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ROUGHLY GEODESIC $B-r-$PREINVEX FUNCTIONS ON CARTAN HADAMARDMANIFOLDS *

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Abstract. In this paper, we introduce a new class of functions called roughly geodesic $B-r-$ preinvex function on a Hadamard manifold and establish some properties of roughly geodesic $B-r-$ preinvex functions on Hadamard manifolds. It is observed that a local minimum point for a scalar optimization problem is also a global minimum point under roughly geodesic B-r- preinvexity on the Hadamard manifolds. The results presented in this paper extend to and generalize the results in the literature.
Keywords: Hadamard manifold, preinvex function, minimum point.

1. Introduction

In mathematics, the concept of convexity is well known and it contributes a fundamental character to engineering, mathematical economics, optimization theory, management science and Riemannian manifolds. One of the most significant applications of the convex function is that any local minimum is also a global minimum. However, convexity does not give accurate results in real world mathematical problems and economic models. For this reason various authors have introduced concepts of generalized convex functions. Initially in 1981 Hanson [11] presented a significant generalization of the convex function which was later known as an invex function. Further, the convex set and the convex function were generalized by Ben-Israel and Mond [10], and called invex set and preinvex function, correspondingly. The characterization of preinvex functions and its applications in optimization theory have been discussed in [14, 27]. Noor [19, 23] studied the equilibrium problems and variational inequalities under these functions. Many articles have appeared in the literature on preinvex functions (see, [1, 2, 3, 5, 9, 12, 16, 20, 28]).

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Few results related to optimization theory and nonlinear analysis have been enhanced on Riemannian manifolds from the Euclidean space. Geodesic convexity proposed by Rapcsak [25] and Udriste [26], which is a natural generalization of convexity in which linear space is exchanged by Riemannian manifolds. Furthermore, on a Riemannian manifold the concept of invexity was introduced by Pini [24] and its generalization was explored by Mititelu [18]. On a Riemannian manifold, the geodesic invex set, geodesic η-preinvex function and geodesic η-invex function have been explained by Barani and Pouryayevali [9] and they have also discussed the relationships between these functions. Moreover, the geodesic α-preinvex function is a generalization of the notion of geodesic η-preinvexity introduced by R.P. Agrawal et al. [1]. Recently, the notions of B-invex set and B-invex function were studied on Riemannian manifolds by Zhou and Huang [28].

Analyzing the discussion of Barani and Pouryayevali [9] and Zhou and Huang [28], we attempt to deliberate the notions of geodesic $B-r$-invex set and geodesic $B-r$-preinvex function on a Riemannian manifold. These functions are a generalization of the preinvex function defined in [9, 5, 28]. Barani [8] presented the convexity and monotonicity of set valued mapping on Hadamard manifolds and presented the mean value theorem. Zou et al. [32] introduced the classical Penot generalized directional derivative and Clarke's generalized gradient and used these to discuss the first and second order necessary and sufficient conditions for a minimum point of the nonlinear programming problem. Recently, Jana and Nahak [13] obtained the optimality conditions for the nonlinear optimization problem under generalized invexities on a Riemannian manifold.

The paper is divided sectionally in the following way. In Section 2, we recall specific preliminaries, definitions and results, which are applied to demonstrate the work of this paper. We derive a new class function, namely, geodesic $B-r$-preinvex and geodesic $B-r$-invex function in Section 3. Some properties and relations between the geodesic log-preinvex and the geodesic $B-r$-invex function on a Riemannian manifold are studied in Section 4. Moreover, in Section 5, we discuss the results based on a lower semi-continuous log-preinvexity function with a proximal sub-differential and observe that for a mathematical optimization problem with log-preinvexity on a Riemannian manifold, its local minimum point is also a global minimum point. Finally, we obtain the mean value inequality on Hadamard manifolds in Section 6 and discuss the conclusions of the paper in Section 7.

