FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 2 (2018), 163–176
https://doi.org/10.22190/FUMI1802163G

ON A GENERALIZATION OF CATALAN POLYNOMIALS

Mouloud Goubi

Abstract. In this paper, we define and study the generalized class of Catalan's polynomials. Thereafter we connect them to the class of Humbert's polynomials and re-found the Humbert recurrence relation [5]. This idea helps us to define a new class of generalized Humbert's polynomials different from those given by H. W. Gould [4] and P. N. Shrivastava [9]. Finally, we establish an explicit formula for a special class of generalized Catalan's polynomials and get two useful combinatorial identities.
Keywords: Catalan's polynomials, Gegenbauer's polynomials,Humbert's polynomials, generating functions.

1. Introduction

We recall that the Catalan numbers C_n are defined for any positive integer n by

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and their generating function is

$$C(u) = \frac{1-\sqrt{1-4u}}{2u} = \sum_{n\geq 0} C_n u^n,\ |u| < \frac{1}{4}.$$

It is useful here to remember the proof. Writing $C(u) = \frac{1}{2u}\left(1-\sqrt{1-4u}\right)$. Using the fact that for $|u| < 1$ and $\alpha \in \mathbb{R}$;

$$(1+u)^{\alpha} = 1 + \sum_{n\geq 1} \begin{bmatrix} \alpha \\ n \end{bmatrix} u^n,\ \text{where}\ \begin{bmatrix} \alpha \\ n \end{bmatrix} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-(n-1))}{n!}.$$

We deduce that

$$C(u) = \frac{1}{2u}\sum_{n\geq 1} \begin{bmatrix} \frac{1}{2} \\ n \end{bmatrix} (-4u)^{n-1}$$

Received December 7, 2017; accepted June 11, 2018
2010 *Mathematics Subject Classification.* Primary 05A15, 11B83; Secondary 05A19

then

$$\begin{aligned} C(u) &= \sum_{n\geq 0} \begin{bmatrix} \frac{1}{2} \\ n \end{bmatrix} (-1)^n 2^{2n+1} u^n \\ &= \sum_{n\geq 0} \frac{1}{n+1} \binom{2n}{n} u^n \end{aligned}$$

For positive integers $a, b \geq 1$, the function $C_n^{a,b}(u) = \frac{1-\sqrt{1-au}}{bu}$ generates numbers $C_n^{a,b}$ of the form $C_n^{a,b} = \frac{a^{n+1}}{2^{2n+1}b} C_n$ and $C_n^{4,2} = C_n$. The idea is to remark that $C_n^{a,b}(u) = \frac{a}{2b} C\left(\frac{au}{4}\right)$. Furthermore $C_{a,b}(u) = \sum_{n\geq 0} \frac{a^{n+1}}{2^{2n+1}b} C_n u^n$.

The class $\{P_n(x)\}_{n\geq 0}$ of Catalan's polynomials [6] is defined by the following linear recurrence relation

$$P_{n+2}(x) = P_{n+1}(x) - xP_n(x) \quad , n \geq 2 \tag{1.1}$$

and the starting values $P_0(x) = P_1(x) = 1$. The closed form of $P_n(x)$ [6] is

$$P_n(x) = \frac{\left(1+\sqrt{1-4x}\right)^{n+1} - \left(1-\sqrt{1-4x}\right)^{n+1}}{2^{n+1}\sqrt{1-4x}} \tag{1.2}$$

and the bivariate generating function is

$$f(x,t) = \frac{1}{1-t+xt^2} = \sum_{n\geq 0} P_n(x) t^n \tag{1.3}$$

To get the proof, just write

$$xf(x,t) = \sum_{n\geq 0} \left[P_{n+1}(x) - P_{n+2}(x)\right] t^n$$

and

$$xf(x,t) = \frac{1}{t}(f(x,t)-1) - \frac{1}{t^2}(f(x,t)-1-t)$$

hence

$$\left(xt^2 - t + 1\right) f(t) = 1.$$

It is well-known that the $(n+1)^{\text{th}}$ Catalan's polynomial $P_n(x)$ is written under the following binomial expression

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-x)^k, \tag{1.4}$$

where $\lfloor x \rfloor$ is the integer part of x.
Explicitly we get

$$P_{2n}(x) = \sum_{k=0}^{n} \binom{2n-k}{k} (-x)^k$$

and

$$P_{2n+1}(x) = \sum_{k=0}^{n} \binom{2n+1-k}{k} (-x)^k .$$

Furthermore, $P_{2n}(x)$ and $P_{2n+1}(x)$ have the same degree and only the first coefficient corresponding to degree zero is 1.
A new proof of this identity is given in Section 3. using Gegenbauer's polynomials [2] and generalized Catalan's polynomials properties.

