FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 34, No 1 (2019), 45–56
https://doi.org/10.22190/FUMI1901045S

η-RICCI SOLITONS IN (ε, δ)-TRANS-SASAKIAN MANIFOLDS

Mohd Danish Siddiqi

Abstract. The objective of the present paper is to study (ε, δ)-trans-Sasakian manifolds admitting η-Ricci solitons. It is shown that a symmetric second order covariant tensor in an (ε, δ)-trans-Sasakian manifold is a constant multiple of the metric tensor. Also, an example of an η-Ricci soliton in a 3-diemsional (ε, δ)-trans-Sasakian manifold is provided in the region where (ε, δ)-Trans Sasakian manifold is expanding.
Keywords: Sasakian manifolds; Ricci soliton; Tensor.

1. Introduction

In 1985, J. A. Oubina [22] introduced a new class of almost contact metric manifolds known as trans-Sasakian manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold $M \times \mathbb{R}$ belongs to the class W_4, where the classification of almost Hermition manifolds appears as a class W_4 of Hermitian manifolds which are closely related to locally conformal Kähler manifolds studied by Gray and Hervella [14]. The class $C_5 \oplus C_6$ [22] coincides with the class of trans-Sasakian structure of type (α, β). This class consists of both Sasakian and Kenmotsu structures. If $\alpha = 1$, $\beta = 0$ then the class turn into Sasakian and when $\alpha = 0$, $\beta = 1$ then it turn into Kenmotsu. The above manifolds are studied by many authors like D. E. Blair and J. C. Marrero [1], K. Kenmotsu [17], C. S. Bagewadi and Venkatesha [8], U. C. De and M. M. Tripathi [12].

The differential geometry of manifolds with indefinite metric plays an interesting role in physics. Manifolds with indefinite metric have been studied by several authors. The concept of (ϵ)-Sasakian manifolds was initiated by A. Bejancu and K. L. Duggal [2] and further investigation was taken up by X. Xufeng and C. Xiaoli [30]. U. C. De and A. Sarkar [11] studied (ε)-Kenmotsu manifolds with indefinite metric. S. S. Shukla and D. D. Singh [25] extended with indefinite metric which is a natural generalization of both (ε)-Sasakian and (ε)-Kenmotsu manifolds. The

Received December 19, 2017; accepted November 20, 2018
2010 *Mathematics Subject Classification.* Primary 53C15, 53C20; Secondary 53C25, 53C44

authors H. G. Nagaraja et al. [20] studied (ε,δ)-trans-Sasakian manifolds which are extensions of (ε)-trans-Sasakian manifolds. M. D. Siddiqi et al. also studied some properties of (ε,δ)-trans-Sasakian manifolds in [26].

In 1982, R. S. Hamilton [15] stated that Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow which is given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

Definition 1.1. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. The Ricci soliton is said to be shrinking, steady or expanding depending on whether $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

In 1925, Levy [18] obtained necessary and sufficient conditions for the existence of such tensors. later, R. Sharma [24] initiated a study of Ricci solitons in contact Riemannian geometry . After that, Tripathi [27], Nagaraja et al. [21] and others like C. S. Bagewadi et al. ([7], [16]) extensively studied Ricci solitons in almost (ϵ)-contact metric manifolds. In 2009, J. T. Cho and M. Kimura [10] introduced the notion of η-Ricci soliton and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. Later η-Ricci solitons in (ε)-almost paracontact metric manifolds were studied by A. M. Blaga et. al. in [5]. Moreover, η-Ricci solitons have been studied by various authors for different structures (see [3], [4], [23], [9], [28]). Recently, K. Venu et al. [29] studied the η-Ricci solitons in trans-Sasakian manifolds. Motivated by these studies in the present paper we investigate η-Ricci solitons in 3-dimensional (ε,δ)-trans-Sasakian manifolds and derive the expression for the scalar curvature.

