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A NEW LOG-LOCATION REGRESSION MODEL WITH INFLUENCE DIAGNOSTICS AND RESIDUAL ANALYSIS

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Abstract. A new four-parameter lifetime model called Odd Log-Logistic Burr XII distribution is defined and investigated. Some of its mathematical properties are derived. Some useful characterization results based on the ratio of two truncated moments, based on the hazard function, as well as on the conditional expectation of certain functions of a random variable, are presented. The maximum likelihood method is used to estimate the model parameters by means of a graphical Monte Carlo simulation study. Moreover, we introduce a new log-location regression model based on the proposed distribution. The Jackknife estimation method as an alternative method is used to estimate the unknown parameters of a new regression model. The generalized cook distance and likelihood distance measures are used to detect possible influential observations. Martingale and modified deviance residuals are defined to detect outliers and evaluate the model assumptions. The potentiality of the new regression model is illustrated by means of a real data set.

Keywords: Regression Model; Burr XII Distribution; Residual Analysis; Influential Diagnostics; Simulation; Jackknife Estimation Method.

1. Introduction

The Pearson system of distributions was originally introduced as an effort for modeling visibly skewed observations. It was well known at the time how to adjust a theoretical model to fit the first two cumulants or moments of observed data. In his original paper and analogously to the Pearson system of densities, Burr [4] proposed another system of distributions that includes twelve types of cdfs (cumulative distribution function) which yield a variety of density shapes. This system is obtained by considering cdfs satisfying a differential equation whose solution is given by

$$G(t) = \frac{1}{\exp\left[-\int \psi(t)\,dt\right] + 1},$$

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where $\psi(t)$ is chosen such that $G(t)$ is a cdf on the real line. Twelve choices for $\psi(t)$ made by Burr resulted in twelve distributions which might be useful for modeling Data. The principal aim in choosing one of these forms of distributions is to facilitate the mathematical analysis to which it is subjected, while attaining a reasonable approximation. Burr ([4], [5], [6]) and others (see Burr and Cislak [7], Hatke [18], Rodriguez [24]) devoted special attention to one of these forms, denoted by type XII, whose distribution function $G(x)$ is

$$G(t;\alpha,\beta,\lambda) = \left\{1 - \left[1 + \left(\frac{t}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}, x \geq 0. \tag{1.1}$$

Both α and β are shape parameters, $\lambda > 0$ is a scale parameter. A location parameter can easily be introduced to make (1.1) a four parameter model. The corresponding probability density function (pdf) of (1.1) is

$$g(t;\alpha,\beta,\lambda) = \alpha\beta\lambda^{-\alpha}t^{\alpha-1}\left[1 + \left(\frac{t}{\lambda}\right)^{\alpha}\right]^{-\beta-1}, x > 0. \tag{1.2}$$

The Burr XII (BXII) model has many applications in different areas including acceptance sampling plans, reliability and failure time modeling. Tadikamalla [28] studied the BXII model and its related models, namely: Pareto type II (Lomax), log-logistic, compound Weibull gamma and Weibull exponential distributions. Zimmer et al. [31] proposed a new three-parameter Burr XII distribution. This distribution, having Weibull and logistic as sub-models, is a very popular distribution for modeling lifetime data and phenomena with monotone failure rates. Shao [29] studied the maximum likelihood estimation for the three-parameter BXII model. Soliman [27] studied the estimation of parameters from progressively censored data using the Burr-XII model. Recently, Silva et al. [25] proposed a new location-scale regression model based on the BXII model; Silva et al. [26] proposed a residual for the log-BXII regression distribution whose empirical model is close to normality; Afify et al. [2] studied the Weibull BXII distribution; Cordeiro et al. [11] proposed a double BXII model with forty special cases; Yousof et al. [30] proposed and studied the Topp Leone generated Burr XII distribution, among others.

Gleaton and Lynch [14] defined the cdf of the so-called odd log-logistic-G (OLL-G) family (for $x > 0$) by

$$F(x;\theta,\xi) = \frac{G(x,\boldsymbol{\xi})^{\theta}}{G(x,\boldsymbol{\xi})^{\theta} + \bar{G}(x,\boldsymbol{\xi})^{\theta}}\ . \tag{1.3}$$

The OLL-G density function is

$$f(x;\theta,\xi) = \frac{\theta g(x,\boldsymbol{\xi})\left[G(x,\boldsymbol{\xi})\bar{G}(x,\boldsymbol{\xi})\right]^{\theta-1}}{\left[G(x,\boldsymbol{\xi})^{\theta} + \bar{G}(x,\boldsymbol{\xi})^{\theta}\right]^{2}}, \tag{1.4}$$

where $\theta > 0$ is the shape parameter and $\xi = \xi_k = (\ \xi_1, \xi_2, ...)$ is a parameters vector. A random variable (rv) X with pdf (1.4) is denoted by $X \sim$OLL-G (θ, ξ). In the last decade, researchers have showed a great interest in introducing a new family of distributions by adding parameter(s) to OLL-G family. Recent extensions of the OLL-G family can be cited as follows: the Zografos-Balakrishnan odd log-logistic family of distributions by Cordeiro et al. [8], the generalized odd log-logistic family by Cordeiro et al. [9], the beta dd log-logistic generalized family of distributions by Cordeiro et al. [10], and a new generalized odd log-logistic family of distributions by Haghbin et al. [16].

Here, a new extension of the BXII distribution is proposed by means of the OLL-G family. Inserting (1.1) and (1.2) in (1.3) and (1.4), the cdf and pdf of the odd log-logistic BXII (OLLBXII) distribution are defined as

$$F(x) = F(x;\theta,\alpha,\beta,\lambda) = \frac{\overbrace{\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{\theta}}^{A_i}}{\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{\theta}+\underbrace{\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\theta\beta}}_{B_i}}, \quad x \geq 0, \tag{1.5}$$

and

$$\begin{aligned} f(x) = f(x;\theta,\alpha,\beta,\lambda) &= \theta\alpha\beta\lambda^{-\alpha}x^{\alpha-1}\left[1+\left(\tfrac{x}{\lambda}\right)^{\alpha}\right]^{-\beta-1} \\ &\times\frac{\left(\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right)^{\theta-1}}{\left(\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{\theta}+\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\theta\beta}\right)^{2}}, \quad x > 0, \end{aligned} \tag{1.6}$$

respectively.

The paper is organized as follows: The graphical presentation and motivation for the new model are presented in Section 2. In Section 3, we derive some mathematical properties of the new model. In Section 4, some useful characterization results based on the ratio of two truncated moments, based on the hazard function, and based on the conditional expectation of certain functions of a random variable are presented. In Section 5, the maximum likelihood method is discussed to estimate the model parameters by means of a Monte Carlo simulation study. A new log-location regression model and its estimation via maximum likelihood method and Jackknife estimation method, sensitivity analysis, and residual analysis are presented and displayed in Section 6. In Section 7, two applications to real data sets are performed to demonstrate the empirically importance of the new model. Finally, some conclusions and future work are given in Section 8.

2. Graphical presentation and motivation

The importance of pdf (1.6) can be summarized as follows: the OLLBXII model contains some well-known models as its sub-models. More clearly, the BXII model is a special sub-model when $\theta = 1$. For $\theta = \lambda = \alpha = 1$ and $\theta = \lambda = \beta = 1$, we have the standard Lomax and standard log-logistic distributions, respectively. For $\lambda = \alpha = 1$ we have the OLL-Lomax distribution. For $\lambda = \beta = 1$ we have the OLL-LL distribution. For $\beta \rightarrow 1$ we have the OLL-Weibull distribution.

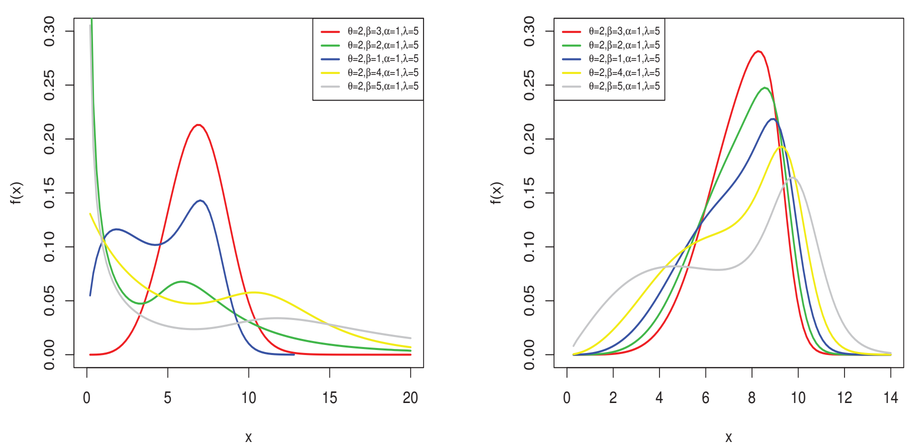

Fig. 2.1: The pdf plots of OLLBXII distribution for several parameter values.

We are motivated to introduce OLLBXII distribution because it contains a number of the aforementioned known lifetime models as illustrated above. The hrf of OLLBXII distribution exhibits decreasing, upside-down, and bathtub hazard rates as illustrated in Figure 2.2. It is shown in Section 3 that OLLBXII distribution can be viewed as a mixture of the two-parameter BXII distribution. It can be viewed as a suitable model for fitting the left-skewed, right-skewed, symmetric and bimodal data sets as illustrated in Figure 2.1.

Moreover, Figure 2.3 displays the hrf regions of OLLBXII distribution for fixed $\alpha = 4$, $\lambda = 0.1$ parameters. The developed computational codes are provided in Appendix. As seen from Figure 2.3, when the parameter $\theta < 0.255$, the hrf of OLLBXII distribution is decreasing, otherwise, it is upside-down. Similar results can be obtained for different parameter combinations by using the computational codes given in Appendix.