2. Generalized class of Catalan's polynomials

Definition 2.1. The generalized class of Catalan's polynomials $\left\{\mathcal{P}_{n,m}^{\lambda,A}(x)\right\}_{n\geq 0}$ is given by the following generating function

$$f_{m,\lambda,A}(x,t) = \frac{1+A(x)t}{(1-mt+xt^m)^{\lambda}} = \sum_{n\geq 0} \mathcal{P}_{n,m}^{\lambda,A}(x)\, t^n, \tag{2.1}$$

where $A(x)$ is any polynomial of $\mathbb{Z}[x]$. With starting values

$$\mathcal{P}_{0,m}^{\lambda,A}(x) = 1 \text{ and } \mathcal{P}_{1,m}^{\lambda,A}(x) = A(x) + \lambda m.$$

To simplify notations let us denote

$$\mathcal{P}_{n,m}^{\lambda,0}(x) = \mathcal{P}_{n,m}^{\lambda}(x) \text{ and } \mathcal{P}_{n,2}^{1,0}(x) = \mathcal{P}_n(x).$$

From the generating function $f_{m,\lambda,A}(x,t)$ we deduce that

$$\mathcal{P}_{n,m}^{\lambda,A}(x) = \mathcal{P}_{n,m}^{\lambda}(x) + A(x)\,\mathcal{P}_{n-1,m}^{\lambda}(x) \tag{2.2}$$

This family generalizes Catalan's polynomials. Using the definition (2.1) we get

$$f_{2,1,0}(x,t) = \frac{1}{(1-2t+xt^2)} = \sum_{n\geq 0} \mathcal{P}_n(x)\, t^n$$

and

$$f_{2,1,0}\left(x,\frac{t}{2}\right) = \frac{1}{\left(1-t+\frac{x}{4}t^2\right)} = f\left(\frac{x}{4},t\right)$$

then

$$2^{-n}\mathcal{P}_n(x) = P_n\left(\frac{x}{4}\right)$$

or

$$\mathcal{P}_n(4x) = 2^n P_n(x)$$

The generalized Catalan's polynomials are related to several polynomial types as Gegenbauer, Humbert-type polynomials. This connection is the subject of Section 3.

The recurrence relation satisfied by the class $\left\{\mathcal{P}_{n,m}^{\lambda,A}(x)\right\}_{n\geq 0}$ according to the positive integers n and m is established in the following theorem

Theorem 2.1. *If* $2 \leq n < m-1$

$$(n+1)\,\mathcal{P}^{\lambda,A}_{n+1,m}(x) = (\lambda+n-2)\,mA(x)\,\mathcal{P}^{\lambda,A}_{n-1,m}(x) - [(n-1)\,A(x) - mn - \lambda m]\,\mathcal{P}^{\lambda,A}_{n,m}(x),$$

if $n \geq m$

$$\begin{aligned}(n+1)\,\mathcal{P}^{\lambda,A}_{n+1,m}(x) &= (1-\lambda m-n+m)\,xA(x)\,\mathcal{P}^{\lambda,A}_{n-m,m}(x) + (n+\lambda-2)\,mA(x)\,\mathcal{P}^{\lambda,A}_{n-1,m}(x) \\ &+ (m-n-\lambda m-1)\,x\mathcal{P}^{\lambda,A}_{n-m+1,m}(x) + [\lambda m-(n-1)\,A(x)+mn]\,\mathcal{P}^{\lambda,A}_{n,m}(x)\end{aligned} \tag{2.3}$$

and for $m \geq 2$

$$\begin{aligned}m\mathcal{P}^{\lambda,A}_{m,m}(x) &= (\lambda+m-3)\,mA(x)\,\mathcal{P}^{\lambda,A}_{m-2,m}(x) \\ &+ \left[\lambda m+m^2-m-(n-2)\,A(x)\right]\mathcal{P}^{\lambda,A}_{m-1,m}(x) - \lambda mx\end{aligned} \tag{2.4}$$

As a consequence of Theorem 2.1 we get the following corollary.