1.1. Preliminaries

Let M be an almost contact metric manifold equipped with the almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ \eta \circ \phi = 0, \ \ \phi\xi = 0, \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \qquad \eta(X) = \varepsilon g(X, \xi), \qquad g(\xi, \xi) = \varepsilon, \tag{1.4}$$

for all X, Y vector fields on M, where ε is 1 or -1 according as ξ is space-like or time-like. In particular, if the metric g is positive definite, then the (ε)-almost contact metric manifold is the usual almost contact metric manifold [25].

An (ε)-almost contact metric metric manifold is called an (ε)-trans Sasakian manifold [25] if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \varepsilon\eta(Y)\phi X) \tag{1.5}$$

holds for some smooth functions α and β on M. According to the characteristic vector field ξ we have two classes of (ε)-trans-Sasakian manifolds. When $\varepsilon = -1$ and index of g is odd, then M is a time-like trans-Sasakian manifold and when $\varepsilon = 1$ and index of g is even, then M is a space-like trans-Sasakian manifold. Further, M is a usual trans-Sasakian manifold for $\varepsilon = 1$ and the index of g is 0 and M is a Lorentzian trans-Sasakian manifold for $\varepsilon = -1$ and the index of g is 1. An ε-almost contact metric manifold is said to be a (ε, δ)-trans-Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X) \tag{1.6}$$

holds for some smooth functions α and β on M, where ε is 1 or -1 according as ξ is space-like or time-like and δ is alike ε.
From (1.6), we have

$$\nabla_X \xi = -\varepsilon\alpha\phi X - \delta\beta\phi^2 X, \tag{1.7}$$

and

$$(\nabla_X \eta)Y = \delta\beta[\varepsilon g(X,Y) - \eta(X)\eta(Y)] - \alpha g(\phi X, Y). \tag{1.8}$$

In (ε, δ)-trans-Sasakian manifold M, we have the following relations [7]:

$$\begin{aligned} R(X,Y)\xi = {} & (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] \\ & + 2\varepsilon\delta\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ & + \varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] \\ & + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \\ & + 2\alpha\beta(\delta - \varepsilon)g(\phi X, Y)\xi, \end{aligned} \tag{1.9}$$

$$\begin{aligned} S(X,\xi) = {} & [((n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\eta(X) \\ & - \varepsilon((\phi X)\alpha) - (n-2)\varepsilon(X\beta)), \end{aligned} \tag{1.10}$$

$$Q\xi = ((n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta))\xi + \varepsilon\phi(grad\alpha) - \varepsilon(n-2)(grad\beta), \tag{1.11}$$

where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

Further in a (ε, δ)-trans-Sasakian manifold, we have

$$\varepsilon\phi(grad\alpha) = \varepsilon(n-2)(grad\beta), \tag{1.12}$$

and

$$\varepsilon(\xi\alpha) + 2\varepsilon\delta\alpha\beta = 0. \tag{1.13}$$

Using (1.9) and (1.12), for constants α and β , we have

$$R(\xi, X)Y = (\alpha^2 - \beta^2)[\varepsilon g(X,Y)\xi - \eta(Y)X], \tag{1.14}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y], \tag{1.15}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{1.16}$$

$$S(X,\xi) = [((n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\eta(X), \tag{1.17}$$

$$Q\xi = [(n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\xi. \tag{1.18}$$

An important consequence of (1.7) is that ξ is a geodesic vector field

$$\nabla_\xi \xi = 0. \tag{1.19}$$

For an arbitrary vector field X, we have that

$$d\eta(\xi, X) = 0. \tag{1.20}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (1.15), we have

$$K_\xi = g(R(\xi, X), \xi, X) = (\alpha^2 - \beta^2) - \delta(\xi\beta). \tag{1.21}$$

It follows from (1.21) that ξ-sectional curvature does not depend on X.