2. Preliminaries

The present paper is based on the concept of generalized convexity on Cartan Hadamard manifolds. The objective of this paper is to present a new notion of roughly geodesic $B-r-$ invex and roughly geodesic $B-r-$ preinvex functions, and some properties and relationships between these functions are discussed. First of all, we apply the smoothness condition on a roughly geodesic $B-r-$ preinvex function with lower semi-continuity and try to obtain the existence condition for a global minimum. Finally, we obtain the mean value inequality for $B-r-$ preinvex function on Cartan Hadamard manifolds.

To recall some basic definitions of and results for Riemannian manifolds for further study, we refer the reader to ([9], [17], [28], [24]) and references therein.

Definition 2.1. A simply connected complete Riemannian manifold with a non-positive sectional curvature is called a Hadamard manifold. On a Hadamard manifold $\bar{M}$, there exists an exponential mapping $exp_p : T_p\bar{M} \to \bar{M}$ such that $exp_p v = \gamma_v(1)$, where γ_v is a geodesic defined by its position p and velocity γ at p.

Lemma 2.1.(Cartan-Hadamard theorem) Suppose X be a connected complete metric space which is locally convex. Then, with respect to the induced length metric d, the universal cover of X is a geodesic convex space. Let $(\bar{M}, \langle ., .\rangle)$ be a Hadamard manifold with a Riemannian metric $\langle ., .\rangle$. For a subset $U \subset \bar{M}$, a mapping $\eta \times \eta \to T\bar{M}$ is a function such that for every $u, v \in U$, $\eta(u, v) \in T\bar{M}$.

Definition 2.2. The geodesic distance $d(u, v)$ is the length of a minimal geodesic segment between any two points u, v on a manifold.

Definition 2.3. For a mapping $\psi : U \to R$, if the following limit

$$limit_{\lambda\to 0} \frac{\psi(exp_v \lambda\eta(u,v)) - \psi(v)}{\lambda\|\eta(u,v)\|},$$

exists, then ψ is said to be a $\eta(u, v)-$differentiable mapping at $v \in \bar{M}$,

Moreover, the $\eta(u, v)-$differential of ψ at v is given by

$$d_{\eta(u,v)}\psi(v) = limit_{\lambda\to 0} \frac{\psi(exp_v \lambda\eta(u,v)) - \psi(v)}{\lambda\|\eta(u,v)\|}.$$

Definition 2.4[9]. On a Riemannian manifold $\bar{M}$ for the function $\eta : \bar{M}\times\bar{M} \to T\bar{M}$ such that $\eta(u, v) \in T_v\bar{M}$, for every $u, v \in \bar{M}$. A non-empty subset U of $\bar{M}$ is said to be a geodesic invex set with respect to η if for every $u, v \in U$, there exists a unique geodesic $\gamma_{u,v} : [0, 1] \to \bar{M}$ such that

$$\gamma_{u,v}(0) = v, \quad \gamma'_{u,v}(0) = \eta(u, v), \quad \gamma_{u,v}(s) \in U, \quad for\ all\ s \in [0, 1].$$

3. Results and Discussion

The present section is divided into three subsections.

3.1. Geodesic B-invex sets and roughly geodesic $B-r-$preinvex functions

Convexity and its generalizations play an important role in the development of optimality conditions and duality theory. Various generalizations of convexity have appeared in literature. In [10], Ben-Israel and Mond introduced a new generalization of the convex function. Craven [29] named the invex function. Antczak [5] presented a generalization of the V-invex function [15] and the $r-$invex function [4], called a V-r-invex function. Bector and Singh [30] and Suneja et al. [31] introduced B-vex and B-preinvex functions. Recently, Zhou and Huang [28] defined the geodesic $B-$invex set as follows:

Definition 3.1 [28]. The set U is said to be a geodesic $B-$invex set on a Hadamard manifold with respect to η and $b(u,v,\lambda) : U \times U \times [0,1] \to R_+$, if for all $u, v \in U$ and $\lambda \in [0,1]$ such that $exp_v \lambda b \eta(u,v) \in U$.