Corollary 2.1. *If* $2 \leq n < m-1$

$$(n+1)\,\mathcal{P}^{\lambda}_{n+1,m}(x) = m(\lambda+n)\,\mathcal{P}^{\lambda}_{n,m}(x), \tag{2.5}$$

if $n \geq m$

$$(n+1)\,\mathcal{P}^{\lambda}_{n+1,m}(x) - m(n+\lambda)\,\mathcal{P}^{\lambda}_{n,m}(x) + (n-m+1+\lambda m)\,x\mathcal{P}^{\lambda}_{n-m+1,m}(x) = 0$$

and for $m \geq 2$*,*

$$(\lambda+m-1)\,\mathcal{P}^{\lambda}_{m-1,m}(x) - \mathcal{P}^{\lambda}_{m,m}(x) = \lambda x \tag{2.6}$$

Proof. The relations (2.5), (2.6) and (2.6) of Corollary 2.1 are immediate from the equalities (2.3), (2.3) and (2.4) of Theorem 2.1 by considering $A(x) = 0$. □

2.1. Proof of Theorem 2.1

$$f_{m,\lambda,A}(x,t) = \frac{1+A(x)\,t}{(1-mt+xt^m)^\lambda} = \sum_{n\geq 0}\mathcal{P}^{\lambda,A}_{n,m}(x)\,t^n$$

Let $\frac{df_{m,\lambda,A}(x,t)}{dt} = f'_{m,\lambda,A}(x,t)$ then

$$f'_{m,\lambda,A}(x,t) = \sum_{n\geq 1} n\mathcal{P}^{\lambda,A}_{n,m}(x)\,t^{n-1} = \sum_{n\geq 0}(n+1)\,\mathcal{P}^{\lambda,A}_{n+1,m}(x)\,t^n.$$

and

$$\begin{aligned}(1+A(x)\,t)(1-mt+xt^m)\,f'_{m,\lambda,A}(x,t) &= A(x)(1-mt+xt^m)\,f_{m,\lambda,A}(x,t) \\ &- \lambda m\left(xt^{m-1}-1\right)(1+A(x)\,t)\,f_{m,\lambda,A}(x,t)\end{aligned}$$

Taking

$$\Delta = (1-\lambda m)\, xA(x)\, t^m - \lambda m x t^{m-1} + (\lambda - 1)\, mA(x)\, t + A(x) + \lambda m$$

then

$$\begin{aligned}\Delta f_{m,\lambda,A}(x,t) &= (1-\lambda m)\, xA(x) \sum_{n\geq m} \mathcal{P}^{\lambda,A}_{n-m,m}(x)\, t^n - \lambda m x \sum_{n\geq m-1} \mathcal{P}^{\lambda,A}_{n-m+1,m}(x)\, t^n \\ &+ (\lambda-1)\, mA(x) \sum_{n\geq 1} \mathcal{P}^{\lambda,A}_{n-1,m}(x)\, t^n + (A(x)+\lambda m) \sum_{n\geq 0} \mathcal{P}^{\lambda,A}_{n,m}(x)\, t^n\end{aligned}$$

and taking

$$\sigma = (1 + A(x)\, t)(1 - mt + xt^m) = 1 + (A(x) - m)\, t - mA(x)\, t^2 + xt^m + xA(x)\, t^{m+1}$$

then

$$\begin{aligned}\sigma f'_{m,\lambda,A}(x,t) &= \sum_{n\geq 0} (n+1)\, \mathcal{P}^{\lambda,A}_{n+1,m}(x)\, t^n + (A(x)-m) \sum_{n\geq 1} n\mathcal{P}^{\lambda,A}_{n,m}(x)\, t^n \\ &+ x \sum_{n\geq m} (n-m+1)\, \mathcal{P}^{\lambda,A}_{n-m+1,m}(x)\, t^n - mA(x) \sum_{n\geq 1} (n-1)\, \mathcal{P}^{\lambda,A}_{n-1,m}(x)\, t^n \\ &+ xA(x) \sum_{n\geq m} (n-m)\, \mathcal{P}^{\lambda,A}_{n-m,m}(x)\, t^n.\end{aligned}$$

Writing the equality

$$\sigma f'_{m,\lambda,A}(x,t) = \Delta f_{m,\lambda,A}(x,t)$$

in expansion series form and comparing the coefficients of t^n we get the result.

