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# ON TZITZEICA CURVES IN EUCLIDEAN 3-SPACE $\mathbb{E}^3$

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**Abstract.** In this study, we consider Tzitzeica curves (Tz-curves) in a Euclidean 3-space $\mathbb{E}^3$. We characterize such curves according to their curvatures. We show that there is no Tz-curve with constant curvatures (W-curves). We consider Salkowski (TC-curve) and anti-Salkowski curves.
**Keywords**: Tz-curves, W-curves, TC-curves

## 1. Introduction

Gheorgha Tzitzeica, a Romanian mathematician (1872-1939), introduced a class of curves, nowadays called Tzitzeica curves, and a class of surfaces of the Euclidean 3-space called Tzitzeica surfaces. A Tzitzeica curve in $\mathbb{E}^3$ is a spatial curve $x = x(s)$ for which the ratio of its torsion $\kappa_2$ and the square of the distance $d_{osc}$ from the origin to the osculating plane at an arbitrary point $x(s)$ of the curve is constant, i.e.,

$$\frac{\kappa_2}{d_{osc}^2} = a \tag{1.1}$$

where $d_{osc} = \langle N_2, x\rangle$ and $a \neq 0$ is a real constant, $N_2$ is the binormal vector of $x$.

In [3] the authors gave the connections between the Tzitzeica curve and the Tzitzeica surface in a Minkowski 3-space and the original ones from the Euclidean 3-space. In [7] the authors determined the elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition in a Euclidean space. In [12], the elliptic cylindrical curves verifying Tzitzeica condition were adapted to the Minkowski 3-space. In [2], the authors gave the necessary and sufficient condition for a space curve to become a Tzitzeica curve. The new classes of symmetry reductions for the Tzitzeica curve equation were determined. In [1], the authors were interested in the curves of Tzitzeica type and they investigated the conditions for non-null general helices, pseudo-spherical curves and pseudo-spherical general helices to become of Tzitzeica type in a Minkowski space $\mathbb{E}_1^3$.

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A Tzitzeica surface in $\mathbb{E}^3$ is a spatial surface $M$ given with the parametrization $X(u,v)$ for which the ratio of its Gaussian curvature $K$ and the distance $d_{\tan}$ from the origin to the tangent plane at any arbitrary point of the surface is constant, i.e.,

$$\frac{K}{d_{\tan}^4} = a_1 \tag{1.2}$$

for a constant $a_1$. The orthogonal distance from the origin to the tangent plane is defined by

$$d_{\tan} = \left\langle X, \overrightarrow{U} \right\rangle \tag{1.3}$$

where $X$ is the position vector of the surface and $\overrightarrow{U}$ is a unit normal vector of the surface.

The asymptotic lines of a Tzitzeica surface with a negative Gausssian curvature are Tzitzeica curves [7]. In [18], the authors gave the necessary and sufficient condition for the Cobb-Douglas production hypersurface to be a Tzitzeica hypersurface. In addition, a new Tzitzeica hypersurface was obtained in parametric, implicit and explicit forms in [8]

In this study, we consider Tzitzeica curves (Tz-curves) in a Euclidean 3-space $\mathbb{E}^3$. Furthermore, we investigate a Tzitzeica curve in a Euclidean 3-space $\mathbb{E}^3$ whose position vector $x = x(s)$ satisfies the parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s), \tag{1.4}$$

for some differentiable functions, $m_i(s)$, $0 \le i \le 2$, where $\{T, N_1, N_2\}$ is the Frenet frame of $x$. We characterize such curves according to their curvatures. We show that there is no Tzitzeica curve in $\mathbb{E}^3$ with constant curvatures (W-curves). We give the relations between the curvatures of the Tz-Salkowski curve (TC-curve) and the Tz-anti-Salkowski curve.

## 2. Basic Notations

Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in a Euclidean 3-space $\mathbb{E}^3$. Let us denote $T(s) = x'(s)$ and call $T(s)$ a unit tangent vector of $x$ at $s$. We denote the curvature of $x$ by $\kappa_1(s) = \|x''(s)\|$. If $\kappa_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve $x$ at $s$ is given by $x^{''}(s) = \kappa_1(s)N_1(s)$. The unit vector $N_2(s) = T(s) \times N_1(s)$ is called the unit binormal vector of $x$ at $s$. Then we have the Serret-Frenet formulae:

$$\begin{aligned} T'(s) &= \kappa_1(s)N_1(s), \\ N_1'(s) &= -\kappa_1(s)T(s) + \kappa_2(s)N_2(s), \\ N_2'(s) &= -\kappa_2(s)N_1(s), \end{aligned} \tag{2.1}$$

where $\kappa_2(s)$ is the torsion of the curve $x$ at $s$ (see, [10]).

