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SOME RESULTS ON GENERALIZED (k, μ)-PARACONTACT METRIC MANIFOLDS

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Abstract. The aim of this paper is to study the Codazzi type of the Ricci tensor in generalized (k, μ)-paracontact metric manifolds. We also study the cyclic parallel Ricci tensor in generalized (k, μ)-paracontact metric manifolds. Further, we characterize generalized (k, μ)-paracontact metric manifolds whose structure tensor ϕ is η-parallel. Finally, we investigate locally ϕ-Ricci symmetric generalized (k, μ)-paracontact metric manifolds.

Keywords: Generalized (k, μ)-paracontact metric manifold, Codazzi type of tensor, cyclic parallel Ricci tensor, η-parallel ϕ-tensor, locally ϕ-Ricci symmetric.

1. Introduction

In 1985, Kaneyuki and Williams [8] introduced the idea of paracontact geometry. A systematic investigation on paracontact metric manifolds was done by Zamkovoy [12]. Recently, Cappelletti-Montano et al [5] introduced a new type of paracontact geometry, the so-called paracontact metric (k, μ) space, where k and μ are constants. This is known [2] about the contact case $k \leq 1$, but in the paracontact case there is no restriction of k. Recently, three-dimensional generalized (k, μ)-paracontact metric manifolds were studied by Kupeli Erken et al [9, 10].
Zamkovoy [12] studied paracontact metric manifolds and some remarkable subclasses named para-Sasakian manifolds. In particular, in recent years, many authors have pointed to the importance of paracontact geometry and, in particular, para-Sasakian geometry. Several papers have established relationships with the theory of para-Kahler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. A normal paracontact metric manifold is a para-Sasakian manifold. An almost paracontact metric manifold is a para-sasakian manifold if and only if [12]

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X. \tag{1.1}$$

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A. Gray [7] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. The Ricci tensor S of type (0,2) is said to be cyclic parallel if it is non-zero and satisfies the condition

$$(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = 0. \tag{1.2}$$

Again, a Riemannian or a pseudo-Riemannian manifold is said to be of Codazzi type if its Ricci tensors of type (0,2) is non-zero and satisfy the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{1.3}$$

for all vector fields X, Y, Z. On a contact metric manifold there is an associated CR-structure which is integrable if and only if the structure tensor ϕ is η-parallel, that is,

$$g((\nabla_X \phi)Y, Z) = 0,$$

for all vector fields X, Y, Z in the contact distribution $D(\eta = 0)$. In 2005, Boeckx and Cho [3] considered a milder condition that h is η-parallel, that is,

$$g((\nabla_X h)Y, Z) = 0,$$

for all vector fields X, Y, Z in the contact distribution D.

The paper is organized in the following way:
In Section 2, we discuss some basic results of paracontact metric manifolds. Further, we characterize the Codazzi type of the Ricci tensor in generalized (k,μ)-paracontact metric manifolds. In Section 4, we investigate the cyclic parallel Ricci tensor in generalized (k,μ)-paracontact metric manifolds. In the next section we study η-parallel ϕ-tensor in a generalized (k,μ)-paracontact metric manifold. Finally, we investigate locally ϕ-Ricci symmetric generalized (k,μ)-paracontact metric manifolds.

2. Preliminaries

An odd dimensional smooth manifold $M^n (n > 1)$ is said to be an almost paracontact manifold [8] if it carries a $(1,1)$-tensor ϕ, a vector field ξ and a 1-form η satisfying :
(i) $\phi^2 X = X - \eta(X)\xi$, for all $X \in \chi(M)$,
(ii)$\eta(\xi) = 1$, $\phi(\xi) = 0$, $\eta \circ \phi = 0$,
(iii) the tensor field ϕ induces an almost paracomplex structure on each fiber of $D = ker(\eta)$, that is, the eigen distributions D^+_ϕ and D^-_ϕ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have an equal dimension n.