3. Generalized class of Humbert's polynomials

The class of Gegenbauer's polynomials [1, 2] $\left\{G^\lambda_n(x)\right\}_{n\geq 0}$ is defined by the following generating function

$$G^\lambda(x,t) = \frac{1}{(1-2xt+t^2)^\lambda} = \sum_{n\geq 0} G^\lambda_n(x)\, t^n \tag{3.1}$$

The corresponding recurrence relation is

$$nG^\lambda_n(x) = 2x(n+\lambda-1)\, G^\lambda_{n-1}(x) - (n+2\lambda-2)\, G^\lambda_{n-2}(x), \quad n\geq 2 \tag{3.2}$$

with starting values $G^\lambda_0(x) = 1$ and $G^\lambda_1(x) = 2\lambda x$. Their explicit form is

$$G^\lambda_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(\lambda)_{n-k}}{k!\,(n-2k)!}\, (2x)^{n-2k} \tag{3.3}$$

where

$$(\lambda)_n = \lambda(\lambda+1)\ldots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}.$$

and Γ is a gamma function.

Gegenbauer's polynomials are a particular case of Humbert's polynomials $\left\{\Pi_{n,m}^{\lambda}(x)\right\}_{n\geq 0}$ were defined in 1921 by Humbert [5]. Their generating function is

$$\frac{1}{(1-mxt+t^m)^{\lambda}} = \sum_{n\geq 0} \Pi_{n,m}^{\lambda}(x)\, t^n \tag{3.4}$$

Here we define a new generalization of Humbert's polynomials in a way similar to that for Catalan's polynomials different from the class given by H. W. Gould [4]:

$$(C-mxt+yt^m)^p = \sum_{n\geq 0} P_n(m,x,y,p,C)\, t^n$$

and the generalization defined by P. N. Shrivastava [9]:

$$\left(C-axt+bx^l t^m\right)^{-v} = \sum_{n\geq 0} P_n^{(l)}(m,x,a,v,b)\, t^n.$$

Definition 3.1. The generalized Humbert's polynomials of type $\Pi_{n,m}^{\lambda,A}(x)$ are given in means of the function.

$$h_{m,\lambda,A}(x,t) = \frac{1+A(x)\,t}{(1-mxt+t^m)^{\lambda}} = \sum_{n\geq 0} \Pi_{n,m}^{\lambda,A}(x)\, t^n.$$

Then the generalized Gegenbauer's polynomials are defined in means of the generating function

$$\frac{1+A(x)\,t}{(1-2xt+t^2)^{\lambda}} = \sum_{n\geq 0} G_{n,A}^{\lambda}(x)\, t^n.$$

It is obvious that the polynomial $\Pi_{n,m}^{\lambda,A}(x)$ is related to Humbert's polynomial $\Pi_{n,m}^{\lambda}(x)$ by the relation

$$\Pi_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}(x) + A(x)\,\Pi_{n-1,m}^{\lambda}(x) \tag{3.5}$$

and are identical for $A(x) = 0$.

Let $A(x)$ and $B(x)$ be two polynomials not forcedly of same degree. Some elementary arithmetic properties of those polynomials are:

$$\Pi_{n,m}^{\lambda,A}(x) + \Pi_{n,m}^{\lambda,-A}(x) = 2\Pi_{n,m}^{\lambda}(x), \tag{3.6}$$

$$\Pi_{n,m}^{\lambda,A}(x) - \Pi_{n,m}^{\lambda,B}(x) = [A(x)-B(x)]\,\Pi_{n-1,m}^{\lambda}(x) \tag{3.7}$$

and

$$\Pi_{n,m}^{\lambda,A+B}(x) = \Pi_{n,m}^{\lambda,A}(x) + \Pi_{n,m}^{\lambda,B}(x) - \Pi_{n,m}^{\lambda}(x). \tag{3.8}$$

The recurrence relation of $\Pi_{n,m}^{\lambda,A}(x)$ in means of $\mathcal{P}_{n,m}^{\lambda,A}(x)$, $\mathcal{P}_{n,m}^{\lambda}(x)$ and $\Pi_{n,m}^{\lambda}(x)$ is stated in the following theorem.