U is said to be a geodesic $B-$invex set with respect to η on a Hadamard manifold, if U is $B-$invex for all $u, v \in U$ on a Hadamard manifold with respect to η.

Definition 3.2 [28]. Let U be a geodesic $B-$invex set. Then a mapping $\psi : U \to T\bar{M}$ is said to be a roughly geodesic B-preinvex function with respect to η with a roughness degree ρ at $v \in U$, if there exists $b(u,v,\lambda) : U \times U \times [0,1] \to R$ such that

$$f(exp_v \lambda b \eta(u,v)) \leq \lambda b \psi(u) + (1-\lambda b)\psi(v),$$

for all $u \in U$ and $\lambda \in [0,1]$ with $d(u,v) \geq \rho$, ψ is said to be a roughly geodesic $B-$preinvex function on U with respect to η, if it is a roughly geodesic B-preinvex function at any $v \in U$ with respect to the same η on U.

Now we introduce a roughly geodesic $B-r-$preinvex function on $\bar{M}$.

Definition 3.3. For a geodesic $B-$invex set U, the mapping $\psi : U \to T\bar{M}$ is said to be a roughly geodesic $B-r-$preinvex function with respect to η with a roughness degree ρ at $v \in U$, if there exists $b(u,v,\lambda) : U \times U \times [0,1] \to R_+$ such that

$$\psi(exp_v \lambda b \eta(u,v)) \leq \begin{cases} log(\lambda b e^{r\psi(u)} + (1-\lambda b)e^{r\psi(v)})^{\frac{1}{r}} & \text{if } r \neq 0, \\ \lambda b \psi(u) + (1-\lambda b)\psi(v) & \text{if } r = 0. \end{cases}$$

for any $u \in S$ and $\lambda \in [0,1]$ with $d(u,v) \geq \rho$. ψ is said to be a roughly geodesic $B-r-$preinvex function on U with respect to η and b, if it is a roughly geodesic $B-r-$preinvex function at any $v \in U$ with respect to η on U.

The function ψ is called a strictly roughly geodesic $B-r-$ preinvex function, if the above inequality holds as a strict inequality.

Remark 3.1. Every roughly geodesic $B-$preinvex function and geodesic $\eta-$preinvex function are a roughly geodesic $B-r-$preinvex function for $r=0$ and $b=1$, respectively. However, the converse does not hold in general.

We provide the following non-trivial example for the geodesic $B-r-$preinvex function but not a geodesic $B-$ preinvex function.

Example 3.1. Let $\bar{M}=\{e^{i\theta}:0<\theta<1\}$ and $\psi:\bar{M}\to R$ defined by $\psi(e^{i\theta})=\cos\theta$ with $u,v\in\bar{M}$, $u=e^{i\alpha}$ and $v=e^{i\beta}$. If $exp_v\lambda b\eta(u,v))=e^{i((1-\lambda b)\beta+\lambda b\alpha)}$, then ψ is a geodesic $B-r-$preinvex function but not a geodesic $B-$preinvex function at $\alpha=\frac{\pi}{2}$, $\beta=\frac{\pi}{4}, b=2$, since $\cos[\frac{\pi}{4}+\frac{\pi}{4}2\lambda]>\frac{1-2\lambda}{\sqrt{2}}$ for $\lambda=\frac{3}{4}$.

Proposition 3.1. If $\psi:U\to R$ is a roughly $B-r-$preinvex function with respect to $\eta:U\times U\to T\bar{M}$ and $v\in U$, then for any real number $k\in R$, the level set $U_k=\{u|u\in U,\psi(u)\leq k\}$ is a geodesic $B-$invex set.