1.2. η-Ricci solitons on (M, ϕ, ξ, η, g)

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇, that is, $\nabla h = 0$. Applying the Ricci commutation identity [20]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{1.22}$$

we obtain the relation

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0. \tag{1.23}$$

Replacing $Z = W = \xi$ in (1.23) and using (1.9) and the symmetry of h, we have

$$\begin{aligned} &2(\alpha^2 - \beta^2)[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] \\ &+2\varepsilon[(Y\alpha)h(\phi X,\xi) - (X\alpha)h(\phi Y,\xi)] + 2\delta[(Y\beta)h(\phi^2 X,\xi) - (X\beta)h(\phi^2 Y,\xi)] \\ &+4\varepsilon\delta\alpha\beta[\eta(Y)h(\phi X,\xi) - \eta(X)h(\phi Y,\xi)] + 4\alpha\beta(\delta - \varepsilon)g(\phi X,Y)h(\xi,\xi) = 0. \end{aligned} \tag{1.24}$$

Putting $X = \xi$ in (1.24) and by virtue of (1.3), we obtain

$$-2[\varepsilon(\xi\alpha) + 2\varepsilon\delta\alpha\beta]h(\phi Y,\xi) \tag{1.25}$$

$$+2[(\alpha^2 - \beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi, \xi) - h(Y, \xi)] = 0.$$

By using (1.13) in (1.25), we have

$$[(\alpha^2 - \beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi, \xi) - h(Y, \xi)] = 0. \tag{1.26}$$

Suppose $(\alpha^2 - \beta^2) - \delta(\xi\beta) \neq 0$; it results in

$$h(Y, \xi) = \eta(Y)h(\xi, \xi). \tag{1.27}$$

Now, we can call a regular (ε, δ)-trans-Sasakian manifold if $(\alpha^2 - \beta^2) - \delta(\xi\beta) \neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of the (ε, δ)-trans-Sasakian manifold.

Differentiating (1.27) covariantly with respect to X, we have

$$(\nabla_X h)(Y, \xi) + h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) \tag{1.28}$$

$$= [\varepsilon g(\nabla_X Y, \xi) + \varepsilon g(Y, \nabla_X \xi)]h(\xi, \xi)$$

$$+\eta(Y)[(\nabla_X h)(Y, \xi) + 2h(\nabla_X \xi, \xi)].$$

By using the parallel condition $\nabla h = 0$, $\eta(\nabla_X \xi) = 0$ and by virtue of (1.27) in (1.28), we get

$$h(Y, \nabla_X \xi) = \varepsilon g(Y, \nabla_X \xi)h(\xi, \xi).$$

Now using (1.7) in the above equation, we get

$$-\varepsilon\alpha h(Y, \phi X) + \delta\beta h(Y, X) = -\alpha g(Y, \phi X)h(\xi, \xi) + \varepsilon\delta\beta g(Y, X)h(\xi, \xi). \tag{1.29}$$

Replacing $X = \phi X$ in (1.29) and after simplification, we get

$$h(X, Y) = \varepsilon g(X, Y)h(\xi, \xi), \tag{1.30}$$

which together with the standard fact that the parallelism of h implies that $h(\xi, \xi)$ is a constant, via (1.27). Now by considering the above equations, we can give the conclusion:

Theorem 1.1. *Let (M, ϕ, ξ, η, g) be a (ε, δ)-trans-Sasakian manifold with a non-vanishing ξ-sectional curvature and endowed with a tensor field $h \in \Gamma T_2^0(M)$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇, then it is a constant multiple of the metric tensor g.*

Let (M, ϕ, ξ, η, g) be an (ε)-almost contact metric manifold. Consider the equation [10]

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.31}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_X \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{1.32}$$

for any $X, Y \in \chi(M)$.

Definition 1.2. The data (g, ξ, λ, μ) which satisfy the equation (3.10) is said to be η- *Ricci soliton* on M [10]; in particular, if $\mu = 0$ then (g, ξ, λ) is the Ricci soliton [10] and it is called shrinking, steady or expanding following $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively [10].