Theorem 3.1.

$$\mathcal{P}_{n-1,m}^{\lambda}(x)\left[\Pi_{n,m}^{\lambda,A}(x) - \Pi_{n,m}^{\lambda}(x)\right] = \Pi_{n-1,m}^{\lambda}(x)\left[\mathcal{P}_{n,m}^{\lambda,A}(x) - \mathcal{P}_{n,m}^{\lambda}(x)\right]. \tag{3.9}$$

This theorem is true for every polynomial $A(x)$ and all positive integers $m, n \geq 2$. At $x = 1$, $\Pi_{n,m}^{\lambda,A}(1)$ is identical to $\mathcal{P}_{n,m}^{\lambda,A}(1)$. The proof of Theorem 3.1 needs the following technical lemma

Lemma 3.1.

$$x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right) + x^{-1/m}A(x)\,\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right) \tag{3.10}$$

and for $A(x) = 0$,

$$\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right) = x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda}(x) \tag{3.11}$$

Remark 3.1. Taking into account the property (3.11), the relation (3.10) is a reformulation of the equality (2.2) in terms of Humbert's polynomials.

Proof. Writing $f_{m,\lambda,A}(x,t)$ under the following form

$$f_{m,\lambda,A}(x,t) = \frac{1}{(1 - mt + xt^m)^{\lambda}} + \frac{A(x)\,t}{(1 - mt + xt^m)^{\lambda}}.$$

Then

$$f_{m,\lambda,A}(x,t) = \frac{1}{\left(1 - mx^{-1/m}\left(x^{1/m}t\right) + \left(x^{1/m}t\right)^m\right)^{\lambda}} + \frac{A(x)\,t}{\left(1 - mx^{-1/m}\left(x^{1/m}t\right) + \left(x^{1/m}t\right)^m\right)^{\lambda}}$$

and

$$f_{m,\lambda,A}(x,t) = \sum_{n\geq 0}\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)x^{n/m}t^n + A(x)\sum_{n\geq 0}\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)x^{n/m}t^{n+1}.$$

Thus

$$\sum_{n\geq 1}\mathcal{P}_{n,m}^{\lambda,A}(x)\,t^n = \sum_{n\geq 1}\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)x^{n/m}t^n + A(x)\sum_{n\geq 1}\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right)x^{n-1/m}t^n.$$

Furthermore

$$\mathcal{P}_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)x^{n/m} + A(x)\,\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right)x^{n-1/m}.$$

Finally

$$x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right) + x^{-1/m}A(x)\,\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right).$$

When $A(x) = 0$ the result (3.11) is deduced. □

From the expression (3.11) Lemma 3.1 we deduce that

$$G_n^1\left(x^{-1/2}\right) = \Pi_{n,2}^1\left(x^{-1/2}\right) = x^{-\frac{n}{2}}\mathcal{P}_n(x)$$

and Catalan's polynomials $P_n(x)$ are joined to Gegenbauer's polynomials $G_n^1(x)$ by the following useful relation

$$P_n\left(\frac{x}{4}\right) = 2^{-n}x^{\frac{n}{2}}G_n^1\left(x^{-1/2}\right)$$

Each relation leads to

$$P_n(x) = x^{\frac{n}{2}}G_n^1\left((4x)^{-1/2}\right)$$

Taking into account the expression (3.3) of the polynomial $G_n^{\lambda}(x)$ and remarking that $(1)_{n-k} = \Gamma(n-k+1) = (n-k)!$ we deduce that

$$x^{\frac{n}{2}}G_n^1\left((4x)^{-1/2}\right) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\frac{(n-k)!}{k!\,(n-2k)!}x^k$$

Since

$$\frac{(n-k)!}{k!\,(n-2k)!} = \binom{n-k}{k}$$

the binomial sum representation (1.4) of $P_n(x)$ is deduced.

Combining the results in Corollary 2.1 and Lemma 3.1 we get the Humbert recurrence relation [5, 7, 8, 3].