If the Frenet curvature $\kappa_1(s)$ and torsion $\kappa_2(s)$ of $x$ are constant functions then $x$ is called a screw line or a helix [9]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations then F. Klein and S. Lie called them *W-curves* [14]. It is known that a curve $x$ in $\mathbb{E}^3$ is called a *general helix* if the ratio $\kappa_2(s)/\kappa_1(s)$ is a nonzero constant [16]. Salkowski (resp. anti-Salkowski) curves in a Euclidean space $\mathbb{E}^3$ are generally known as the family of curves with A constant curvature (resp. torsion) but non-constant torsion (resp. curvature) with an explicit parametrization [15, 17] (for T.C-curve see also [13]).

For a space curve $x : I \subset \mathbb{R} \to \mathbb{E}^3$, the planes at each point of $x(s)$ spanned by $\{T, N_1\}$, $\{T, N_2\}$ and $\{N_1, N_2\}$ are known as the *osculating plane*, the *rectifying plane* and *normal plane*, respectively. If the position vector $x$ lies on its rectifying plane, then $x(s)$ is called *rectifying curve* [5]. Similarly, the curve for which the position vector $x$ always lies in its osculating plane is called *osculating curve.* Finally, $x$ is called *normal curve* if its position vector $x$ lies in its normal plane.

Rectifying curves characterized by the simple equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s), \tag{2.2}$$

where $\lambda(s)$ and $\mu(s)$ are smooth functions and $T(s)$ and $N_2(s)$ are tangent and binormal vector fields of $x$, respectively [5, 6].

For a regular curve $x(s)$, the position vector $x$ can be decomposed into its tangential and normal components at each point:

$$x = x^T + x^N. \tag{2.3}$$

A curve in $\mathbb{E}^3$ is called *N-constant* if the normal component $x^N$ of its position vector $x$ is of constant length [4, 11]. It is known that a curve in $\mathbb{E}^3$ is congruent to an $N$-constant curve if and only if the ratio $\frac{\kappa_2}{\kappa_1}$ is a non-constant linear function of an arc-length function $s$, i.e., $\frac{\kappa_2}{\kappa_1}(s) = c_1 s + c_2$ for some constants $c_1$ and $c_2$ with $c_1 \neq 0$ [4]. Further, an $N$-constant curve $x$ is called first kind if $\left\|x^N\right\| = 0$, otherwise second kind [11].

## 3. Tzitzeica Curves in $\mathbb{E}^3$

In the present section we characterize Tzitzeica curves in $\mathbb{E}^3$ in terms of their curvatures.

**Definition 3.1.** Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve with curvatures $\kappa_1(s) > 0$ and $\kappa_2(s) \neq 0$. If the torsion of $x$ satisfies the condition

$$\kappa_2(s) = a.d_{osc}^2, \tag{3.1}$$

for some real constant $a$ then $x$ is called Tzitzeica curve (Tz-curve), where

$$d_{osc} = \langle N_2, x\rangle \tag{3.2}$$

is the orthogonal distance from the origin to the osculating plane of $x$.

We have the following result.

**Proposition 3.1.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$. If $x$ is a Tz-curve, then the equation*

$$\kappa_2' \langle x, N_2 \rangle + 2\kappa_2^2 \langle x, N_1 \rangle = 0 \tag{3.3}$$

*holds.*

*Proof.* Let $x$ be a unit speed curve in $\mathbb{E}^3$, then by the use of the equations (3.1) and (3.2) we get

$$\frac{\kappa_2(s)}{\langle N_2, x \rangle^2} = a \neq 0. \tag{3.4}$$

Further, differentiating the equation (3.4), we obtain the result. □

**Definition 3.2.** Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve with curvatures $\kappa_1(s) > 0$ and $\kappa_2(s) \neq 0$. Then $x$ is a spherical curve if and only if

$$\frac{\kappa_2(s)}{\kappa_1(s)} = \left( \frac{\kappa_1'(s)}{\kappa_2(s)\kappa_1^2(s)} \right)' \tag{3.5}$$

holds [9].