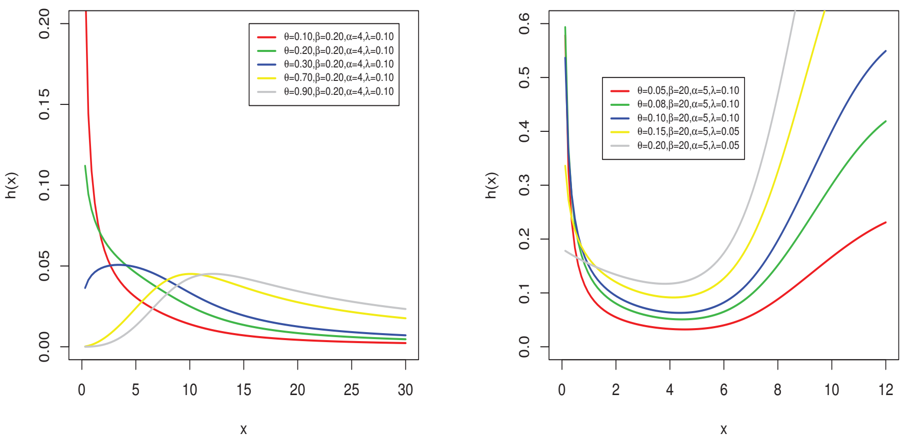

FIG. 2.2: The hrf plots of OLLBXII distribution for several parameter values.

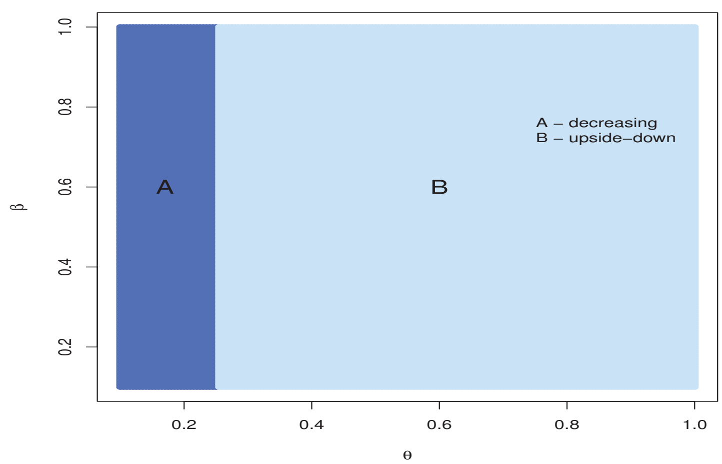

FIG. 2.3: The hrf regions of OLLBXII distribution for fixed $\alpha = 4$, $\lambda = 0.1$ parameters.

3. Mathematical Properties

3.1. Quantile function

Let U have a uniform $U(0,1)$ distribution, the quantile function (qf) of OLLBXII distribution is defined by

$$Q\left(u\right)=\lambda\left\{\left[\frac{\left(1-u\right)^{\frac{1}{\theta}}}{u^{\frac{1}{\theta}}+\left(1-u\right)^{\frac{1}{\theta}}}\right]^{-\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}, \tag{3.1}$$

follows the density function (1.6). The following algorithm can be used to generate random variables from density (1.6).

Algorithm 3.1. Algorithm

1. Generate $U\sim U(0,1)$
2. Set $X=\lambda\left[\left\{\frac{(1-U)^{1/\theta}}{U^{1/\theta}+(1-U)^{1/\theta}}\right\}^{-1/\beta}\right]^{1/\alpha}$

The effects of the shape parameters of the new model can be measured by the skewness and kurtosis using the qf (3.1). These measures, called Bowley's skewness and Moors's kurtosis, are given respectively by

$$Skewness=\frac{Q(1/4)+Q(3/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}$$

and

$$Kurtosis=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

The plots of Bowley's skewness and Moors's kurtosis of the BOLL-GHN distribution are displayed in Figure 3.1. Figures 3.1(a) and (b) display the effects of the parameters β and θ on skewness and kurtosis measures for fixed $\alpha=10,\lambda=0.5$. Figures 3.1(c) and (d) display the effects of the parameters α and θ on skewness and kurtosis measures for fixed $\beta=10,\lambda=0.5$. As seen in Figure 3.1; when the parameters α,β and θ increase, skewness and kurtosis decrease.

3.2. Mixture representation

We provide a very useful linear representation for the OLLBXII cdf. First, we use a power series for the quantity A_i ($\theta>0$ real) given by

$$A_i=\sum_{k=0}^{\infty}a_k\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{k}, \tag{3.2}$$

where $a_k = \sum_{j=k}^{\infty} (-1)^{k+j} \binom{\theta}{j}\binom{j}{k}$. For any real $\theta > 0$, we consider the generalized binomial expansion

$$B_i = \sum_{k=0}^{\infty} (-1)^k \binom{\theta}{k} \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^k. \tag{3.3}$$

Inserting (3.2) and (3.3) in equation (1.5), we obtain

$$F(x) = \frac{\sum_{k=0}^{\infty} a_k \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^k}{\sum_{k=0}^{\infty} b_k \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^k},$$

where $b_k = a_k + (-1)^k \binom{\theta}{k}$. The ratio of the two power series can be expressed as

$$F(x) = \sum_{k=0}^{\infty} c_k \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^k = \sum_{k=0}^{\infty} c_k \Pi_k\left(x; \alpha, \beta, \lambda\right), \tag{3.4}$$

where $\Pi_k\left(x; \alpha, \beta, \lambda\right) = \left[G(x, \alpha, \beta, \lambda)\right]^k$ is the exponentiated BXII cdf with power parameter k, and the coefficients c_k's (for $k \geq 0$) are determined from the recurrence equation

$$c_k = b_0^{-1}\left(a_k + b_0^{-1} \sum_{w=0}^{\infty} b_w c_{k-w}\right).$$

By differentiating (3.4), the pdf of X can be expressed as

$$f(x) = \sum_{k=0}^{\infty} c_{1+k} \pi_{1+k}\left(x; \alpha, \beta, \lambda\right) = \sum_{r=0}^{\infty} v_r g(x; \alpha, (1+r)\,\beta, \lambda), \tag{3.5}$$

where $\pi_{1+k}\left(x; \alpha, \beta, \lambda\right)$ is the exponentiated BXII density with power parameter $k+1$, $g(x; \alpha, (1+r)\,\beta, \lambda)$ is the BXII density with parameters $\alpha, (1+r)\,\beta$ and λ and

$$v_r = \sum_{k=0}^{\infty} (-1)^r \frac{(1+k)}{(1+r)} c_{k+1} \binom{k}{r}$$

3.3. Moments and cumulants

Let W be a random variable having BXII distribution (1.2) with parameters α and β and λ. For $n < \alpha\beta \Leftrightarrow \frac{n}{\alpha} < \beta$, the n^{th} ordinary and incomplete moments of W are given respectively, by

$$\mu_n' = \beta\lambda^n \,\mathrm{B}\left(\beta - n\alpha^{-1}, 1 + n\alpha^{-1}\right)$$

and

$$\varphi_n(z) = \beta\,\lambda^n \mathrm{B}\left(z^{\alpha}; \beta - n\alpha^{-1}, 1 + n\alpha^{-1}\right),$$

where

$$\mathrm{B}(a,b)=\int_0^\infty t^{a-1}\left(1+t\right)^{-(a+b)}dt$$

and

$$\mathrm{B}(z;a,b)=\int_0^z t^{a-1}\left(1+t\right)^{-(a+b)}dt$$

are beta and incomplete beta functions of the second type, respectively. So, several structural properties of the OLLBXII model can be obtained from (3.4) and those properties of BXII distribution.

The n^{th} ordinary moment of X is given by

$$\mu_n'=E(X^n)=\sum_{r=0}^{\infty} v_r \int_{-\infty}^{\infty} x^n\, g(x;\alpha,(1+r)\,\beta,\lambda)dx,$$

and (for $0<(1+r)\,\beta-n\alpha^{-1}$)

$$\mu_n'=\sum_{r=0}^{\infty} v_r\,(1+r)\,\beta\lambda^n\,\mathrm{B}\left((1+r)\,\beta-n\alpha^{-1},1+n\alpha^{-1}\right). \tag{3.6}$$

By setting $n=1$ in (3.6), we have the mean of X. The last integration can be computed numerically for most parent distributions. The n^{th} central moment of X, say μ_n, is given by

$$\mu_n=E\left(X-\mu_1'\right)^n=\sum_{m=0}^{n}\binom{n}{m}\left(-\mu_1'\right)^{n-m}\,\mu_{n-m}'.$$

The cumulants (κ_s) of X are determined from the ordinary moments as (for $s\geq 2$)

$$\kappa_s=\mu_s'-\sum_{k=1}^{s-1}\binom{s-1}{k-1}\kappa_k\,\mu_{s-k}',$$

where $\kappa_1=\mu_1'$. The skewness ($\gamma_1=\kappa_3/\kappa_2^{3/2}$) and kurtosis ($\gamma_2=\kappa_4/\kappa_2^2$) of X are just the third and fourth standardized cumulants. They are important to derive Edgeworth expansions for the cdf and pdf of the standardized sum and mean of independent and identically distributed random variables with OLLBXII distribution.