Corollary 3.1. *If* $2 \leq n < m-1$

$$(n+1)\,\Pi_{n+1,m}^{\lambda}(x) = m(\lambda+n)\,x\Pi_{n,m}^{\lambda}(x), \tag{3.12}$$

If $n \geq m$

(3.13)

$$(n+1)\,\Pi_{n+1,m}^{\lambda}(x) - mx(n+\lambda)\,\Pi_{n,m}^{\lambda}(x) + (n-m+1+\lambda m)\,\Pi_{n-m+1,m}^{\lambda}(x) = 0$$

and

$$(\lambda+m-1)\,x\Pi_{m-1,m}^{\lambda}(x) - \Pi_{m,m}^{\lambda}(x) = \lambda \tag{3.14}$$

The author is thankful to Professor G. V. Milovanović for the information that there is a misprint for the recurrence relation of $\Pi_{n,m}^{\lambda}(x)$ in the works [3], [5] and [7]. The proper one is in the relation 8 [8] rewritten for polynomials $\Pi_{n,m}^{\lambda}(2x/m)$.

Proof. Substituting the value of $\mathcal{P}_{n,m}^{\lambda}(x)$ token from the expression (3.11), Lemma 3.1 in the recurrence formulae (2.5), (2.6) and (2.6) Corollary 2.1, we deduce the recurrence relations (3.12), (3.13) and (3.14) of $\Pi_{n,m}^{\lambda}(x)$. □

For $m = 2$, all the formulae (3.12), (3.13) and (3.14) are reduced to one formula because n is only greater than 1 for $m = 2$. This formula is the well-known recurrence relation [7] of Gegenbauer's polynomials.

$$(n+1)\,G_{n+1}^{\lambda}(x) - 2x\,(n+\lambda)\,G_{n}^{\lambda}(x) + (n-1+2\lambda)\,G_{n-1}^{\lambda}(x) = 0\ , n \geq 1$$

3.1. Proof of Theorem 3.1

The relation 3.10 states that

$$A(x) = \frac{x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda,A}(x) - \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)}{x^{-1/m}\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right)}$$

Substitute this value in the equality (3.5) we get

$$\Pi_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}(x) + \frac{x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda,A}(x) - \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)}{x^{-1/m}\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right)}\Pi_{n-1,m}^{\lambda}(x)\,.$$

Using the relation $\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right) = x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda}(x)$ the result (3.9) of Theorem 3.1 holds.

4. Special class of generalized Catalan's polynomials

In this section we study the special class $M_n(x)$ of generalized Catalan's polynomials defined by the generating function

$$g(x,t) = \frac{1+(1-x)\,t}{1-xt+t^2} = \sum_{n\geq 0} M_n(x)\,t^n$$

and starting values $M_0(x) = M_1(x) = 1$.

The polynomials $M_n(x)$ are an interesting example of generalized Gegenbauer's polynomials. It is enough to note that

$$g(2x,t) = \sum_{n\geq 0} G_{n,A}^{1}(x)\,t^n \text{ with } A(x) = 1-2x$$

and then $M(2x) = G_{n,A}^{1}(x)$.

Proposition 4.1. *The generalized Catalan's polynomials $M_n(x)$ depend on Catalan's polynomials. More precisely, we get the following expression*

$$M_n(x) = x^n P_n\left(x^{-2}\right) + x^{n-1}(1-x)\,P_{n-1}\left(x^{-2}\right) \tag{4.1}$$

We note that for $x = 1$ only $M_n(1) = P_n(1)$ for any positive integer n.

4.1. Proof of the Proposition 4.1

Since

$$f_{2,0,1}(x,t)=\frac{1}{1-2t+xt^2}$$

then

$$f_{2,0,1}\left(x,\frac{t}{2x}\right)=\frac{1}{1-2x^{-1/2}\left(\frac{x^{-1/2}t}{2}\right)+\left(\frac{x^{-1/2}t}{2}\right)^2}$$

and

$$\left(1+\left(\frac{1}{2}-x^{-1/2}\right)x^{-1/2}t\right)f_{2,0,1}\left(x,\frac{t}{2x}\right)=g\left(2x^{-1/2},\frac{x^{-1/2}t}{2}\right).$$

Replacing the variable $x^{-1/2}$ by x we get

$$\left(1+\left(\frac{1}{2}-x\right)xt\right)f_{2,0,1}\left(1/x^2,\frac{x^2}{2}t\right)=g\left(2x,\frac{xt}{2}\right)$$