Now, from (1.7) , the equation (1.31) becomes:

$$S(X,Y) = -(\lambda + \delta\beta)g(X,Y) + (\varepsilon\delta\beta - \mu)\eta(X)\eta(Y). \tag{1.33}$$

The above equations yields

$$S(X,\xi) = -[(\lambda + \mu) + (1-\varepsilon)\delta\beta]\eta(X) \tag{1.34}$$

$$QX = -(\lambda + \beta\delta)X + (\varepsilon\delta\beta - \mu)\xi \tag{1.35}$$

$$Q\xi = -[(\lambda + \mu) + (1-\varepsilon)\delta\beta]\xi \tag{1.36}$$

$$r = -\lambda n - (n-1)\varepsilon\delta\beta - \mu, \tag{1.37}$$

where r is the scalar curvature. Off the two natural situations regrading the vector field V: $V \in Span\xi$ and $V \perp \xi$, we investigate only the case $V = \xi$.

Our interest is in the expression for $\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$. A direct computation gives

$$\mathcal{L}_\xi g(X,Y) = 2\delta\beta[g(X,Y) - \varepsilon\eta(X)\eta(Y)]. \tag{1.38}$$

In a 3-dimensional (ε, δ)-trans-Sasakian manifold the Riemannian curvature tensor is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y$$
$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{1.39}$$

Putting $Z = \xi$ in (1.39) and using (1.9) and (1.10) for 3-dimensional (ε, δ)-trans-Sasakian manifold, we get

$$(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\varepsilon\delta\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y]$$
$$+\varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \tag{1.40}$$

$$+2(\delta-\varepsilon)\alpha\beta g(\phi X, Y)$$

$$=\varepsilon[(\varepsilon\alpha^2-\delta\beta^2)-(\xi\beta)]\eta(Y)X-\eta(X)Y]$$

$$+\varepsilon\eta(Y)QX-\varepsilon\eta(X)QY-\varepsilon[((\phi Y)\alpha)X+(Y\beta)X]+\varepsilon[((\phi X)\alpha)Y+(X\beta)Y].$$

Again, putting $Y=\xi$ in (1.40) and using (1.3) and (1.13), we obtain

$$QX=\left[\frac{r}{2}-2(\varepsilon\alpha^2-\delta\beta^2)+\varepsilon(\alpha^2-\beta^2)\right]X. \tag{1.41}$$

$$+\left[4(\varepsilon\alpha^2-\delta\beta^2)-\frac{r}{2}-(\alpha^2-\beta^2)\right]\eta(X)\xi$$

From (1.41), we have

$$S(X,Y)=\left[\frac{r}{2}-2(\varepsilon\alpha^2-\delta\beta^2)+\varepsilon(\alpha^2-\beta^2)\right]g(X,Y) \tag{1.42}$$

$$+\left[4(\varepsilon\alpha^2-\delta\beta^2)-\frac{r}{2}-(\alpha^2-\beta^2)\right]\varepsilon\eta(X)\eta(Y).$$

Equation (1.42) shows that a 3-dimensional (ε, δ)-trans-Sasakian manifold is η-*Einstein*.
Next, we consider the equation

$$h(X,Y)=(\mathcal{L}_\xi g)(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y). \tag{1.43}$$

By Using (1.48) and (1.42) in (1.43), we have

$$h(X,Y)=\left[r-4(\varepsilon\alpha^2-\delta\beta^2)+2\varepsilon(\alpha^2-\beta^2)+2\delta\beta\right]g(X,Y) \tag{1.44}$$

$$+\left[8(\varepsilon\alpha^2-\delta\beta^2)-2\varepsilon(\alpha^2-\beta^2)-2\delta\beta-r\right]\varepsilon\eta(X)\eta(Y)+2\mu\eta(X)\eta(Y).$$

Putting $X=Y=\xi$ in (1.5), we get

$$h(\xi,\xi)=2[2\varepsilon(\varepsilon\alpha^2-\delta\beta^2)-2\mu]. \tag{1.45}$$

Now, (1.30) becomes

$$h(X,Y)=2[2\varepsilon(\varepsilon\alpha^2-\delta\beta^2)-2\mu]\varepsilon g(X,Y). \tag{1.46}$$

From (1.43) and (1.46), it follows that (g, ξ, μ) is an η-Ricci soliton.
Therefore, we can state as:

Theorem 1.2. *Let (M, ϕ, ξ, η, g) be a 3-dimensional (ε, δ)-trans-Sasakian manifold. Then (g, ξ, μ) yields an η-Ricci soliton on M.*

Let V be pointwise collinear with ξ, i.e., $V = b\xi$, where b is a function on the 3-dimensional (ε, δ)-trans-Sasakian manifold. Then

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0$$

or

$$bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X)$$

$$+2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (1.7), we obtain

$$bg(-\varepsilon\alpha\phi X - \delta\beta(-X + \eta(X)\xi, Y) + (Xb)\eta(Y) + bg(-\varepsilon\alpha\phi Y - \delta\beta(-Y + \eta(Y)\xi, X)$$

$$+(Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

which yields

(1.47) $$2b\delta\beta g(X,Y) - 2b\delta\beta\eta(X)\eta(Y) + (Xb)\eta(Y)$$

$$+(Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ in (1.47), we obtain

(1.48) $$(Xb) + (\xi b)\eta(X) + 2[2(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta) + \lambda + \mu]\eta(X) = 0.$$

Again putting $X = \xi$ in (1.48), we obtain

$$\xi b = -2(\varepsilon\alpha^2 - \delta\beta^2) + (\xi\beta) - \lambda - \mu.$$

Plugging this in (1.48), we get

$$(Xb) + 2[2(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta) + \lambda + \mu]\eta(X) = 0,$$

or

(1.49) $$db = -\left\{\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2)\right\}\eta = 0.$$

Applying d on (1.49), we get $\left\{\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2)\right\} d\eta$. Since $d\eta \neq 0$ we have

(1.50) $$\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2) = 0.$$

Equation (1.50) in (1.49) yields b as a constant. Therefore from (1.47), it follows that

$$S(X,Y) = -(\lambda + \delta\beta)g(X,Y) + (\varepsilon\delta b\beta - \mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for the constant $\delta\beta$. This leads to the following:

Theorem 1.3. *If in a 3-dimensional (ε,δ)-trans-Sasakian manifold the metric g is an η-Ricci soliton and V is pointwise collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided $\delta\beta$ is a constant.*

Tanking $X = Y = \xi$ in (1.30) and (1.42) and comparing, we get

$$\lambda = -2(\epsilon\alpha^2 - \delta\beta^2) + (\xi\beta) + \mu = -2K_\xi - \mu. \tag{1.51}$$

From (1.37) and (1.51), we obtain

$$r = 6(\epsilon\alpha^2 - \delta\beta^2) + 3(\xi\beta) - 2\varepsilon\delta\beta + 2\mu. \tag{1.52}$$

Since λ is a constant, it follows from (1.51) that K_ξ is a constant.

Theorem 1.4. *Let (g,ξ,μ) be an η-Ricci soliton in the 3-dimensional (ε,δ)-trans Sasaakian manifold (M,ϕ,ξ,η,g). Then the scalar $\lambda+\mu = -2K_\xi$, $r = 6K_\xi + 2\mu + 3(\xi\beta) - 2\varepsilon\delta\beta$.*

Remark 1.1. For $\mu = 0$, (1.51) reduces to $\lambda = -2K_\xi$, so the Ricci soliton in a 3-dimensional (ε,δ)-trans-Sasakian manifold is shrinking.

2. Example of η-Ricci solitons on (ε,δ)-Trans-Sasakian manifolds

Example 2.1. Consider the three dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 z \neq 0\}$, where (x,y,z) are the cartesian coordinates in $\mathbb{R}^3$ and let the vector fields

$$e_1 = \frac{e^x}{z^2}\frac{\partial}{\partial x}, \qquad e_2 = \frac{e^y}{z^2}\frac{\partial}{\partial y}, \qquad e_3 = \frac{-(\epsilon+\delta)}{2}\frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Riemannian metric defined by
$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = \varepsilon$, $g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$,
where $\epsilon = \pm 1$.

