Psi_{\phi X} Y = \nabla_{\phi X} Y - \phi \nabla_X Y \tag{3.11}$$

for any $X, Y \in \tau_0^1(M)$. A map $\Psi_\phi : \tau^*(M) \to \tau(M)$, which satisfies conditions (a), (b), (c), (e) of the Tachibana operator and the condition (D), is called the Vishnevskii operator on M[9, 11].

Theorem 3.2. *For* Ψ_ϕ *the Vishnevskii operator on* M *and* ∇^H *the horizontal lift of an affine connection* ∇ *in* M *to* $T(M), \tilde{J} \in \tau_1^1(T(M))$ *defined by (3.3), we have*

$$\begin{aligned}
&\text{(a)} \quad \Psi_{\tilde{J}X^V} Y^H = -\textstyle\sum_{p=1}^r ((\eta_p(X)\nabla_\xi)_p Y^H \\
&\text{(b)} \quad \Psi_{\tilde{J}X^H} Y^V = ((\hat{\nabla}_Y \phi)X)^V - ((L_X \phi)X)^V + \textstyle\sum_{p=1}^r (\eta_p \hat{\nabla}_Y X)^V \xi_p^H - \sum_{p=1}^r (\eta_p L_Y X)^V \xi_p^H \\
&\text{(c)} \quad \Psi_{\tilde{J}X^V} Y^V = -\textstyle\sum_{p=1}^r (\eta_p(X))^V)\nabla_{\xi_p^H}^H Y^V \\
&\text{(d)} \quad \Psi_{\tilde{J}X^H} Y^H = ((\hat{\nabla}_Y \phi)X)^H - ((L_X \phi)X)^H - \textstyle\sum_{p=1}^r (\eta_p \hat{\nabla}_Y X)^V \xi_p^V + \sum_{p=1}^r (\eta_p L_Y X)^V \xi_p^V
\end{aligned} \tag{3.12}$$

where $X, Y \in \tau_0^1(M)$, *a tensor field* $\phi \in \tau_1^1(T(M))$, *vector fields* ξ_p *and a 1-form* $\eta_p \in \tau_1^0, p = 1 \ldots r$.

Proof.

$$\text{(a)} \Psi_{\tilde{J}X^V} Y^H = \nabla_{\tilde{J}X^V}^H Y^H - \tilde{J}\nabla_{X^V}^H Y^H$$

$$= \nabla^H_{(\phi^H + \sum_{p=1}^r (\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H))X^V} Y^H - \left(\phi^H + \sum_{p=1}^r \left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right) \nabla_{X^V}^H Y^H$$

$$= \nabla_{(\phi X)^V - \sum_{p=1}^{r}(\eta_p X)^V \sigma_p^H} Y^H \quad as \nabla^H_{X^V} Y^H = 0$$

$$= -\sum_{p=1}^{r} (\eta_p X)^V (\nabla_{\xi_p} Y)^H \quad as \nabla^H_{(\phi X)^V} Y^H = 0$$

$$= -\sum_{p=1}^{r} (\eta_p(X) \nabla_{\xi_p} Y)^H. \tag{3.13}$$

$$(b) \Psi_{\tilde{J} X^H} Y^V = \nabla^H_{\tilde{J} X^H} Y^H - \tilde{J} \nabla^H_{X^H} Y^V$$

$$= \nabla^H_{\left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right) X^H} Y^V - \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right) \nabla^H_{X^H} Y^V$$

$$= \nabla^H_{(\phi X)^H} Y^V - \phi^H (\nabla_X Y)^V + \sum p = 1^r \eta_p^H (\nabla_X Y)^V \xi_p^H$$

$$= (\hat{\nabla_Y} \phi X)^V + [\phi X, Y]^V - \phi^H ((\hat{\nabla_Y} X)^V + [X, Y]^V) + \sum_{p=1}^{r} \eta_p^H ((\hat{\nabla_Y} X)^V + [X, Y]^V) \xi_p^H$$

$$= ((\hat{\nabla_Y} \phi) X)^V - ((L_Y \phi) X)^V + \sum_{p=1}^{r} (\eta_p \hat{\nabla_Y} X)^V \xi_p^H - \sum_{p=1}^{r} (\eta_p \hat{L_Y} X)^V \xi_p^H \tag{3.14}$$

$$(c) \Psi_{\tilde{J} X^V} Y^V = \nabla^H_{\tilde{J} X^V} Y^V - \tilde{J} \nabla^H_{X^V} Y^V$$

$$= \nabla^H_{\left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right) X^V} Y^V - \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right) \nabla^H_{X^V} Y^V$$

$$= \nabla^H_{(\phi X)^V} Y^V - \sum_{p=1}^{r} \eta_p(X))^V) \nabla^H_{\xi_p^H} Y^V$$

$$= -\sum p = 1^r \eta_p(X))^V) \nabla^H_{\xi_p^H} Y^V \quad as \nabla^H_{(\phi X)^V} Y^V = 0. \tag{3.15}$$

$$(d) \Psi_{\tilde{J} X^H} Y^H = \nabla^H_{\tilde{J} X^H} Y^H - \tilde{J} \nabla^H_{X^H} Y^H$$

$$= \nabla^H_{\left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right) X^H} Y^H - \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right) \nabla^H_{X^H} Y^H$$

$$= \nabla^H_{(\phi X)^H} Y^H - \phi^H (\nabla_X Y)^H - \sum_{p=1}^{r} \eta_p^V (\nabla_X Y)^H \xi_p^V$$

$$= (\hat{\nabla_Y} \phi X)^H + [\phi X, Y]^H - \phi^H ((\hat{\nabla_Y} X)^H + [X, Y]^H) - \sum_{p=1}^{r} \eta_p^V ((\hat{\nabla_Y} X)^H + [X, Y]^H) \xi_p^H$$

$$= ((\hat{\nabla_Y} \phi) X)^H - ((L_Y \phi) X)^H - \sum_{p=1}^{r} (\eta_p \hat{\nabla_Y} X)^V \xi_p^V - \sum_{p=1}^{r} (\eta_p L_Y X)^V \xi_p^V \tag{3.16}$$

□

Corollary 3.2. *If we put* $Y = \xi_p$ *i.e.* $\eta_p^H(\xi_p^H) = \eta_p^V(\xi_p^V) = 0$, $\eta_p^H(\xi_p^V) = \eta_p^V(\xi_p^H) = 1$, *then we have*

(a) $$\Psi_{\tilde{J}\xi_p^V} Y^H = -(\nabla_\xi)_p Y^H$$

(b) $$\Psi_{\tilde{J}\xi_p^H} Y^V = -\phi^H(\hat{\nabla}_Y \xi_p)^V + (\phi L_Y \xi_p)^V + \sum_{p=1}^r (\eta_p(\hat{\nabla}_Y \xi_p)^V \xi_p^H - \sum_{p=1}^r (\eta_p(L_Y \xi_p)^V)\xi_p^H$$

(c) $$\Psi_{\tilde{J}\xi_p^V} Y^V = -(\nabla_{\xi_p} Y)^V$$

(d) $$\Psi_{\tilde{J}\xi_p^H} Y^H = ((\hat{\nabla}_Y \phi)\xi_p)^H + (\phi[Y, \xi_p])^H + \sum_{p=1}^r ((\hat{\nabla}_Y \eta_p)\xi_p)^V \xi_p^V - \sum_{p=1}^r ((L_Y \eta_p)\xi_p)^V \xi_p^V.$$

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