Thus

$$\sum_{n\geq 0}M_n(2x)\,2^{-n}x^nt^n=\left(1+\left(\frac{1}{2}-x\right)xt\right)\sum_{n\geq 0}\mathcal{P}_n\left(x^{-2}\right)2^{-n}x^{2n}t^n$$

and

$$\sum_{n\geq 0}M_n(2x)\,2^{-n}x^nt^n=\sum_{n\geq 0}\mathcal{P}_n\left(x^{-2}\right)2^{-n}x^{2n}t^n+\left(\frac{1}{2}-x\right)x\sum_{n\geq 1}\mathcal{P}_{n-1}\left(x^{-2}\right)2^{-n+1}x^{2n-2}t^n$$

After getting the series expansion and comparing the coefficients of t^n by using the relationship between $\mathcal{P}_n(x)$ and $P_n(x)$ we conclude that

$$M_n(2x)=(2x)^n\left(P_n\left(\frac{x^{-2}}{4}\right)-P_{n-1}\left(\frac{x^{-2}}{4}\right)\right)+(2x)^{n-1}P_{n-1}\left(\frac{x^{-2}}{4}\right)$$

Replacing $2x$ by x we get

$$M_n(x)=x^n\left[P_n\left(x^{-2}\right)-P_{n-1}\left(x^{-2}\right)\right]+x^{n-1}P_{n-1}\left(x^{-2}\right)$$

and the result 4.1 follows.

4.2. Explicit form of the class $\{M_n)\}_{n\geq 0}$ and application to combinatorics

Applying the formula 4.1 Proposition 4.1 and the recurrence formula 1.1 to $P_n\left(x^{-2}\right)$, we easily found the following recurrence formula of the class $\{M_n(x)\}_{n\geq 0}$.

$$M_{n+2}(x)=xM_{n+1}(x)-M_n(x) \tag{4.2}$$

Table 4.1: First few polynomials

n	$M_n(x)$
0	1
1	1
2	$x-1$
3	x^2-x-1
4	x^3-x^2-2x+1
5	$x^4-x^3-3x^2+2x+1$
6	$x^5-x^4-4x^3+3x^2+3x-1$

In means of this relation the first few polynomials are given in the table 4.1.

Their binomial sum expression is given in the following lemma

Lemma 4.1.

$$M_{2n}(x) = \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n+k-1}{n-k-1} x^{2k} - \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n+k}{n-k-1} x^{2k+1} \tag{4.3}$$

$$M_{2n+1}(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} x^{2k} + \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n+k}{n-k-1} x^{2k+1}. \tag{4.4}$$

Proof. From the formula (4.1) and the expression (1.4) of $P_n(x)$ we deduce that

$$P_n\left(x^{-2}\right) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{-2k},$$

and

$$\begin{aligned} M_n(x) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} (-1)^k x^{n-1-2k} \\ &+ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-1)^k x^{n-2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} (-1)^k x^{n-2k} \end{aligned}$$

After simplification, following the parity of n we get the results (4.3) and (4.4) of Lemma 4.1.

□

The coincidence $M_n(1) = P_n(1)$, the explicit formula of $M_n(x)$ found in (4.3) of Lemma 4.1 and the binomial form of $P_n(x)$ include the following two useful combinatorial identities.

Proposition 4.2.

$$\sum_{k=0}^{n-1}(-1)^k\left(n^2+3k^2+2k\right)\frac{(n+k-1)!}{(n-k)!\,(2k+1)!}=(-1)^{n+1} \tag{4.5}$$

$$\sum_{j=k}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2j+1}\binom{j}{k}=2^{n-2k}\binom{n-k}{k} \tag{4.6}$$

4.2.1. Proof of Proposition 4.2

The expression 4.3 Lemma 4.1 of the polynomial $M_{2n}(x)$ conducts to

$$M_{2n}(1)=\sum_{k=0}^{n-1}(-1)^{n-k}\left(\binom{n+k-1}{n-k-1}-\binom{n+k}{n-k-1}\right)$$

Then

$$M_{2n}(1)=-(-1)^n\sum_{k=0}^{n-1}(-1)^k\binom{n+k-1}{n-k-2}$$

But $P_{2n}(1)$ can be written in this form

$$P_{2n}(1)=1+(-1)^n\sum_{k=0}^{n-1}(-1)^k\binom{n+k}{n-k}.$$

Since $M_{2n}(1)=P_{2n}(1)$ then

$$1=(-1)^{n+1}\sum_{k=0}^{n-1}(-1)^k\left(\binom{n+k-1}{n-k-2}+\binom{n+k}{n-k}\right).$$