Let η be the 1-form defined by $\eta(X) = \varepsilon g(X,\xi)$, for any vector field X on M, let ϕ be the (1,1)-tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then by using the linearity of ϕ and g, we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y)$, for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (ε)-almost contact structure in $\mathbb{R}^3$.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \\ & - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

which is known as Koszul's formula.

$\nabla_{e_1}e_3 = -\frac{(\varepsilon+\delta)}{z}e_1, \quad \nabla_{e_2}e_3 = -\frac{(\varepsilon+\delta)}{z}e_2, \quad \nabla_{e_1}e_2 = 0,$
using the above relation, for any vector X on M, we have $\nabla_X\xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X$, where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence (ϕ, ξ, η, g) structure defines the (ε, δ)-tran-Sasakian structure in $\mathbb{R}^3$.

Here ∇ is the Levi-Civita connection with respect to the metric g , so we have $[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{(\varepsilon+\delta)}{z}e_1, \; [e_2, e_3] = -\frac{(\varepsilon+\delta)}{z}e_2.$

Thus we have

$$\nabla_{e_1}e_3 = -\frac{(\varepsilon+\delta)}{z}e_1 + e_2, \nabla_{e_1}e_2 = 0$$

$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -\frac{(\varepsilon+\delta)}{z}e_2, \quad \nabla_{e_2}e_3 = -\frac{(\varepsilon+\delta)}{z}e_2e_1$$

$$\nabla_{e_3}e_1 = 0, \qquad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = -\frac{(\varepsilon+\delta)}{z}e_1 + e_2.$$

The manifold M satisfies (1.7) with $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence M is a (ε, δ)-trans-Sasakian manifolds. Then the non-vanishing components of the curvature tensor fields are computed as follows:

$$R(e_1, e_3)e_3 = \tfrac{(\varepsilon+\delta)}{z^2}e_1, \quad R(e_3, e_1)e_3 = -\tfrac{(\varepsilon+\delta)}{z^2}e_1, \qquad R(e_1, e_2)e_2 = \tfrac{(\varepsilon+\delta)}{z^2}e_1$$

$$R(e_2, e_3)e_3 = \tfrac{(\varepsilon+\delta)}{z^2}e_1, \quad R(e_3, e_2)e_3 = -\tfrac{(\varepsilon+\delta)}{z^2}e_1, \; R(e_2, e_1)e_1 = -\tfrac{(\varepsilon+\delta)}{z^2}e_1.$$

From the above expression of the curvature tensor we can also obtain

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = \frac{(\varepsilon^2 + \delta\varepsilon)}{z^2}$$

since $g(e_1, e_3) = g(e_1, e_2) = 0$.
Therefore, we have

$$S(e_i, e_i) = -\frac{(\varepsilon+\delta)}{z^2}g(e_i, e_i),$$

for $i = 1, 2, 3$, and $\alpha = \frac{1}{z}$, $\beta = -\frac{1}{z}$. Hence M is also an *Einstein* manifold. In this case, from (1.32), we have

(2.1) $\quad 2\delta\beta[g(e_i, e_i - \varepsilon\eta(e_i)\eta(e_i)] + 2S(e_i, e_i) + 2\lambda g(e_i, e_i) + 2\mu\eta(e_i)\eta(e_i) = 0.$

Now, from (2.1), we get $\lambda = \frac{\varepsilon[\delta(1+z)-\varepsilon]}{z^2}$ (i.e, $\lambda > 0$) and $\mu = -\frac{\varepsilon[\varepsilon^2-\varepsilon-\delta(1+\varepsilon+\varepsilon z)]}{z^2}$, the data (g, ξ, λ, μ) is an η-Ricci soliton on (M, ϕ, ξ, η, g) i. e., expanding.

Acknowledgement. The author is thankful to the referees for their valuable comments and suggestions towards the improvement of the paper.

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Mohd Danish Siddiqi
Department of Mathematics
Faculty of Science
P. O. Box 114
Jazan University, Jazan, Kingdom of Saudi Arabia
Emails:anallintegral@gmail.com, msiddiqi@jazanu.edu.sa