An almost paracontact structure is said to be normal [8] if and only if the $(1,2)$ type torsion tensor $N_\phi = [\phi,\phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. A para-Sasakian manifold is a normal paraconatact metric manifold. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.1}$$

for X, $Y \in \chi(M)$, then we say that (M,ϕ,ξ,η,g) is an almost paracontact metric manifold. Any such pseudo-Riemannian metric is of signature $(n+1,n)$. An almost paracontact structure is said to be a paracontact structure if $g(X,\phi Y) = d\eta(X,Y)$ [12]. In a paracontact metric manifold we define $(1,1)$-type tensor fields h by $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds_\xi\phi$ is the Lie derivative of ϕ along the vector field ξ. Then we observe that h is symmetric and anti-commutes with ϕ. Also h satisfies the following conditions [12]:

$$h\xi = 0,\ tr(h) = tr(\phi h) = 0, \tag{2.2}$$

$$\nabla_X\xi = -\phi X + \phi hX, \tag{2.3}$$

for all $X \in \chi(M)$, where ∇ denotes the Levi-Civita connection of the pseudo-Riemannian manifold.
Moreover, h vanishes identically if and only if ξ is a Killing vector field. In this case, (M,ϕ,ξ,η,g) is said to be a K-paracontact manifold [11].

Generalized (k,μ)-paracontact metric manifolds were studied by Erken et al. [10] and Erken [9]. A generalized (k,μ)-paracontact metric manifold means a three-dimensional paracontact metric manifold which satisfies the curvature condition

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{2.4}$$

where k and μ are smooth functions.

In a generalized $(k \neq -1,\mu)$-paracontact manifold the following results hold [4, 5, 9, 10]

$$h^2 = (1+k)\phi^2, \tag{2.5}$$

$$\xi(k) = 0, \tag{2.6}$$

$$Q\xi = 2k\xi, \tag{2.7}$$

$$(\nabla_\xi h)(Y) = \mu h(\phi Y), \tag{2.8}$$

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X &= -(1+k)[2g(X,\phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X] \\ &\quad +(1-\mu)(\eta(X)\phi hY - \eta(Y)\phi hX),\end{aligned} \tag{2.9}$$

$$(\nabla_X\phi)Y = -g(X-hX,Y)\xi + \eta(Y)(X-hX),\ \text{for}\ \ k \neq -1 \tag{2.10}$$

$$h\ grad\mu = gradk, \tag{2.11}$$

$$(\nabla_X \eta)(Y) = -g(\phi X, Y) + g(\phi h X, Y), \tag{2.12}$$

$$QX = (\frac{r}{2} - k)X + (-\frac{r}{2} + 3k)\eta(X)\xi + \mu h X, k \neq -1, \tag{2.13}$$

where X is any vector fields on M, Q is the Ricci operator of M, r denotes the scalar curvature of M.
From (2.13), we have

$$S(X,Y) = (\frac{r}{2} - k)g(X,Y) + (-\frac{r}{2} + 3k)\eta(X)\eta(Y) + \mu g(hX, Y), k \neq -1. \tag{2.14}$$

3. The Codazzi type of the Ricci tensor in generalized (k,μ)-paracontact metric manifolds

In this section we characterize generalized (k,μ)-paracontact metric manifolds whose Ricci tensor is of Codazzi type.
Then we have

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{3.1}$$

which implies r =constant.

Now from (2.14) we have

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \{\frac{(Xr)}{2} - (Xk)\}g(Y,Z) + \{-\frac{(Xr)}{2} + 3(Xk)\}\eta(Y)\eta(Z) \\ &\quad + \{-\frac{r}{2} + 3k\}\{(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)\} + (X\mu)g(hY,Z) \\ &\quad + \mu g((\nabla_X h)(Y), Z)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(\nabla_Y S)(X,Z) &= \{\frac{(Yr)}{2} - (Yk)\}g(X,Z) + \{-\frac{(Yr)}{2} + 3(Yk)\}\eta(X)\eta(Z) \\ &\quad + \{-\frac{r}{2} + 3k\}\{(\nabla_Y \eta)(X)\eta(Z) + \eta(X)(\nabla_Y \eta)(Z)\} + (Y\mu)g(hX,Z) \\ &\quad + \mu g((\nabla_Y h)(X), Z).\end{aligned} \tag{3.3}$$