**Theorem 3.1.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed spherical curve in $\mathbb{E}^3$. If $x$ is a Tz-curve then the equation*

$$\frac{\kappa_2'(s)}{2\kappa_2^3(s)} = \frac{\kappa_1(s)}{\kappa_1'(s)} \tag{3.6}$$

*holds between the curvatures of $x$.*

*Proof.* Let $x$ be a unit speed spherical curve in $\mathbb{E}^3$. Then we have

$$\|x\| = r \tag{3.7}$$

where $r$ is the radius of the sphere. Differentiating the equation (3.7) with respect to $s$, we get

$$\langle x, T \rangle = 0. \tag{3.8}$$

Further, differentiating the equation (3.8), we have

$$\langle x, N_1 \rangle = -\frac{1}{\kappa_1}. \tag{3.9}$$

By differentiating the equation (3.9), we obtain

$$\langle x, N_2 \rangle = \frac{\kappa_1'}{\kappa_1^2 \kappa_2}. \tag{3.10}$$

Finally, substituting (3.9) and (3.10) into (3.3), we get the result. □

**Corollary 3.1.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed spherical Tz-curve in $\mathbb{E}^3$. Then the torsion of $x$ satisfies the equation*

$$\kappa_2 = \sqrt{\frac{\kappa_1''\kappa_1 - 2(\kappa_1')^2}{3\kappa_1^2}}. \tag{3.11}$$

*Proof.* Substituting (3.6) into (3.5), we get the result. □

**Corollary 3.2.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed anti-Salkowski spherical Tz-curve in $\mathbb{E}^3$. Then the curvature of $x$ is given by*

$$\kappa_1 = \frac{\sqrt{3}\kappa_2}{c_1 \sin\left(\sqrt{3}\kappa_2 s\right) - c_2 \cos\left(\sqrt{3}\kappa_2 s\right)} \tag{3.12}$$

*where $c_1$, $c_2$ are integral constants and $\kappa_2$ is the constant torsion of $x$.*

*Proof.* Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed anti-Salkowski spherical Tz-curve in $\mathbb{E}^3$. Then from (3.11), we obtain the differential equation

$$\kappa_1''\kappa_1 - 2(\kappa_1')^2 - 3\kappa_1^2\kappa_2^2 = 0 \tag{3.13}$$

which has the solution (3.12). □

**Lemma 3.1.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$ whose position vector satisfies the parametric equation*

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s) \tag{3.14}$$

*for some differentiable functions, $m_i(s)$, $0 \leq i \leq 2$. If $x$ is a Tz-curve then we get*

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' + \kappa_2 m_1 &= 0, \\ \kappa_2' m_2 + 2\kappa_2^2 m_1 &= 0. \end{aligned} \tag{3.15}$$

*Proof.* Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$. Then, by taking the derivative of (3.14) with respect to the parameter $s$ and using the Frenet formulae, we obtain

$$\begin{aligned} x'(s) = \; & (m_0'(s) - \kappa_1(s)m_1(s))T(s) \\ & + (m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) \\ & + (m_2'(s) + \kappa_2(s)m_1(s))N_2(s). \end{aligned} \tag{3.16}$$

Further, using the equations (3.3) and (3.16), we get (3.15). □

**Theorem 3.2.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed anti-Salkowski Tz-curve in $\mathbb{E}^3$ (with the curvatures $\kappa_1 > 0$ and $\kappa_2 \neq 0$) given with the parametrization (3.14). Then $x$ is congruent to a rectifying curve with the parametrization*

$$x(s) = (s + c_1)\, T(s) + c_2 N_2(s) \tag{3.17}$$

*where $c_1$ and $c_2$ are integral constants.*

*Proof.* Let $x$ be a unit speed anti-Salkowski Tz-curve in $\mathbb{E}^3$. Then, the torsion $\kappa_2$ of $x$ is constant. From the equation (3.15), we get

$$\begin{aligned} m_0 &= s + c_1 \\ m_1 &= 0 \\ m_2 &= c_2 \end{aligned} \tag{3.18}$$

where $c_1$ and $c_2$ are integral constants. Finally, substituting (3.18) into (3.14), we get the result. $\square$

**Corollary 3.3.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed anti-Salkowski Tz-curve in $\mathbb{E}^3$ (with curvatures $\kappa_1 > 0$ and $\kappa_2 \neq 0$) given with the parametrization (3.14). Then $x$ is congruent to $N$-constant curve of second kind.*