3.4. Moment generating function

Let X have OLLBXII$(\theta,\alpha,\beta,\lambda)$ distribution. The mgf of X, say $M\,(t)$, using the Maclaurin series expansion of an exponential function (Abramowitz and Stegun [3]), can be written as

$$M\,(t)=E\left[\exp\left(tX\right)\right]=\sum_{m=0}^{\infty}\left(-1\right)^m E(X^m)/m!.$$

Another representation for M(t) can be obtained from (3.4) as an infinite weighted sum

$$M\left(t\right)=\sum_{r=0}^{\infty}v_{r}M_{1+r}\left(t\right),$$

where $M_{1+r}\left(t\right)$ is the mgf of the BXII density with parameters $\alpha,\left(1+r\right)\beta$ and λ. Paranaíba et al. [20] introduced a simple exemplification for the mgf of the three-parameter BXII distribution. In a similar manner, we provide another exemplification for the mgf, say $M_{1+r}(t)$, of the BXII$(\alpha,\left(1+r\right)\beta,\lambda)$ model. For $0>t$, we can write

$$M\left(t\right)=\alpha\beta\left(1+r\right)\lambda^{-\alpha}\int_{0}^{\infty}\exp\left(yt\right)y^{\alpha-1}\left[1+\left(\frac{y}{\lambda}\right)^{\alpha}\right]^{-\beta\left(1+r\right)-1}dy.$$

Next, we require the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x|\begin{array}{c}a_{1},...,a_{p}\\ b_{1},...,b_{q}\end{array}\right)=\left(2\pi i\right)^{-1}\int_{L}\frac{\prod_{j=1}^{m}\Gamma\left(b_{j}+t\right)\prod_{j=1}^{n}\Gamma\left(1-a_{j}-t\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_{j}+t\right)\prod_{j=m+1}^{p}\Gamma\left(1-b_{j}-t\right)}x^{-t}dt,$$

where $i=\sqrt{-1}$ is the complex unit and L denotes an integration path (Gradshteyn and Ryzhik [15], Section 9.3). The Meijer G-function contains, as particular cases, many integrals with elementary and special functions (see Prudnikov et al. [21]). We now assume that $\alpha=m\beta^{-1}$, where m and β are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. We have the following result, which holds for m and β positive integers, $-1<\mu$ and $0>p$ (Prudnikov et al. [22], p. 21),

$$I\left(p,\mu,m\beta^{-1},v\right)=\int_{0}^{\infty}\exp\left(-px\right)x^{\mu}\left(1+x^{m\beta^{-1}}\right)^{v}dx,$$

or

$$I\left(p,\mu,m\beta^{-1},v\right)=VG_{\beta+m,\beta}^{\beta,\beta+m}\left(\left(mp^{-1}\right)^{m}|\begin{array}{c}\triangle\left(m,-\mu\right),\triangle\left(\beta,v+1\right)\\ \triangle\left(\beta,0\right)\end{array}\right),$$

where

$$V=\beta^{-v}\left[\Gamma\left(-v\right)\right]^{-1}m^{v+\frac{1}{2}}p^{-\left(\mu+1\right)}\left(2\pi\right)^{-\frac{m-1}{2}}$$

and

$$\triangle\left(\beta,a\right)=a\beta^{-1},\left(a+1\right)\beta^{-1},...,\left(a+\beta\right)\beta^{-1}.$$

The mgf of of the BXII(α,β,λ) can be written as

$$M\left(t\right)=mI\left(-\lambda t,m\beta^{-1}-1,m\beta^{-1},-\beta-1\right),t<0.$$

Hence, the mgf of of the OLLBXII$(\theta,\alpha,\left(1+r\right)\beta,\lambda)$ can be expressed as

$$M_{X}\left(t\right)=m\sum_{r=0}^{\infty}v_{r}I\left(-\lambda t,m\left[\beta\left(r+1\right)\right]^{-1}-1,m\left[\beta\left(r+1\right)\right]^{-1},-\left[\beta\left(r+1\right)+1\right]\right).$$

3.5. Incomplete moment

The s^{th} incomplete moment, say $\varphi_s(t)$, of OLLBXII distribution is given by $\varphi_s(t) = \int_0^t x^s\, f(x)dx$. From the equation (3.4),

$$\varphi_s(t) = \sum_{r=0}^{\infty} v_r \int_0^t x^s\, g(x;\alpha,\beta\,(r+1)\,,\lambda)dx,$$

and using the lower incomplete gamma function, we have (for $s < \alpha\beta$)

$$\varphi_s(t) = \sum_{r=0}^{\infty} v_r\,\beta\,(r+1)\,\lambda^s\,\mathrm{B}\left(t^{\alpha};\beta\,(r+1)\,-s\alpha^{-1},1+s\alpha^{-1}\right).$$

The 1^{st} incomplete moment of X, denoted by $\varphi_1\,(t)$, is simply determined from $\varphi_s(t)$ by taking $s = 1$. The 1^{st} incomplete moment has important applications related to the residual life, the mean waiting time and Bonferroni and Lorenz curves.

3.6. Moments of reversed residual life and mean waiting time

The s^{th} moment of the reversed residual life, say $R_s(t) = E\left[(t-X)^s \mid X \leq t\right]$ for $t > 0$ and $s = 1, 2, \ldots$, uniquely determines $F(x)$. Then, $R_s(t)$ is defined by

$$R_s(t) = \frac{1}{F(t)}\int_0^t (t-x)^s dF(x).$$

The s^{th} moment of the reversed residual life of X is

$$R_s(t) = \frac{1}{F(t)}\sum_{i=0}^{n}\sum_{r=0}^{\infty}\frac{(-1)^i s!}{i!(s-i)!} v_r\,\beta\,(r+1)\,\lambda^s\;\mathrm{B}\left(t^{\alpha};\beta\,(r+1)\,-s\alpha^{-1},1+s\alpha^{-1}\right).$$

The mean waiting time (MWT) or the mean inactivity time (MIT), also named the mean reversed residual life function, $R_1(t) = E[(t-X) \mid X \leq t]$, represents the waiting time elapsed since the failure of a component on condition that this failure has occurred in $(0, x)$. The MIT of X can be obtained by setting $s = 1$ in the above equation.

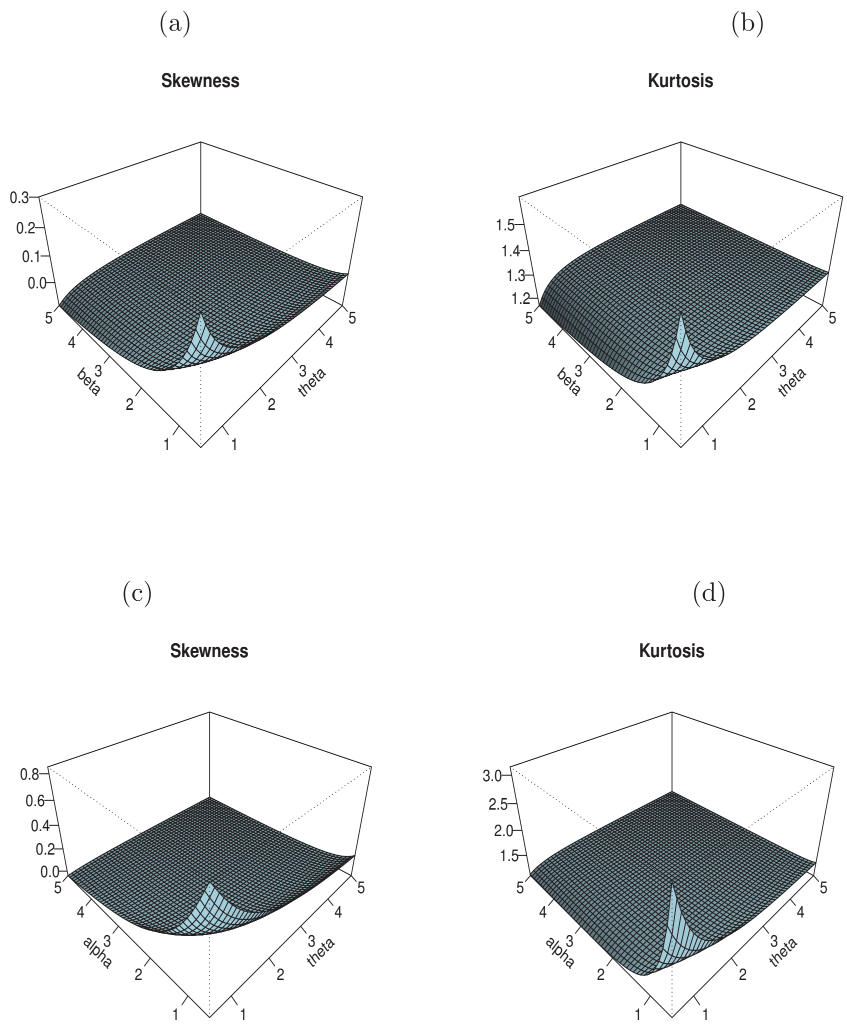

Fig. 3.1: The skewness and kurtosis plots of OLLBXII distribution for several parameter values.

4. Characterizations

In this section we present certain characterizations of OLLBXII distribution. These characterizations are in terms of: (i) the ratio of two truncated moments; (ii) the hazard function and (iii) conditional expectations of functions of the random variable. One of the advantages of characterization (i) is that the cdf is not required to have a closed form. We present our characterizations $(i)-(iii)$ in three subsections.

4.1. Characterizations based on the ratio of two truncated moments

In this subsection we present characterizations of OLLBXII distribution in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to Glänzel [12], see Theorem 1 of Appendix A. Note that the result also holds when the interval H is not closed. Moreover, as mentioned above, it could also be applied when the cdf F does not have a closed form. As shown in Glänzel [13], this characterization is stable in the sense of weak convergence.

Proposition 4.1. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)=\frac{\left(\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{\theta}+\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta\theta}\right)^{2}}{\left(\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right)^{\theta-1}}$ and $q_2(x)=q_1(x)\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}$ for $x>0$. The random variable X has pdf (1.6) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta},\quad x>0.$$

Proof. Let X be a random variable with pdf (1.6), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\theta\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta},\quad x>0,$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\frac{\theta}{2}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-2\beta},\quad x>0,$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{1}{2}q_1(x)\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}<0\quad for\ \ x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha\beta\lambda^{-\alpha}x^{\alpha-1}}{1+\left(\frac{x}{\lambda}\right)^{\alpha}}\quad x>0,$$

and hence

$$s(x) = \log\left\{\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{\beta}\right\}, \quad x>0.$$

Now, in view of Theorem 1, X has density (1.6).