Using (3.2) and (3.3) in (3.1) yields

$$\begin{aligned}&\{\frac{(Xr)}{2} - (Xk)\}g(Y,Z) + \{-\frac{(Xr)}{2} + 3(Xk)\}\eta(Y)\eta(Z) \\ &+ \{-\frac{r}{2} + 3k\}\{(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)\} + (X\mu)g(hY,Z) \\ &+ \mu g((\nabla_X h)(Y), Z) = \{\frac{Yr}{2} - Yk\}g(X,Z) \\ &+ \{-\frac{(Yr)}{2} + 3(Yk)\}\eta(X)\eta(Z) + \{-\frac{r}{2} + 3k\}\{(\nabla_Y \eta)(X)\eta(Z) \\ &+ \eta(X)(\nabla_Y \eta))(Z)\} + (Y\mu)g(hX,Z) + \mu g((\nabla_Y h)(X), Z).\end{aligned} \tag{3.4}$$

Substituting $Z=\xi$ in (3.4) gives

$$\begin{aligned}&\{\frac{(Xr)}{2}-(Xk)\}\eta(Y)+\{-\frac{(Xr)}{2}+3(Xk)\}\eta(Y)+\{-\frac{r}{2}+3k\}\{(\nabla_X\eta)(Y)\\&+\eta(Y)(\nabla_X\eta))(\xi)\}+\mu\eta((\nabla_X h)(Y))=\{\frac{Yr}{2}-Yk\}\eta(X)\\&+\{-\frac{(Yr)}{2}+3(Yk)\}\eta(X)+\{-\frac{r}{2}+3k\}\{(\nabla_Y\eta)(X)+\eta(X)(\nabla_Y\eta))(\xi)\}\\&+\mu\eta((\nabla_Y h)(X)).\end{aligned}\tag{3.5}$$

Putting $X=\xi$ in (3.5) and using r =constant, we obtain

$$\mu\eta((\nabla_\xi h)(Y))=2(Yk)+\mu\eta((\nabla_Y h)(\xi))=0.\tag{3.6}$$

Applying (2.8) in (3.6), we have $(Yk)=0$, which implies k =constant. Hence from (2.11), we get either $h=0$ or μ =constant. Thus, we can state the following

Theorem 3.1. *If in a generalized (k,μ)-paracontact metric manifold with $k\neq -1$ the Ricci tensor is of Codazzi type, then the manifold is either a (k,μ)-paracontact metric manifold or a K-paracontact manifold.*

4. The cyclic parallel Ricci tensor in generalized (k,μ)-paracontact metric manifolds

This section is devoted to the study of the cyclic parallel Ricci tensor in generalized (k,μ)-paracontact metric manifolds
If the Ricci tensors is cyclic parallel, then we have

$$(\nabla_Z S)(X,Y)+(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)=0,\tag{4.1}$$

which implies $r=constant$.

Now from the equation (2.14), we obtain

$$\begin{aligned}&\{\frac{(Zr)}{2}-(Zk)\}g(X,Y)+\{-\frac{(Zr)}{2}+3(Zk)\}\eta(X)\eta(Y)\\&+\{-\frac{r}{2}+3k\}\{(\nabla_Z\eta)(X)\eta(Y)+\eta(X)(\nabla_Z\eta)(Y)\}+(Z\mu)g(hX,Y)\\&+\mu g((\nabla_Z h)(X)Y)+\{\frac{Xr}{2}-Xk\}g(Y,Z)+\{-\frac{(Xr)}{2}+3(Xk)\}\eta(Y)\eta(Z)\\&+\{-\frac{r}{2}+3k\}\{(\nabla_X\eta)(Y)\eta(Z)+\eta(Y)(\nabla_X\eta)(Z)\}+(X\mu)g(hY,Z)\\&+\mu g((\nabla_X h)(Y),Z)+\{(Yr)-(Yk)\}g(Z,X)+\{-\frac{(Yr)}{2}+3(Yk)\}\eta(Z)\eta(X)\\&+\{-\frac{r}{2}+3k\}\{(\nabla_Y\eta)(Z)\eta(X)+\eta(Z)(\nabla_Y\eta)(X)\}+(Y\mu)g(hZ,X)\\&+\mu g((\nabla_Y h)(Z),X)=0.\end{aligned}\tag{4.2}$$

Substituting $X = Y = \xi$ and applying (2.8) in (4.2) yields

$$2(Zk) + (-\frac{r}{2} + 3k)(\nabla_\xi \eta)(Z) + (-\frac{r}{2} + 3k)(\nabla_\xi \eta)(Z) = 0. \tag{4.3}$$

Now using (2.12) in (4.3), we have

$$(Zk) = 0. \tag{4.4}$$

Therefore, k =constant. Hence from (2.11), we have either $h = 0$ or μ =constant. This leads to the following:

Theorem 4.1. *If in a generalized (k,μ)-paracontact metric manifold with $k \neq -1$ the Ricci tensor is cyclic parallel, then the manifold is either a (k,μ)-paracontact metric manifold or a K-paracontact manifold.*

5. The η-parallel ϕ-tensor in generalized (k,μ)-paracontact metric manifolds

In this section we study the η-parallel ϕ-tensor in generalized (k,μ)-paracontact metric manifolds
If the $(1,1)$ tensor ϕ is η-parallel, then we have [1]

$$g((\nabla_X \phi)Y, Z) = 0. \tag{5.1}$$

From (2.10) and (5.1), we get

$$-g(X,Y)\eta(Z) + g(hX,Y)\eta(Z) + g(X,Z)\eta(Y) - g(hX,Z)\eta(Y) = 0. \tag{5.2}$$

Putting $Z = \xi$ in (5.2) yields

$$-g(X,Y) + g(hX,Y) + \eta(X)\eta(Y) = 0. \tag{5.3}$$

Substituting $X = hX$ in (5.3), we have

$$-g(hX,Y) - (k+1)g(X,Y) + (k+1)\eta(X)\eta(Y) = 0. \tag{5.4}$$

Adding (5.3) and (5.4), we obtain

$$(k+2)\{g(X,Y) - \eta(X)\eta(Y)\} = 0. \tag{5.5}$$

Thus we have $k = -2$, that is, k = constant. Using (2.11) we have $h\ grad\mu = 0$. Therefore, either $h = 0$ or μ =constant.
Thus we can state the following:

Theorem 5.1. *If in a generalized (k,μ)-paracontact metric manifold with $k \neq -1$, the tensor ϕ is η-parallel, then the manifold is either a (k,μ)-paracontact metric manifold or a K-paracontact manifold.*

6. Locally ϕ-Ricci symmetric generalized (k,μ)-paracontact manifolds

A paracontact metric manifold is said to be locally ϕ-Ricci symmetric [6] if it satisfies

$$\phi^2(\nabla_X Q)(Y) = 0, \tag{6.1}$$

for all vector fields X, Y orthogonal to ξ, where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

Taking the covariant derivative of (2.13) with respect to Y and applying ϕ^2 we get

$$-\{\frac{(Yr)}{2} - Yk\}X - (Y\mu)hX + \mu\phi^2((\nabla_Y h)X) = 0. \tag{6.2}$$

Interchanging X and Y in (6.2), we have

$$-\{\frac{(Xr)}{2} - Xk\}Y - (X\mu)hY + \mu\phi^2((\nabla_X h)Y) = 0. \tag{6.3}$$

Subtracting (6.3) from (6.2), we obtain

$$\begin{aligned}&\{\frac{(Yr)}{2} - Yk\}X - \{\frac{(Xr)}{2} - Xk\}Y + (Y\mu)hX - (X\mu)hY \\ &+\mu\phi^2((\nabla_X hY) - (\nabla_Y hX)) = 0.\end{aligned} \tag{6.4}$$

Applying (2.9) in (6.4), we get

$$\{\frac{(Yr)}{2} - Yk\}X - \{\frac{(Xr)}{2} - Xk\}Y + (Y\mu)hX - (X\mu)hY = 0. \tag{6.5}$$

Substituting $X = \xi$ in (6.5) yields

$$-\frac{1}{2}(\xi r)Y - (\xi\mu)hY + \{\frac{Yr}{2} - Yk\}\xi = 0. \tag{6.6}$$

Taking the inner product with Z from (6.6), we have

$$-\frac{1}{2}(\xi r)g(Y,Z) - (\xi\mu)g(hY,Z) = 0. \tag{6.7}$$

Let $\{e_i\}$, $i = 1,2,3$ be a local orthonormal basis in the tangent space $T_P M$ at each point p $\in M$. Substituting $Y = Z = e_i$ in (6.7) and summing over $i = 1$ to 3, we infer that $\xi r = 0$, since $k \neq -1$.
This leads to the following:

Theorem 6.1. *If a generalized (k,μ)-paracontact metric manifold with $k \neq -1$, is locally ϕ-Ricci symmetric, then the characteristic vector field ξ leaves the scalar curvature invariant.*

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