Using the relation

$$\binom{n+k}{n-k}=\frac{(2k+1)(n+k)}{(n-k)(n-k-1)}\binom{n+k-1}{n-k-2}$$

and the fact that

$$\binom{n+k-1}{n-k-2}+\binom{n+k}{n-k}=\left(n^2+3k^2+2k\right)\frac{(n+k-1)!}{(n-k)!\,(2k+1)!}$$

the result (4.5) of Proposition 4.2 holds.

For the second identity let us denote $f(x)=\sqrt{1-4x}$, then from the closed form (1.2) we deduce that

$$f(x)\,P_n(x)=\frac{1}{2^{n+1}}\left[(1+f(x))^{n+1}-(1-f(x))^{n+1}\right].$$

Using the binomial formula we get

$$f\left(x\right)P_n\left(x\right)=\frac{1}{2^{n+1}}\left[\sum_{j=0}^{n+1}\binom{n+1}{j}f^j\left(x\right)-\sum_{j=0}^{n+1}\binom{n+1}{j}\left(-1\right)^j f^j\left(x\right)\right].$$

Then

$$f\left(x\right)P_n\left(x\right)=\frac{1}{2^{n+1}}\sum_{j=0}^{n+1}\left(1-\left(-1\right)^j\right)\binom{n+1}{j}f^j\left(x\right).$$

Furthermore

$$P_n\left(x\right)=\frac{1}{2^n}\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n+1}{2j+1}f^{2j}\left(x\right)$$

and then

$$P_n\left(x\right)=\frac{1}{2^n}\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n+1}{2j+1}\left(1-4x\right)^j.$$

Using again the binomial formula for the power $(1-4x)^j$ we obtain

$$P_n\left(x\right)=\frac{1}{2^n}\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{k=0}^{j}\binom{n+1}{2j+1}\binom{j}{k}\left(-4x\right)^k.$$

hence

$$P_n\left(x\right)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{1}{2^{n-2k}}\left[\sum_{j=k}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n+1}{2j+1}\binom{j}{k}\right]\left(-x\right)^k.$$

After After comparison with the binomial form (1.4) of $P_n(x)$, the result (4.6) of Proposition (4.2) holds.

REFERENCES

1. G. B. Djordjević and G. V. Milovanović: *Special Classes of Polynomials.* University of Niš, Faculty of Technology, Leskovac 2014.
2. L. Gegenbauer: *Zur Theorie der Functionen $C_n^v(x)$.* Osterreichische Akademie der Wissenschaften Mathematisch Naturwissen Schaftliche Klasse Denkscriften **48** (1884), 293-316
3. O. Golsah, Y. Simsek and G. V. Milovanović: *Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials.* Mideterr. J. Math. 14, 3, **117** (2017), 17 p.
4. H. W. Gould: *Inverse series relations and other expansions involving Humbert polynomials.* Duke Math. J. **32**, 4 (1965), 697-712

5. P. Humbert: *Some extensions of Pincherle's polynomials*. Proc. Edinburgh Math. Soc. **39**, 1 (1921), 21–24.
6. A. F. Jarvis, P. J. Larcombe and E. J. Fennessey: *Some Factorization and Divisibility Properties of Catalan Polynomials*. Bulletin of the ICA. **71** (2014), 36–56.
7. G. V. Milovanović and G. B. Djordjević: *On some properties of Humbert's polynomials II*. Facta Univ. Ser. Math. Inform. **6** (1991), 23–30.
8. G. V. Milovanović and G. B. Djordjević: *On some properties of Humberts polynomials*. Fibonacci Quart. **25**, Number 4 (1987), 356-360
9. P. N. Shrivastava: *On a generalization of Humbert polynomials*. Publication de d'Institut Mathmatiques, Nouvelle, série. Tome 22, **36** (1997), 245–253

Mouloud Goubi
Faculty of Science
Department of Mathematics
University of UMMTO
P. O. Box 17
15000 Tizi-ouzou, Algeria
mouloud.ummto@hotmail.fr