Proof. For any $u,v\in U_k$, we have $\psi(u)\leq k$, $\psi(v)\leq k$. Since ψ is a roughly geodesic $B-r-$preinvex function, then we have

$$\psi(exp_v\lambda b\eta(u,v))\leq log(\lambda be^{r\psi(u)}+(1-\lambda b)e^{r\psi(v)})^{\frac{1}{r}},$$

or

$$\begin{aligned}e^{r\psi(exp_v\lambda b\eta(u,v))}&\leq\lambda be^{r\psi(u)}+(1-\lambda b)e^{r\psi(v)}\\&\leq\lambda be^{rk}+(1-\lambda b)e^{rk}.\end{aligned}$$

Equivalently,

$$e^{r\psi(exp_v\lambda b\eta(u,v))}\leq e^{rk},$$

or

$$\psi(exp_v\lambda b\eta(u,v)\leq k.$$

Therefore, $exp_v\lambda b\eta(u,v)\in U_k$ for all $\lambda\in[0,1]$, and the result is proved.

Theorem 3.1. Let U be a geodesic $B-$invex set and let $\psi:U\to R$ be a roughly geodesic $B-r-$preinvex with respect to $\eta:U\times U\to T\bar{M}$ with a roughness degree ρ on U. Then $epi(\psi)$ is a $B-$invex set on $U\times R$.

Proof. Let ψ be a roughly geodesic $B-r-$preinvex function with respect to $\eta:U\times U\to T\bar{M}$ with a roughness degree ρ on U. Then there exists $b(u,v,\lambda):U\times U\times[0,1]\to R_+$ such that

$$\psi(exp_v\lambda b\eta(u,v))\leq log(\lambda be^{r\psi(u)}+(1-\lambda b)e^{r\psi(v)})^{\frac{1}{r}},$$

where $exp_v\lambda b\eta(u,v)\in U$ and $d(u,v)\geq\rho$. Assume that $(u,\alpha),(v,\beta)\in epi(\psi)$. Then it is easy to see that $\psi(u)\leq\alpha,\psi(v)\leq\beta$, from these observations we have

$$\begin{aligned}\psi(exp_v\lambda b\eta(u,v))&\leq log(\lambda be^{r\psi(\alpha)}+(1-\lambda b)e^{r\psi(\beta)})^{\frac{1}{r}}\\&=log(e^{r\beta}+(e^{r\alpha}-e^{r\beta})\lambda b)^{\frac{1}{r}}.\end{aligned}$$

Therefore,

$$(exp_v \lambda b\eta(u,v), log(e^{r\beta} + (e^{r\alpha} - e^{r\beta})\lambda b)^{\frac{1}{r}}) \in epi(\psi),$$

which implies that $epi(\psi)$ is a $B-$invex set on $U \times R$.

Theorem 3.2. If $\phi_i : U \to R, i = 1,2,\ldots,m$ are roughly geodesic $B-r-$preinvex functions with respect to the same $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ with a roughness degree ρ on U, then the set defined by $\bar{M} = \{u \in U : \phi_i(u) \leq 0, i = 1,2,\ldots,m\}$ is a geodesic $B-$ invex set with respect to η.

Proof. Since $\phi_i(u), i = 1,2,\ldots,m$, are roughly geodesic $B-r-$preinvex functions, then there exists $b(u,v,\lambda) : \bar{M} \times \bar{M} \times [0,1] \to R_+$, such that

$$\phi_i(exp_v \lambda b\eta(u,v)) \leq log(\lambda b e^{r\phi_i(u)} + (1-\lambda b)e^{r\phi_i(v)})^{\frac{1}{r}} \leq log(\lambda b e^0 + (1-\lambda b)e^0)^{\frac{1}{r}},$$

or equivalently,

$$\phi_i(exp_v \lambda b\eta(u,v)) \leq 0, \quad i = 1,2,\ldots,m,$$

and so $exp_v \lambda b\eta(u,v) \in \bar{M}$. Thus, $\bar{M}$ is a geodesic $B-$ invex set.