Corollary 4.1. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4.1. The pdf of X is (6) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha\beta\lambda^{-\alpha}x^{\alpha-1}}{1+\left(\frac{x}{\lambda}\right)^{\alpha}} \quad x>0.$$

The general solution to the differential equation in Corollary 4.1 is

$$\eta(x) = \left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{\beta}\left[-\int \alpha\beta\lambda^{-\alpha}x^{\alpha-1}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}(q_1(x))^{-1}\, q_2(x) + D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1 with $D=0$. However, it should be also noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

4.2. Characterization based on hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization of OLLBXII distribution, for $\theta = 1$, which is not of the above trivial form.

Proposition 4.2. Let $X : \Omega \to (0,\infty)$ be a continuous random variable. The pdf of X is (1.6), for $\theta = 1$, if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \frac{\alpha-1}{x}h_F(x) = -\frac{\alpha^2\beta\lambda^{-2\alpha}x^{2(\alpha-1)}}{\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^2}, \quad x>0.$$

Proof. If X has pdf (1.6), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{x^{-(\alpha-1)}h_F(x)\right\} = \alpha\beta\lambda^{-\alpha}\frac{d}{dx}\left\{\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-1}\right\}, \quad x>0,$$

from which, we obtain

$$h_F(x) = \frac{\alpha\beta\lambda^{-\alpha}x^{\alpha-1}}{1+\left(\frac{x}{\lambda}\right)^{\alpha}}, \quad x>0,$$

which is the hazard function of OLLBXII distribution.

4.3. Characterization based on the conditional expectation of certain functions of the random variable

In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work [17], so we will just state it here as a proposition, which can be used to characterize OLLBXII distribution.

Proposition 4.3. Let $X : \Omega \to (d,e)$ be a continuous random variable with *cdf* F. Let $\psi(x)$ be a differentiable function on (d,e) with $\lim_{x\to e^-}\psi(x) = 1$. Then for $\delta \neq 1$,

$$E[\psi(X) \mid X \geq x] = \delta\psi(x), \quad x \in (d,e)$$

if and only if

$$\psi(x) = (1-F(x))^{\frac{1}{\delta}-1}, \quad x \in (d,e).$$

Remark 4.3. (A) For $(d,e) = (0,\infty)$, $\psi(x) = \left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-1}$ and $\delta = \frac{\beta}{\beta+1}$, Proposition 4.3 provides a characterization of OLLBXII distribution.

5. Estimation

If X follows the OLLBXII distribution with vector of parameters $\mathbf{\Phi} = (\theta,\alpha,\beta,\lambda)^T$, the log-likelihood for $\mathbf{\Phi}$ from a single observation x of X is given by

$$\begin{aligned}\ell(\mathbf{\Phi}) \quad = \quad & \log\theta + \log\alpha + \log\beta - \alpha\log\lambda - (\beta+1)\log s \\ & + (\theta-1)\log\left[\left(1-s^{-\beta}\right)s^{-\beta}\right] - 2\log\left[\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}\right],\end{aligned}$$

where $s = \left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]$. The components of the unit score vector $U = U(\mathbf{\Phi}) = \left(\frac{\partial\theta}{\ell(\mathbf{\Phi})}, \frac{\partial\alpha}{\ell(\mathbf{\Phi})}, \frac{\partial\beta}{\ell(\mathbf{\Phi})}, \frac{\partial\lambda}{\ell(\mathbf{\Phi})}\right)^{T} = \left(U\left(\theta\right), U\left(\alpha\right), U\left(\beta\right), U\left(\lambda\right)\right)^{T}$ are given by

$$U\left(\theta\right) = \theta^{-1} + \log\left[\left(1-s^{-\beta}\right)s^{-\beta}\right] - 2\frac{\left(1-s^{-\beta}\right)^{\theta}\log\left(1-s^{-\beta}\right) - \beta s^{-\theta\beta}\log s}{\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}},$$

$$\begin{aligned} U\left(\alpha\right) &= \alpha^{-1} - \log\lambda - \frac{\left(\beta+1\right)p}{s} \\ &\quad + \left(\theta-1\right)\frac{\beta p s^{-2\beta-1} - \beta p\left(1-s^{-\beta}\right)s^{-\beta-1}}{\left(1-s^{-\beta}\right)s^{-\beta}} \\ &\quad -2\frac{\theta\beta p s^{-\beta-1}\left(1-s^{-\beta}\right)^{\theta-1} - \theta\beta p s^{-\theta\beta-1}}{\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}}, \end{aligned}$$

$$\begin{aligned} U\left(\beta\right) &= \beta^{-1} - \log s + \left(\theta-1\right)\frac{s^{-2\beta}\log s - s^{-\beta}\left(1-s^{-\beta}\right)\log s}{\left(1-s^{-\beta}\right)s^{-\beta}} \\ &\quad -2\frac{\theta s^{-\beta}\left(1-s^{-\beta}\right)^{\theta-1}\log s - \theta s^{-\theta\beta}\log s}{\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}} \end{aligned}$$

and

$$\begin{aligned} U\left(\lambda\right) &= -\alpha\lambda^{-1} - \frac{\left(\beta+1\right)q}{s} + \left(\theta-1\right)\frac{\beta q s^{-2\beta-1} - \beta q\left(1-s^{-\beta}\right)s^{-\beta-1}}{\left(1-s^{-\beta}\right)s^{-\beta}} \\ &\quad -2\frac{\theta q s^{-\beta-1}\left(1-s^{-\beta}\right)^{\theta-1} - \theta\beta q s^{-\theta\beta-1}}{\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}}, \end{aligned}$$

where $p = \left(\frac{x}{\lambda}\right)^{\alpha}\log\left(\frac{x}{\lambda}\right)$ and $q = \alpha x^{\alpha}\lambda^{-\alpha-1}$. For a random sample $x = \left(x_1, ..., x_n\right)^{T}$ of size n from X, the total log-likelihood is $\ell_n(\mathbf{\Phi}) = \sum_{i=0}^{n}\ell^{(i)}(\mathbf{\Phi})$, where $\ell^{(i)}(\mathbf{\Phi})$ is the log-likelihood for the ith observation. The total score function is $U_n = \sum_{i=0}^{n}U^{(i)}$, where $U^{(i)}$ has the form given before. Maximization of $\ell(\mathbf{\Phi})$ (or $\ell_n(\mathbf{\Phi})$) can be easely performed using well-established routines such as the nlm or optimize in the R statistical package. Setting these equations to zero, $U(\mathbf{\Phi}) = 0$, and solving them simultaneously gives the MLE $\widehat{\mathbf{\Phi}}$ b of $\mathbf{\Phi}$. These equations cannot be solved analytically and statistical software can be used to evaluate them numerically using iterative techniques such as the Newton-Raphson algorithm.

The parameter estimation procedure of the OLLBXII model can be summarized as follows:

- The **optim** function of R software is used to minimize the minus log-likelihood function of the BXII model by means of the Nelder-Mead (NM) optimization method. There is no need to provide the derivatives of the objective function for the NM method.

- The estimated parameters of BXII distribution is used as the initial values of the OLLBXII model. The initial value of the additional parameter θ is chosen as 1. Then, the parameter estimations of the OLLBXII model are obtained with the **optim** function as given in first step.
- The inverse of the estimated Hessian matrix is used to obtain the corresponding standard errors.

5.1. Simulation Study

In this section, the parameter estimation efficiency of the MLE method is evaluated for the parameters of OLLBXII distribution by means of the Monte Carlo simulation. The following simulation procedure is implemented:

1. Set the sample size n and the vector of parameters $\boldsymbol{\theta} = (\theta, \beta, \alpha, \lambda)$
2. Generate random observations of size n from OLLBXII$(\theta, \beta, \alpha, \lambda)$ distribution
3. Using the generated random observations in Step 2, estimate $\hat{\boldsymbol{\theta}}$ by means of MLE method
4. Repeat steps 2 and 3, N times
5. Using $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$ compute the mean relative estimates (MREs) and mean square errors (MSEs) via the following equations:

$$MRE = \sum_{j=1}^{N} \frac{\hat{\boldsymbol{\theta}}_{i,j} \big/ \boldsymbol{\theta_i}}{N} \text{ and } MSE = \sum_{j=1}^{N} \frac{\left(\hat{\boldsymbol{\theta}}_{\boldsymbol{i,j}} - \boldsymbol{\theta_i}\right)^2}{N}, \; i = 1, 2, 3, 4.$$

The statistical software R is used to obtain simulation results. The chosen parameter values for simulation study are $\boldsymbol{\theta} = (0.5, 5, 5, 0.5)$, $N = 10,000$ and $n = (50, 55, 60, ..., 500)$. We expect that MREs are closer to one when the MSEs are near zero. Figures 4 and 5 display the estimated biases, MSEs and MREs. As seen from these figures, the estimated MSEs for all parameters tend to zero for large sample sizes and the values of MREs tend to one. The biases for the parameters θ, β and α are positive whereas the biases for the parameter λ is negative. The biases for all the parameters tend to zero for large sample sizes. It is clear that the estimates of parameters are asymptotically unbiased. Therefore, the MLE is an appropriate method for estimating parameters of the OLLBXII distribution. Similar results can be obtained for different parameter vectors.