**Corollary 3.4.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed Salkowski Tz-curve in $\mathbb{E}^3$ (with the curvatures $\kappa_1 > 0$ and $\kappa_2 \neq 0$) given with the parametrization (3.14). Then we have*

$$m_1'' + \left(\kappa_1^2 + 3\kappa_2^2\right) m_1 + \kappa_1 = 0 \tag{3.19}$$

*where the curvature $\kappa_1$ of $x$ is a real constant.*

*Proof.* Let $x$ be a unit speed Salkowski Tz-curve in $\mathbb{E}^3$. Hence, the curvature $\kappa_1$ of $x$ is constant, from the equation (3.15), we get the result. $\square$

**Corollary 3.5.** *There is no Tz-curve with a constant curvature and a constant torsion. (i.e. Tz-W-curve)*

*Proof.* Let $x$ be a unit speed Tz-curve in $\mathbb{E}^3$ with a constant curvature and a constant torsion. (i. e. Tz-W-curve). Then, using (3.15), we obtain

$$\frac{\kappa_1(s)}{\kappa_2(s)} = \frac{c_2}{s + c_1} \tag{3.20}$$

which is a contradiction. $\square$

**REFERENCES**


1. M.E. Aydin, M. Ergüt: *Non-null curves of Tzitzeica type in Minkowski 3-space.* Romanian J. of Math. and Comp. Science **4(1)** (2014), 81-90.
2. N. Bila: *Symmetry raductions for the Tzitzeica curve equation.* Math. and Comp. Sci. Workin Papers **16** (2012).
3. A. Bobe, W. G. Boskoff and M. G. Ciuca: *Tzitzeica type centro-affine invariants in Minkowski space.* An. St. Univ. Ovidius Constanta **20(2)** (2012), 27-34.
4. B. Y. Chen: *Geometry of warped products as Riemannian submanifolds and related problems.* Soochow J. Math. **28** (2002), 125-156.
5. B. Y. Chen: *Convolution of Riemannian manifolds and its applications.* Bull. Aust. Math. Soc. **66** (2002), 177-191.
6. B.Y. Chen: *When does the position vector of a space curve always lies in its rectifying plane?.* Amer. Math. Monthly **110** (2003), 147-152.
7. M. Craşmareann: *Cylindrical Tzitzeica curves implies forced harmonic oscillators.* Balkan J. of Geom. and Its App. **7(1)** (2002), 37-42.
8. O. Constantinescu, M. Craşmareann.: *A new Tzitzeica hypersurface and cubic Finslerian metrics of Berwall type* Balkan J. of Geom. and Its App. **16(2)** (2011), 27-34.
9. A. Gray: *Modern differential geometry of curves and surface*, CRS Press, Inc. 1993.
10. H. Gluck: *Higher curvatures of curves in Euclidean space* Amer. Math. Monthly **73** (1966), 699-704.
11. S. Gürpinar, K. Arslan, G. Öztürk: *A Characterization of Constant-ratio Curves in Euclidean 3-space $E^3$.* Acta Universitatis Apulensis **44** (2015), 39–51.
12. M. K. Karacan, B. Bükçü: *On the elliptic cylindrical Tzitzeica curves in Minkowski 3-space.* Sci. Manga **5** (2009), 44-48.
13. B. Kiliç, K. Arslan and G. Öztürk: *Tangentially cubic curves in Euclidean spaces.* Differential Geometry-Dynamical Systems **10** (2008), 186-196.
14. F. Klein, S. Lie: *Uber diejenigen ebenenen kurven welche durch ein geschlossenes system von einfach unendlich vielen vartauschbaren linearen Transformationen in sich übergehen* Math. Ann. **4** (1871), 50-84.
15. J. Monterde: *Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion.* Computer Aided Geometric Design. **26** (2009) 271–278.
16. G. Öztürk, K. Arslan and H. Hacisalihoğlu: *A characterization of ccr-curves in $\mathbb{R}^n$.* Proc. Estonian Acad. Sciences **57** (2008), 217-224.
17. E. Salkowski: *Zur transformation von raumkurven.* Mathematische Annalen. **66(4)** (1909) 517–557.
18. G. E. Vilcu: *A geometric perspective on the generalized Cobb-Douglas production function.* Appl. Math. Lett. **24** (2011), 777-783.

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