Theorem 3.3. Let U be a geodesic B-invex set . If $\psi : U \to R$ is an $\eta-$differentiable roughly geodesic $B-r-$preinvex function with respect to $\eta : U \times U \to R$ with a roughness degree ρ at $v \in U$. Then there exists a function $\bar{b}(u,v) : U \times U \to R_+$ such that

$$\|\eta(u,v)\| d_{\eta(u,v)}\psi(v) \leq \frac{\bar{b}(u,v)}{r} e^{-r\psi(v)}(e^{r\psi(u)} - e^{r\psi(v)}),$$

for each $u \in U$ with $d(u,v) \geq \rho$ and $\bar{b}(u,v) = lim_{\lambda\to 0} b(u,v,\lambda)$.

Proof. If ψ is a roughly geodesic $B-r-$preinvex function at v, then there exists $b(u,v,\lambda) : U \times U \times [0,1] \to R_+$ such that

$$\psi(exp_v \lambda b\eta(u,v)) \leq log[\lambda b e^{r\psi(u)} + (1-\lambda b)e^{r\psi(v)}]^{\frac{1}{r}},$$

for each $u \in U$ and $\lambda \in [0,1]$ with $d(u,v) \geq \rho$. Since ψ is $\eta-$ differentiable at v, we have

$$d_{\eta(u,v)}\psi(v) = lim_{\lambda\to 0} \frac{\psi(exp_v \lambda b\eta(u,v)) - \psi(v)}{\lambda\|\eta(u,v)\|},$$

and so

$\psi(v) + d_{\eta(u,v)}\psi(v)\lambda\|\eta(u,v)\| + O^2(\lambda b) = \psi(exp_v \lambda b\eta(u,v)) \leq log[\lambda b e^{r\psi(u)} + (1 - \lambda b)e^{r\psi(v)}]^{\frac{1}{r}}$,

or

$$e^{r\psi(v) + r d_{\eta(u,v)}\psi(v)\lambda\|\eta(u,v)\| + rO^2(\lambda b)} - e^{r\psi(v)} \leq \lambda b(e^{r\psi(u)} - e^{r\psi(v)}).$$

Dividing by λ and taking the limit $\lambda \to 0$, we get

$$\|\eta(u,v)\| d_{\eta(u,v)}\psi(v) \leq \frac{\bar{b}(u,v)}{r} e^{-r\psi(v)}(e^{r\psi(u)} - e^{r\psi(v)}).$$

Remark 3.2. If $r = 0$, then the result in the above Theorem is similar to the result of Theorem 4.4 [28].

3.2. Roughly Geodesic $B-r-$ Preinvexity and semi continuity

Now we discuss geodesic $B-r-$preinvexity on a Cartan Hadamard manifold under a proximal subdifferential of a lower semi-continuous function. First, we recall the definition of the proximal subdifferentiable of a function defined on a Riemannian manifold [9].

Definition 3. 4. Let $\bar{M}$ be a Riemannian manifold and let $\psi : \bar{M} \to (-\infty, \infty]$ be a lower semi-continuous function. A point $\xi \in T_v\bar{M}$ is said to be the proximal subgradient of ψ at $v \in dom(\psi)$, if there exist positive numbers δ and σ such that

$$\psi(u) \geq \psi(v) + < \xi, exp_v^{-1}u >_v - \sigma d^2(u,v),$$

for all $u \in B(v,\delta)$, where $dom\psi = \{u \in \bar{M} : \psi(u) < \infty\}$. The set of all proximal subgradients of $v \in \bar{M}$ is denoted by $\partial_p\psi(v)$.

Theorem 3.4. Let $\bar{M}$ be a Hadamard manifold and U be a geodesic $B-$invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ and $b(u,v,\lambda) : U \times U \times [0,1] \to R$. Let $\psi : U \to R$ be a roughly geodesic $B-r-$ preinvex function. If $\bar{u} \in U$ is a local minimum of the problem

$$\text{(P)} \qquad \text{Minimize } \psi(u), \quad \text{subject to } u \in U,$$

then $\bar{u}$ is a global minimum of (P).