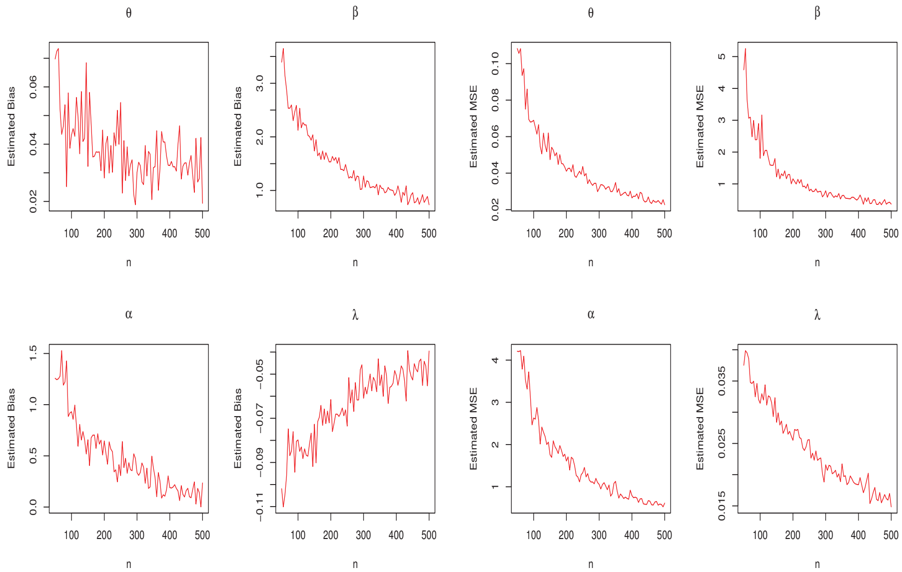

FIG. 5.1: Estimated biases and MSEs for the chosen parameter values.

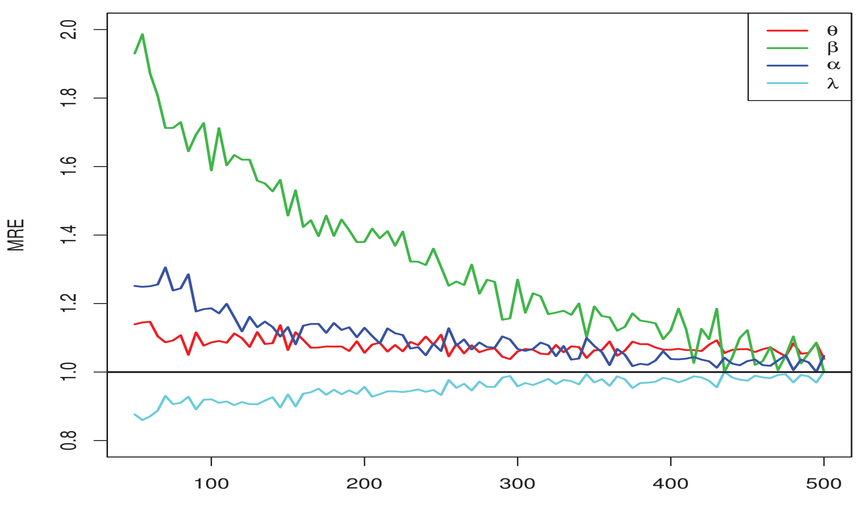

FIG. 5.2: Estimated MREs for the chosen parameter values.

6. Log-OLLBXII regression model

Let X denotes a random variable with OLLBXII distribution and let $Y = \log(X)$. The density function of Y (for $y \in \text{Re}$) for $\alpha = 1/\sigma$ and $\lambda = \exp(\mu)$, can be expressed as

(6.1)

$$f(y)=\frac{\frac{\theta\beta}{\sigma}\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-(\beta+1)}\exp\left(\frac{y-\mu}{\sigma}\right)\left[\left[1-\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]^{\theta-1}}{\left[\left[1-\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]^{\theta}+\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta^{\theta}}\right]^{2}},$$

where $\mu \in \text{Re}$ is the location parameter, $\sigma > 0$ is the scale parameter, $\theta > 0$ and $\beta > 0$ are the shape parameters. We refer to equation (6.1) as the Log-OLLBXII (LOLLBXII) distribution, say $Y \sim \text{LOLLBXII}(\theta, \beta, \sigma, \mu)$. The plots in Figure 6.1 show shapes of density function (6.1) for selected parameter values. They reveal that this distribution is a good candidate to model left and right skewed and symmetric lifetime data sets. The survival function corresponding to (6.1) is given by

$$S(y) = 1 - \frac{\left[1-\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]^{\theta}}{\left[1-\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]^{\theta}+\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta^{\theta}}} \tag{6.2}$$

and the hrf is simply $h(y) = f(y)/S(y)$. The standardized random variable $Z = (Y - \mu)/\sigma$ has density function

(6.3)

$$f(z) = \frac{\theta\beta(1+\exp(z))^{-(\beta+1)}\exp(z)\left[\left[1-(1+\exp(z))^{-\beta}\right](1+\exp(z))^{-\beta}\right]^{\theta-1}}{\left[\left[1-(1+\exp(z))^{-\beta}\right]^{\theta}+(1+\exp(z))^{-\beta^{\theta}}\right]^{2}}.$$

6.1. Estimation

6.1.1. Maximum Likelihood Estimation

Based on the LOLLBXII density, we propose a linear location-scale regression model linking the response variable y_i and the explanatory variable vector $\mathbf{v}_i^{\intercal} = (v_{i1}, \dots, v_{ip})$ given by

$$y_i = \mathbf{v}_i^{\intercal}\boldsymbol{\beta} + \sigma z_i,\ i = 1, \dots, n, \tag{6.4}$$

where the random error z_i has density function (6.3), $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^{\intercal}$, and $\sigma > 0$, $\theta > 0$ and $\beta > 0$ are unknown parameters. The parameter $\mu_i = \mathbf{v}_i^{\intercal}\boldsymbol{\beta}$ is the location of y_i. The location parameter vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^{\intercal}$ is represented by a linear model $\boldsymbol{\mu} = \mathbf{V}\boldsymbol{\beta}$, where $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)^{\intercal}$ is a known model matrix. The LOLLBXII

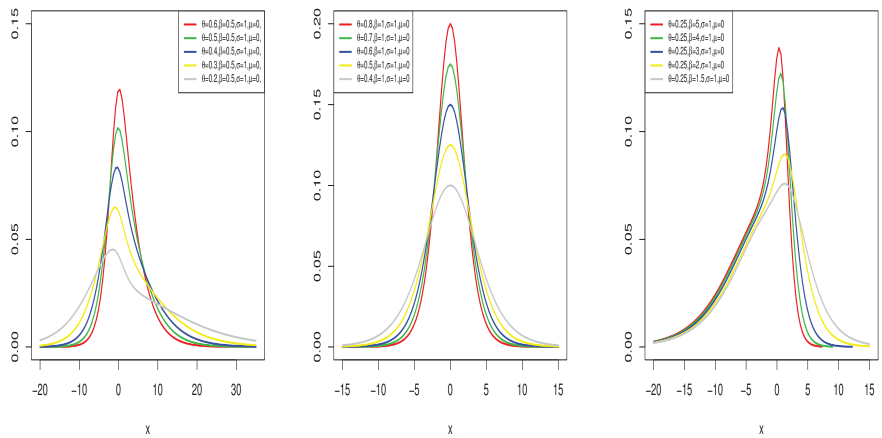

Fig. 6.1: Plots of the LOLLBXII density function for some parameter values.

model (6.4) provides new avenues for modeling several types of data sets. Note that when the parameter $\theta = 1$, the LOLLBXII regression model reduces to the Log-BXII (LBXII) regression model introduced by Silva et al. [25].

Consider a sample $(y_1, \mathbf{v}_1), \ldots, (y_n, \mathbf{v}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(x_i), \log(c_i)\}$ where x_i and c_i are lifetime and censoring times, respectively. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. The log-likelihood function for the vector of parameters $\boldsymbol{\tau} = (\theta, \beta, \sigma, \boldsymbol{\beta}^\intercal)^\intercal$ from model (6.4) has the form $l(\boldsymbol{\tau}) = \sum\limits_{i \in F} l_i(\boldsymbol{\tau}) + \sum\limits_{i \in C} l_i^{(c)}(\boldsymbol{\tau})$, where $l_i(\boldsymbol{\tau}) = \log[f(y_i)]$, $l_i^{(c)}(\boldsymbol{\tau}) = \log[S(y_i)]$, $f(y_i)$ is the density (6.1) and $S(y_i)$ is the survival function (6.2) of Y_i. The total log-likelihood function for $\boldsymbol{\tau}$ is given by

$$
\begin{aligned}
\ell(\tau) &= r \log\left(\frac{\theta\beta}{\sigma}\right) - (\beta+1) \sum_{i \in F} \log\left(1 + \exp\left(z_i\right)\right) + \sum_{i \in F} z_i \\
&+ (\theta - 1) \sum_{i \in F} \log\left[\left[1 - \left(1 + \exp\left(z_i\right)\right)^{-\beta}\right] \left(1 + \exp\left(z_i\right)\right)^{-\beta}\right] \\
&-2 \sum_{i \in F} \log\left[\left[1 - \left(1 + \exp\left(z_i\right)\right)^{-\beta}\right]^{\theta} + \left(1 + \exp\left(z_i\right)\right)^{-\beta\theta}\right] \\
&+ \sum_{i \in C} \log\left[1 - \frac{\left[1 - (1 + \exp(z_i))^{-\beta}\right]^{\theta}}{\left[1 - (1 + \exp(z_i))^{-\beta}\right]^{\theta} + (1 + \exp(z_i))^{-\beta\theta}}\right],
\end{aligned}
\tag{6.5}
$$

where $z_i = (y_i - \mathbf{v}_i^\intercal \boldsymbol{\beta})/\sigma$ and r is the number of uncensored observations (failures). The MLE $\widehat{\boldsymbol{\tau}}$ of the vector of unknown parameters can be evaluated by maximizing the log-likelihood function (6.5). The optim function of R software is used to estimate $\widehat{\boldsymbol{\tau}}$. Under the standard regularity conditions, the asymptotic distribution of $(\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau})$ is multivariate normal $N_{p+3}(0, K(\boldsymbol{\tau})^{-1})$, where $K(\boldsymbol{\tau})$ is the expected information matrix. The asymptotic covariance matrix $K(\boldsymbol{\tau})^{-1}$ of $\widehat{\boldsymbol{\tau}}$ can be approximated by the inverse of the $(p+3) \times (p+3)$ observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\tau})$, whose elements are evaluated numerically. The approximate multivariate normal distribution $N_{p+3}(0, -\ddot{\mathbf{L}}(\boldsymbol{\tau})^{-1})$ for $\widehat{\boldsymbol{\tau}}$ can be used, in the classical way, to construct approximate confidence intervals for the parameters in $\boldsymbol{\tau}$.

The likelihood ratio (LR) statistic can be used for comparing the sub-model of the LOLLBXII regression model. For example, the LR statistic can be used to discriminate between the LOLLBXII and LBXII regression models since they are nested models, or equivalently to test $H_0 : \theta = 1$. The LR statistic reduces to $w = 2[\ell(\hat{\theta}, \hat{\beta}, \sigma, \hat{\boldsymbol{\beta}}) - \ell(1, \tilde{\beta}, \tilde{\sigma}, \tilde{\boldsymbol{\beta}})]$, where $(\hat{\theta}, \hat{\beta}, \hat{\sigma}, \hat{\boldsymbol{\beta}})$ are the unrestricted MLEs and $(1, \tilde{\beta}, \tilde{\sigma}, \tilde{\boldsymbol{\beta}})$ are the restricted estimates under H_0. The statistic w is asymptotically (as $n \to \infty$) distributed as χ^2_k, where k is difference of two parameter vectors of nested models. For example, $k = 1$ for the above hypothesis test.

6.1.2. Jackknife Estimation Method

We used the Jackknife estimation method as an alternative method to estimate the unknown parameters of LOLLBXII regression model. This method is based on "leave one out" procedure. Let $\hat{\boldsymbol{\tau}}$ be the parameter estimation of whole sample and $\hat{\boldsymbol{\tau}}_{-i}$ be the parameter estimation when the i_{th} observation is dropped from the sample. The pseudo-value of i_{th} observation is given by

$$\tilde{\boldsymbol{\tau}}_i = n\hat{\boldsymbol{\tau}} - (n-1)\,\hat{\boldsymbol{\tau}}_{-i}. \tag{6.6}$$

Then, Jackknife estimation of $\boldsymbol{\tau}$, is given by

$$\hat{\boldsymbol{\tau}}_{jack} = \frac{1}{n}\sum_{i=1}^{n} \tilde{\boldsymbol{\tau}}_i. \tag{6.7}$$

It is clear that Jackknife estimation of $\boldsymbol{\tau}$ is the average of pseudo-values. Confidence intervals of Jackknife estimates are

$$\hat{\boldsymbol{\tau}}_{jack} \pm t_{\alpha/2,(n-1)} \frac{s}{\sqrt{n}}, \tag{6.8}$$

where $t_{\alpha/2,(n-1)}$ is the value that is exceeded with probability $\alpha/2$ for the t distribution with $n-1$ degrees of freedom. The parameter estimation of the LOLLBXII regression model can be obtained by means of the theory described above.

6.2. Sensitivity analysis

A first tool to perform the sensitivity analysis, as stated before, is by means of global influence starting from case deletion. Case deletion is a popular method to investigate the influence of taking out the i_{th} case from the data on the parameters estimates. This method compares the $\hat{\boldsymbol{\tau}}$ with $\hat{\boldsymbol{\tau}}_{-i}$ where $\hat{\boldsymbol{\tau}}_{-i}$ is the estimated parameters when the i_{th} case is dropped from the data. If there is a big differences between $\hat{\boldsymbol{\tau}}_{-i}$ and $\hat{\boldsymbol{\tau}}$, the dropped observation could be considered as an influential observation.

Here, generalized cook distance and likelihood distance measures are used to detect the possible influential observations. These measures are described below.

6.2.1. Generalized cook distance

Generalized Cook distance (GD) is given by

$$GD_i\left(\boldsymbol{\tau}\right) = \left(\hat{\boldsymbol{\tau}}_{-i} - \hat{\boldsymbol{\tau}}\right)^T \left[\, -\ddot{L}\left(\hat{\boldsymbol{\tau}}\right)\right] \left(\hat{\boldsymbol{\tau}}_{-i} - \hat{\boldsymbol{\tau}}\right), \tag{6.9}$$

where $-\ddot{L}\left(\hat{\boldsymbol{\tau}}\right)$ is the observed information matrix.

6.2.2. Likelihood Distance

The Likelihood Distance (LD) is given by

$$LD_i\left(\hat{\boldsymbol{\tau}}\right) = 2\left\{\ell\left(\hat{\boldsymbol{\tau}}\right) - \ell\left(\hat{\boldsymbol{\tau}}_{-i}\right)\right\}, \tag{6.10}$$

where $\ell\left(\hat{\boldsymbol{\tau}}\right)$ is the estimated log likelihood value of whole data set and $\ell\left(\hat{\boldsymbol{\tau}}_{-i}\right)$ is the estimated log likelihood value when the i_{th} observations is dropped.

6.3. Residual analysis

Residual analysis has critical role in checking the adequacy of the fitted model. In order to analyse departures from error assumption, two types of residuals are considered: martingale and modified deviance residuals.

6.3.1. Martingale residual

The martingale residuals are defined in the counting process and takes the values between $+1$ and$-\infty$ (see, Fleming and Harrington(1994) for details). The martingale residuals for the LOLLBXII model are,

$$r_{M_i} = \begin{cases} 1 + \log\left(1 - \frac{\left[1-(1+\exp(z_i))^{-\beta}\right]^{\theta}}{\left[1-(1+\exp(z_i))^{-\beta}\right]^{\theta} + (1+\exp(z_i))^{-\beta\theta}}\right) & \text{if} i \in F, \\ \log\left(1 - \frac{\left[1-(1+\exp(z_i))^{-\beta}\right]^{\theta}}{\left[1-(1+\exp(z_i))^{-\beta}\right]^{\theta} + (1+\exp(z_i))^{-\beta\theta}}\right) & \text{if} i \in C, \end{cases} \tag{6.11}$$

where $z_i = (y_i - \mu)/\sigma$.

6.3.2. Modified deviance residual

The main drawback of the martingale residual is that when the fitted model is correct, it is not symmetrically distributed about zero. To overcome this problem, a modified deviance residual was proposed by Therneau et al. (1990). The modified deviance residual is given by

$$r_{D_i} = \begin{cases} sign\left(r_{M_i}\right)\left\{\ -2\left[r_{M_i} + \log\left(1 - r_{M_i}\right)\right]\right\}^{1/2}, \ \text{if} i \in F \\ sign\left(r_{M_i}\right)\left\{\ -2r_{M_i}\right\}^{1/2}, \ \text{if} i \in C, \end{cases} \tag{6.12}$$

where $\hat{r}_{M_i}$ is the martingale residual.

7. Applications

In this section, we provide two applications to real data sets to illustrate the flexibility of the OLLBXII distribution and the LOLLBXII regression model. The statistical software R is used for all numerical computations. The following goodness-of-fit measures are used to compare the OLLBXII model with the BXII model: Cramer von Mises (W*), Anderson Darling (A*), estimated $-\ell$. In general, the smaller the values of these statistics, the better the fit to the data. Moreover, LR test is also used to compare the models.

7.1. Turbocharger data set

We compare the fitting performance of the OLLBXII model with its sub-model. The first data set comes from Xu et al. [32] and it represents the time to failure (103 h) of turbocharger of one type of engine. The data are as follows: 1.6 3.5 4.8 5.4 6.0 6.5 7.0 7.3 7.7 8.0 8.4 2.0 3.9 5.0 5.6 6.1 6.5 7.1 7.3 7.8 8.1 8.4 2.6 4.5 5.1 5.8 6.3 6.7 7.3 7.7 7.9 8.3 8.5 3.0 4.6 5.3 6.0 8.7 8.8 9.0.

The total-time-test (TTT) plot, introduced by Aarset [1], is used to obtain the empirical behavior of the hazard rate of used data set. When the shape of TTT plot has a straight diagonal line, the hazard rate is constant. When the shapes of TTT plot have a convex or concave, the hazard rates are monotonically increasing or decreasing, respectively. Figure 8 displays the TTT plot of the used data set. Based on Figure 8, it is clear that the empirical hazard rate of the used data set is monotonically increasing.

Table 7.1 gives W* and A* statistics and log-likelihood values. Based on Tabl 7.1, it is clear that OLLBXII distribution provides superior fit and therefore could be chosen as a more adequate model than BXII for used data set.

Moreover, the profile log-likelihood functions of OLLBXII distribution are displayed in Figure 7.2. Figure 7.2 reveals that the likelihood equations of OLLBXII distribution have solutions that are maximizers.

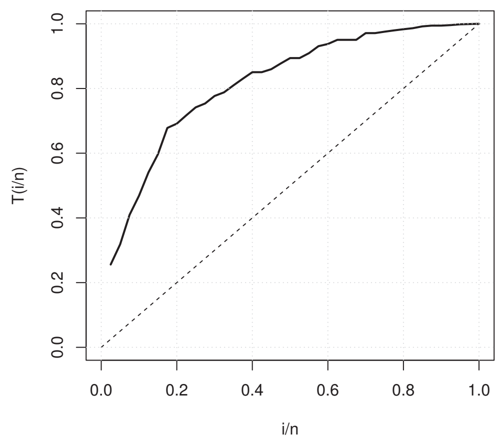

Fig. 7.1: The TTT plot of used data set.

Table 7.1: Fitting summary of the models: MLE estimates and their standard errors, $A^\star$, $W^\star$ and estimated $-\ell$

Models	θ	α	β	λ	$A^\star$	$W^\star$	$-\ell$
BXII		118.7304	3.879613	0.042166	0.582	0.0783	82.53635
		181.7223	0.5222	0.0181			
OLLBXII	0.3051	118.058	10.2619	0.09023	0.1365	0.02005	78.34025
	0.1053	368.047	3.04463	0.03201			

Table 7.2 shows the LR statistics and the corresponding p-values. From Table 7.2, the computed p-value is smaller than 0.05, so the null hypotheses are rejected. We conclude that the OLLBXII model fits the first data better than the its sub-model according to the LR test results.