Proof. If $\bar{u} \in U$ is a local minimum, then there exists a neighbourhood $N_\epsilon(\bar{u})$ such that

$$\psi(\bar{u}) \leq \psi(u), \tag{3.1}$$

for all $u \in U \cap N_\epsilon(\bar{u})$.

If $\bar{u}$ is not a global minimum of ψ, then there exists a point $u^* \in U$ such that

$$\psi(u^*) < \psi(\bar{u}),$$

or

$$e^{r\psi(u^*)} < e^{r\psi(\bar{u})}.$$

As U is a geodesic $B-$invex set with respect to η and b, there exists a unique geodesic $\gamma(\lambda b) = exp_v(\lambda b\eta(u^*, \bar{u}))$ such that $\gamma(0) = \bar{u}$, $\gamma'(0) = \bar{b}(u^*, \bar{u})\eta(u^*, \bar{u})$ where $lim_{\lambda\to 0} b(u^*, \bar{u}, \lambda) = \bar{b}(u^*, \bar{u})$, $exp_v(\lambda b\eta(u^*, \bar{u})) \in U$, for all $\lambda \in [0, 1]$. If we choose $\epsilon > 0$ such that $d(\gamma(\lambda b), \bar{u}) < \epsilon$, then $\gamma(\lambda b) \in N_\epsilon(\bar{x})$. From the roughly geodesic $B - r-$ preinvexity of ψ, we have

$$\psi(exp_v \lambda b\eta(u^*, \bar{u})) \leq log(\lambda b e^{r\psi(u^*)} + (1 - \lambda b)e^{r\psi(\bar{u})})^{\frac{1}{r}}.$$

Equivalently, we have

$$e^{r\psi(exp_v \lambda b\eta(u^*,\bar{u}))} \leq \lambda b e^{r\psi(u^*)} + (1 - \lambda b)e^{r\psi(\bar{u})} < \lambda b e^{r\psi(\bar{u})} + (1 - \lambda b)e^{r\psi(\bar{u})} = e^{r\psi(\bar{u})},$$

or

$$\psi(exp_v \lambda b\eta(u^*, \bar{u})) < \psi(\bar{u}), \quad for\ all\ \lambda \in (0, 1].$$

Therefore, for each $exp_v \lambda b\eta(u^*, \bar{u})) \in U \cap N_\epsilon(\bar{u})$, $\psi(exp_v \lambda b\eta(u^*, \bar{u}))) < \psi(\bar{u})$, which is a contradiction of (3.1). Hence the result.

Theorem 3.5. Let $\bar{M}$ be a Cartan-Hadamard manifold and U be a geodesic $B-$invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ with $\eta(u, v) \neq 0$ for all $u \neq v$. Assume that $\psi : U \to (-\infty, \infty]$ is lower semi-continuous roughly geodesic $B - r-$ preinvex function and $v \in dom(\psi)$, $\xi \in \partial_p\psi(v)$. Then there exists a positive number δ such that

$$e^{r\psi(u)} - e^{r\psi(v)} \geq e^{r\psi(v)} < \xi, \eta(u, v) >_v, \quad for\ all\ u \in U \cap B(v, \delta).$$

Proof. From the definition of $\partial_p\psi(v)$, there are positive numbers δ and σ such that

$$\psi(u) \geq \psi(v) + < \xi, exp_v^{-1} u >_v - \sigma d^2(u, v), \quad for\ all\ u \in B(v, \delta). \tag{3.2}$$

Now, fix $u \in U \cap B(v, \delta)$. Since U is a geodesic $B-$invex set with respect to η, there exists a unique geodesic $\gamma_{u,v}(\lambda b) = exp_v(\lambda b\eta(u, v)) : [0, 1] \to \bar{M}$ such that $\gamma_{u,v}(0) = v$, $\gamma'_{u,v}(0) = b\eta(u, v)$, $\gamma_{u,v}(\lambda b) \in U$, for all $\lambda \in [0, 1]$. If we choose $\lambda_0 = \frac{\delta}{\|\eta(u,v)\|_v}$, then $exp_v(\lambda b\eta(u, v)) \in U \cap B(v, \delta)$ for all $\lambda \in [0, \lambda_0)$.
From the roughly geodesic $B - r-$preinvexity of ψ, we get

$$\psi(exp_v(\lambda b(\eta(u, v)) \leq log(\lambda b e^{r\psi(u)} + (1 - \lambda b)e^{r\psi(v)})^{\frac{1}{r}},$$

or

$$e^{r\psi(exp_v(\lambda b(\eta(u,v))} \leq \lambda b e^{r\psi(u)} + (1 - \lambda b)e^{r\psi(v)}. \tag{3.3}$$

Using (3.1) for each $\lambda \in (0, \lambda_0)$, we get

$$\psi(exp_v(\lambda b(\eta(u, v)) \geq \psi(v) + < \xi, exp_v^{-1} exp_v(\lambda b\eta(u, v)) >_v - \sigma d^2(exp_v(\lambda b\eta(u, v), v)$$
$$= \psi(v) + \langle \xi, \lambda b\eta(u, v)\rangle_v - \sigma d^2(exp_v(\lambda b(\eta(u, v)), v).$$

Since $\bar{M}$ is a Cartan-Hadamard manifold for each $\lambda \in (0, \lambda_0)$, we have

$$d^2(exp_v(\lambda b\eta(u,v),v) = \|\lambda b\eta(u,v)\|_v^2 = \lambda^2 b^2 \|\eta(u,v)\|_v^2.$$

Thus, we have

$$\psi(exp_v(\lambda b(\eta(u,v)) \geq \psi(v) + \langle \xi, \lambda b\eta(u,v)\rangle_v - \sigma\lambda^2 \|b\eta(u,v)\|_v^2,$$

or

$$e^{r\psi(exp_v(\lambda(\eta(u,v))} \geq e^{r\psi(v)} e^{\langle \xi, \lambda b\eta(u,v)\rangle_v - \sigma\lambda^2\|b\eta(u,v)\|_v^2}. \tag{3.4}$$

Inequalities (3.3) and (3.4) give

$$\lambda b e^{r\psi(u)} + (1-\lambda b)e^{r\psi(v)} \geq e^{r\psi(v)} e^{\langle \xi, \lambda b\eta(u,v)\rangle_v - \sigma\lambda^2\|b\eta(u,v)\|_v^2}.$$

By further calculation we arrive at

$$b(e^{r\psi(u)} - e^{r\psi(v)}) \geq e^{r\psi(v)} \frac{1}{\lambda}[e^{\langle \xi, \lambda b\eta(u,v)\rangle_v - \sigma\lambda^2\|b\eta(u,v)\|_v^2} - 1],$$

taking the limit $\lambda \to 0$

$$e^{r\psi(u)} - e^{r\psi(v)} \geq e^{r\psi(v)} \langle \xi, \eta(u,v)\rangle_v.$$

Which proves the theorem completely.

3.3. Mean value inequality

In the present sub-section, we obtain the mean value inequality for $B-r-$ preinvex function defined on a Cartan Hadamard manifold.

In the continuation of Definition 2.3, we have the following definition.

A set P_{xy} is said to be a closed $\eta-$path joining the points x and $y = \gamma(1)$, if

$$P_{xy} = \{v : v = \gamma(s), s \in [0,1]\}.$$

An open $n-$path connecting the points x and y is a set of the type

$$P_{xy}^0 = \{v : v = \gamma(s), s \in (0,1)\}.$$

If $x = y$, then $P_{xy}^0 = \phi$.