More information can be provided in Figure 7.3 by a histogram of the data with fitted lines of the pdfs for all distributions. Figure 7.3(a) suggests that the OLLBXII fits left-skewed data very well. Then, we present the plots of the fitted density, cumulative and survival functions with the probability-probability (P-P) plot for the OLLBXII model in Figure 7.3(b). They reveal a good adjustment for the data of the estimated density, cumulative and survival functions of OLLBXII distribution.

Table 7.2: The LR test results for third data set.

	Hypotheses	LR	p-value
OLLBXII versus BXII	$H_0 : \theta = 1$	8.392	0.004

7.2. HIV data set

The hypothetical dataset contains 100 observations on HIV+ subjects belonging to an Health Maintenance Organization (HMO). The HMO wants to evaluate the survival time of these subjects. In this hypothetical data set, subjects were enrolled from January 1, 1989 until December 31, 1991. Study follow up then ended on December 31, 1995. This data set is reported in Hosmer and Lemeshow [19] and can also be found in R package *Bolstad2*. We adopt the LOLLBXII regression model to analyze this dataset. The variables involved in the study are: y_i - observed survival time (in months); $cens_i$ - censoring indicator (0= alive at study end or lost to follow-up,1=death due to AIDS or AIDS related factors), $x_{i1}(1 = yes, 0 = no)$ represents the history of drug use and x_{i2} represents the ages of patients.

We consider the following regression model

$$y_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1} x_{i1} + \boldsymbol{\beta_2} x_{i2} + \sigma z_i,$$

where y_i has the LOLLBXII density, for $i = 1, \ldots, 100$.

7.2.1. Maximum Likelihood Estimation

The MLE method is used to estimate unknown parameters of the LOLLBXII and LBXII regression models. Table 7.3 lists the MLEs of the model parameters of the LBXII and LOLLBXII regression models fitted to the current data and the log-likelihood and AIC values. These results indicate that the LOLLBXII regression model has the lowest values of these statistics, and so the LOLLW-W model provides better fitting than LBXII model for current data. For the fitted regression models, note that β_0, β_1 and β_2 is marginally significant at the 5% level.

LR test is used to compare the LOLLBXII and LBXII regression models. Table 7.4 shows the LR statistic and the corresponding p-value for the used data set. Based on the figures in Table 7.4, the computed p-value is smaller than 0.05, so the null hypotheses are rejected. We conclude that the LOLLBXII regression model provides better fits than its sub-model according to the LR test results.

7.2.2. Jackknife Estimation Method

Here, the Jackknife estimation method is used to estimate the unknown parameters of LOLLBXII regression model. In Table 7.5, the jackknife estimates for the parameters of the LOLLBXII regression model are reported. From Table 7.5, we conclude

Table 7.3: MLEs of the parameters, their standard errors and p-values, the estimated $-\ell$ and AIC statistic.

	LOLL-BXII			LBXII		
Parameters	Estimates	Std. Errors	p-values	Estimates	Std. Errors	p-values
θ	0.977	1.356	-	-	-	-
β	2.940	6.389	-	0.867	0.361	-
σ	0.705	1.112	-	0.566	0.080	-
$\boldsymbol{\beta_0}$	6.675	3.227	0.039	4.755	0.804	<0.001
$\boldsymbol{\beta_1}$	-0.090	0.014	<0.001	-0.070	0.017	<0.001
$\boldsymbol{\beta_2}$	-0.974	0.210	<0.001	-0.902	0.220	<0.001
$-\ell$	127.942			130.152		
AIC	267.885			270.304		

Table 7.4: The LR test results for HIV+ data set.

	Hypotheses	LR	p-value
LOLLBXII versus LBXII	$H_0 : \alpha = 1$	4.4198	0.035

that the parameters β_0 and β_1 are significant when the jackknife estimation method is used.

7.2.3. Sensitivity Analysis

Here, possible influential observations are analysed with measures described in Section 6.2.. Figure 7.4 displays the results of the generalized Cook distance,$GD_i\left(\boldsymbol{\tau}\right)$. Based on Figure 7.4, cases 41, 48 and 92 can be considered as possible influential observations.

Moreover, the effects of i_{th} observation on parameters of LOLLBXII regression model is analysed and displayed in Figure 7.5. Based on this figure, it is clear that the most influential observations are 41 and 48.

Table 7.5: Jackknife estimates for the parameters of LOLLBXII regression model

Parameters	Estimates	Std. Errors	95% confidence intervals
θ	0.933	0.147	[0.641; 1.224]
β	2.862	0.060	[2.743; 2.980]
σ	0.659	0.165	[0.331; 0.987]
$\boldsymbol{\beta_0}$	6.647	0.203	[6.243; 7.050]
$\boldsymbol{\beta_1}$	-0.092	0.015	[-0.121; -0.063]
$\boldsymbol{\beta_2}$	-0.926	0.538	[-1.994; 0.142]

7.2.4. Residual Analysis

Figure 7.6 displays the index plot of the modified deviance residuals and its Q-Q plot against to $N(0,1)$ quantiles for Stanford heart transplant data set. Based on Figure 7.6 we conclude that none of observed values appears as possible outliers. Therefore, the fitted model is appropriate for these data set.

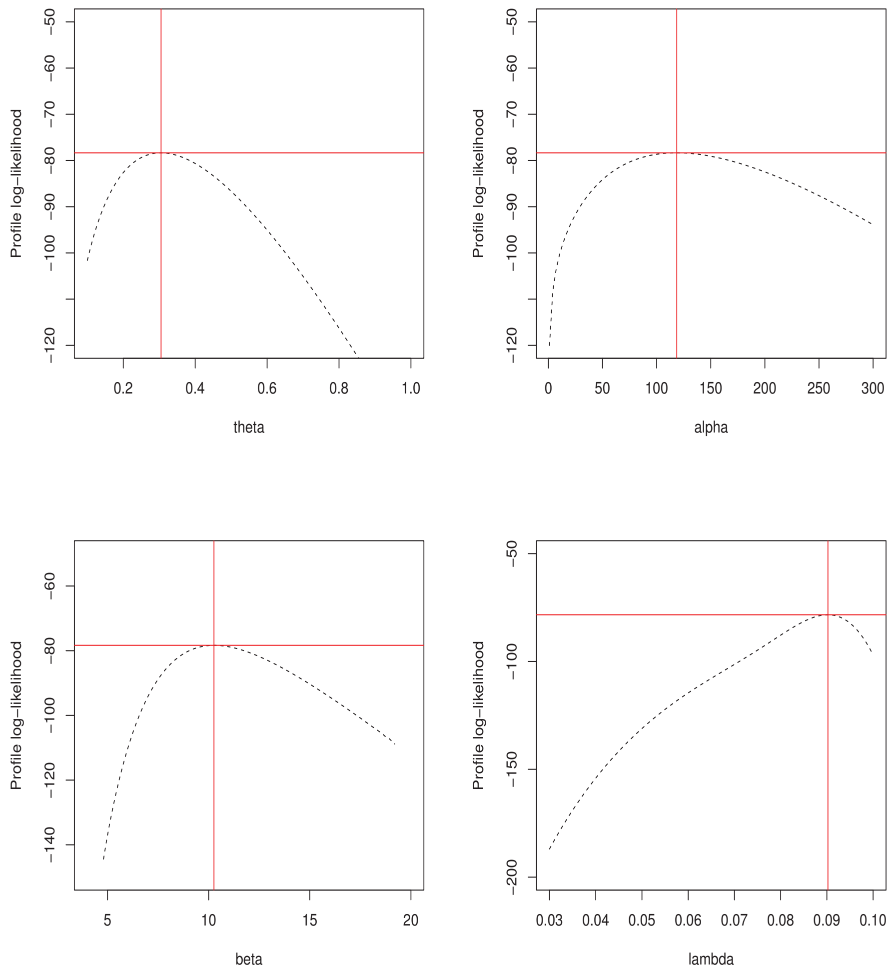

Fig. 7.2: Profile log-likelihood plots of OLLBXII distribution

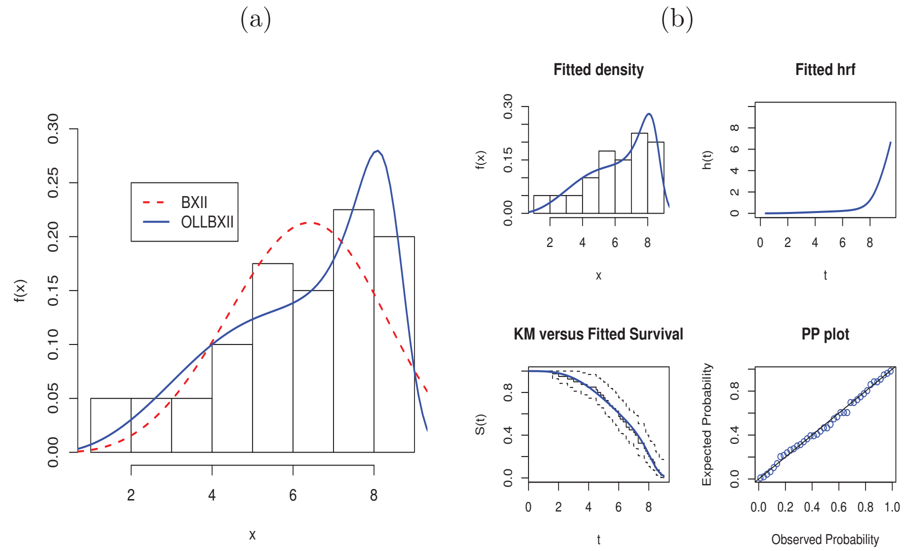

Fig. 7.3: Fitted densities of distributions for the first data set

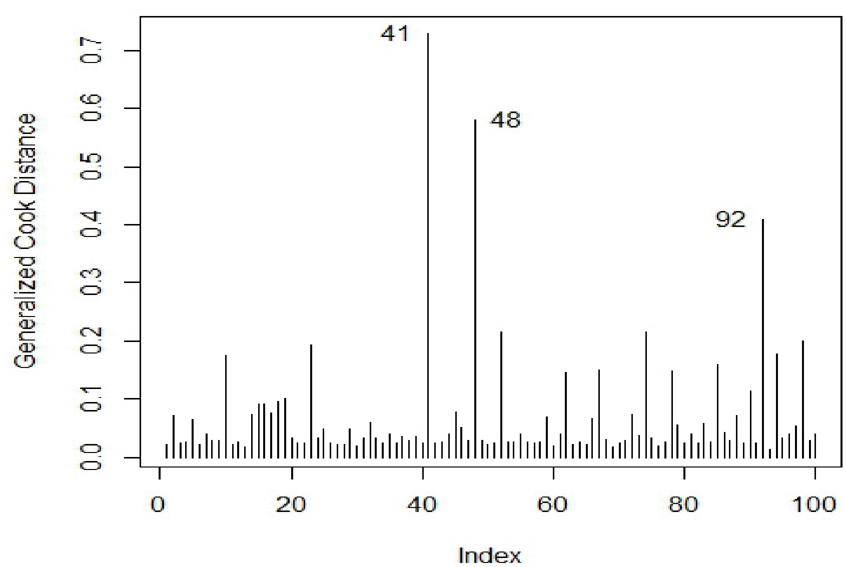

Fig. 7.4: Index plot of generalized cook distance.