Theorem 3.6. Let $\bar{M}$ be a Cartan Hadamard manifold and U be a geodesic $B-$invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ such that $\eta(x,y) \neq 0$ for all

$x, y \in U, x \neq y$. Let $\gamma_{y,x}(s) = exp_x(sb\eta(y,x))$ for all $x, y \in U$, $s \in [0,1]$ and $z = \gamma_{x,y}(1)$. Then the function $\psi : U \to R$ is to be a geodesic $B - r-$preinvex if and only if the following inequality

$$e^{r\psi(u)} \leq e^{r\psi(x)} + \frac{e^{r\psi(y)} - e^{r\psi(x)}}{\langle \eta(y,x), \eta(y,x) \rangle_x} \langle exp_x^{-1} u, \eta(y,x) \rangle_x. \tag{3.5}$$

holds, for all $u \in P_{zx}$.

Proof. Let $\psi : U \to R$ be a $B - r-$preinvex function. If $u = x$ or $u = z$. Then the inequality (3.5) is true trivially. If $u \in P_{zx}$ then $u = exp(sb\eta(y,x))$, for $s \in [0,1]$. Since U is a $B-$invex set, then for $u \in U$, we have

$$s = \frac{\langle exp_x^{-1} u, \eta(y,x) \rangle_x}{b \langle \eta(y,x), \eta(y,x) \rangle_x}.$$

By the $B - r-$invexity of ψ, we have

$$\psi(u) = \psi(exp_x(sb\eta(y,x))) \leq log(sbe^{r\psi(y)} - (1-sb)e^{r\psi(x)})^{\frac{1}{r}},$$

or

$$\begin{aligned} e^{r\psi(u)} &\leq sbe^{r\psi(y)} + (1-sb)e^{r\psi(x)} \\ &= e^{r\psi(x)} + sb(e^{r\psi(y)} - e^{r\psi(x)}). \end{aligned}$$

Utilizing the value of s we get the required inequality.

Conversely, suppose the inequality (3.5) is true. Let $x, y \in U$ and $u = exp_x(sb\eta(y,x))$, for some $s \in [0,1]$. Then for $u \in U$, we have $\psi(u) = \psi(exp_x(sb\eta(y,x)))$. From (**??**)

$$\begin{aligned} e^{r\psi(u)} &\leq e^{r\psi(x)} + \frac{e^{r\psi(y)} - e^{r\psi(x)}}{\langle \eta(y,x), \eta(y,x) \rangle_x} \langle exp_x^{-1} u, \eta(y,x) \rangle_x \\ &= e^{r\psi(x)} + \frac{e^{r\psi(y)} - e^{r\psi(x)}}{\langle \eta(y,x), \eta(y,x) \rangle_x} \langle exp_x^{-1} exp_x(sb\eta(y,x), \eta(y,x) \rangle_x \\ &= sbe^{r\psi(y)} + (1-sb)e^{r\psi(x)}, \end{aligned}$$

or

$$\psi(u) \leq log(sbe^{r\psi(y)} + (1-sb)e^{r\psi(x)})^{\frac{1}{r}}.$$

Equivalently,

$$\psi(exp_x(sb\eta(y,x)) \leq log(sbe^{r\psi(y)} + (1-sb)e^{r\psi(x)})^{\frac{1}{r}},$$

which shows that ψ is a geodesic $B - r-$preinvex function on U.

Remark 3.2. If $r = 0$ and $b = 1$, then the mean value inequality becomes the inequality proved in [9].

4. Conclusion

In the present paper, we have defined the concept of the roughly geodesic $B-r-$ preinvex function on a Riemannian manifold. This function generalizes the preinvex functions defined in ([1], [2], [3], [4], [9], [16], [24]). Further, we have proved that a local minimum point is also a global minimum point for a scalar optimization problem under the aforesaid function. Finally, the mean value inequality is also proved involving a geodesic $B-r$ preinvex function. This inequality generalizes the inequality obtained in ([1], [9]). As a future work, the findings of this paper can be utilized for multiobjective mathematical problems on Riemannian manifolds.

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