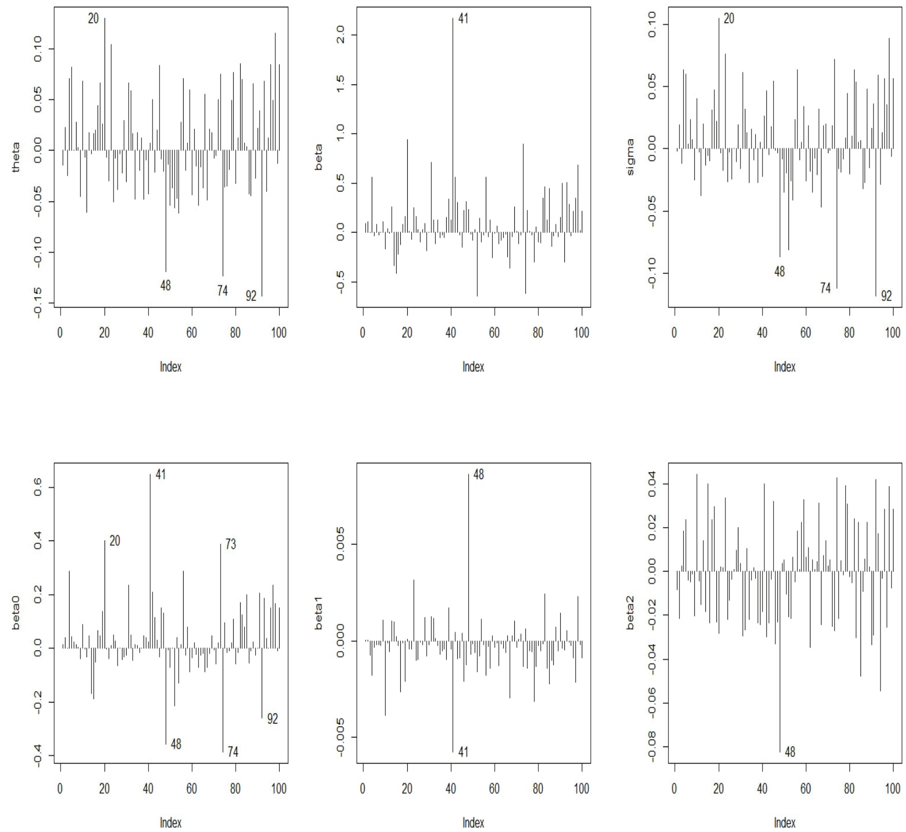

FIG. 7.5: The effects of observations on parameter values.

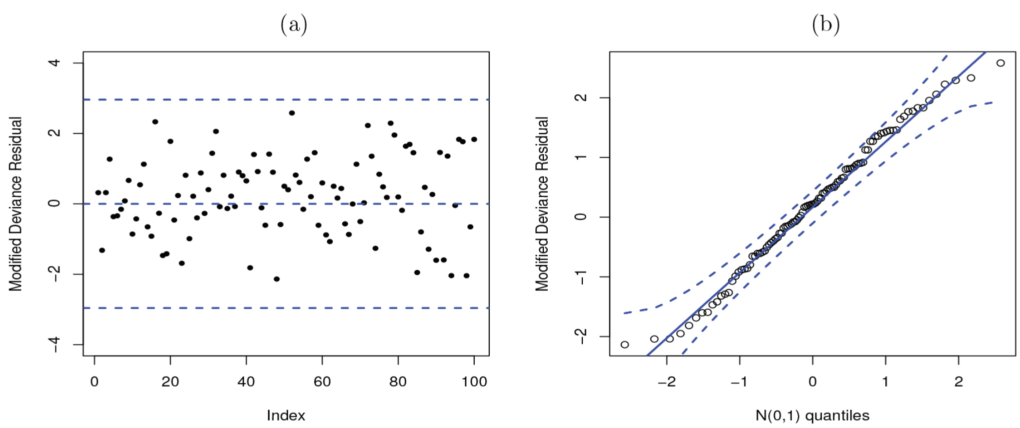

FIG. 7.6: (a) Index plot of the modified deviance residual and (b) Q-Q plot for modified deviance residual.

8. Concluding remarks

We propose a new lifetime model called Odd Log-logistic Burr XII distribution. Some of its mathematical properties are derived. Some useful characterization results based on the ratio of two truncated moments, based on the hazard function, and based on the conditional expectation of certain functions of the random variable are presented. The maximum likelihood method is used to estimate the model parameters by means of a graphical Monte Carlo simulation study. Moreover, we introduce a new log-location regression model based on the proposed distribution. The Jackknife estimation method is employed as an alternative method to estimate the unknown parameters of the new regression model. The generalized cook distance and likelihood distance measures are used to detect possible influential observations. Martingale and modified deviance residuals are defined to detect outliers and evaluate the model assumptions. The potentiality of the new regression model is illustrated by means of real data sets. Additionally, the index plot of the generalized cook distance and the plots for the effects of observations on the model parameters are presented.

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Appendix A

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ $(a = -\infty,\ b = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right) \mid X \geq x\right] = \mathbf{E}\left[q_1\left(X\right) \mid X \geq x\right]\eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η , particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)\, q_1(u) - q_2(u)} \right| \exp(-s(u))\ du \ ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' \, q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Appendix B

```
library(numDeriv)
rm(list=ls(all=TRUE))


f=function(x,theta,beta,alpha,lambda,a,b)
{


f=G(x,beta,alpha,lambda,a,b)**theta/(G(x,beta,alpha,lambda,a,b)
**theta+(1-G(x,beta,alpha,lambda,a,b))**theta)
ff=theta*g(x,beta,alpha,lambda,a,b)*(G(x,beta,alpha,lambda,a,b)
*(1-G(x,beta,alpha,lambda,a,b)))**(theta-1)/
((G(x,beta,alpha,lambda,a,b)
**theta+(1-G(x,beta,alpha,lambda,a,b))**theta))**2


fff=ff/(1-f)
return(fff)
}


g=function(x,beta,alpha,lambda,a,b){dburr(x,beta,alpha,lambda)}
G=function(x,beta,alpha,lambda,a,b){pburr(x,beta,alpha,lambda)}


pdf=function(x){f(x[1],theta,beta,alpha,lambda)}
pdf2=function(y,theta,beta,alpha,lambda){f(y,theta,beta,alpha,
    lambda)}

troca=function(){
  y=seq(0.1,15,0.1); mod=c(); deriv=c()
  ate=pdf2(y,theta,beta,alpha,lambda)
```

```
  ate=ate[ate!=Inf] ;  n=length(ate)
  for(i in 1:n){deriv=c(deriv,grad(func=pdf,x=c(y[i])))}
  sinal=sign(deriv)
  change=c()
  for(j in 1:n-1){
    change1=ifelse(sinal[j]==sinal[j+1],0,1); change=c(change,
       change1)}
  position=which(change %in% c(1))

  if (sum(change)==0) mod<-ifelse(sinal[1]>0,"+","-")
  if (sum(change)>0) mod<-ifelse(sinal[position]>sinal[position
      +1],"+-","-+")

  if (identical(mod,c("+"))) mod<<-"crescente"
  if (identical(mod,c("+-"))) mod<<-"modal"
  if (identical(mod,c("+-","-+"))) mod<<-"n"
  if (identical(mod,c("+-","-+","+-"))) mod<<-"m"
  if (identical(mod,c("-"))) mod<<-"decrescente"
  if (identical(mod,c("-+"))) mod<<-"banheira"
  if (identical(mod,c("-+","+-"))) mod<<-"inv(n)"
  if (identical(mod,c("-+","+-","-+"))) mod<<-"w"
  return(c(sum(change)))}

#fixing parameters
alpha=4;lambda=0.1; alphax=c(); betax=c(); a2=c(); a3=c()

for(theta in seq(0.1,1,0.005)){
  for(beta in seq(0.1,1,0.005)){
  alphax=c(alphax,theta);betax=c(betax,beta);a=troca();a2=c(a2,
      a); a3=c(a3,mod)}}

ff=factor(a3,labels=1:2)
ff1=as.numeric(ff)
ff1[ff1==1]='royalblue1'       #decres
ff1[ff1==2]='slategray1'     # inv (n)
ff1[ff1==3]='darkslategray3' #m bimo
ff1[ff1==4]='slategray1'     #mod
plot(alphax,betax,col=ff1,pch=16,cex=1,ylab =expression(beta)
,xlab=expression(theta))


text(0.17,0.6,'A',col=1,cex=1.5)
text(0.6,0.6,'B',col=1,cex=1.5)
legend(0.7,0.8,c("A␣-␣decreasing","B␣-␣upside-down"
                   ),bty="n",cex=1)
```