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AN ANALOGUE OF COWLING-PRICE'S THEOREM FOR THE Q-FOURIER-DUNKL TRANSFORM

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Abstract. The Q-Fourier-Dunkl transform satisfies some uncertainty principles in a similar way to the Euclidean Fourier transform. By using the heat kernel associated to the Q-Fourier-Dunkl operator, we have established an analogue of Cowling-Price, Miyachi and Morgan theorems on $\mathbb{R}$ by using the heat kernel associated to the Q-Fourier-Dunkl transform.
Keywords: Cowling-Price's theorem; Miyachi's theorem; Uncertainty Principles; Q-Fourier-Dunkl transform.

1. Introduction

There are many theorems which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot simultaneously be very small at infinity. This principle has several versions which were proved by M.G. Cowling and J.F. Price [3] and Miyachi [6]. In this paper, we will study an analogue of Cowling-Price's theorem and Miyachi's theorem for the Q-Fourier-Dunkl transform. Many authors have established the analogous of Cowling-Price's theorem in other various setting of harmonic analysis (see for instance [5]) The outline of the content of this paper is as follows.
Section 2 is dedicated to some properties and results concerning the Q-Fourier-Dunkl transform. In Section 3 we give an analogue of Cowling-Price's theorem, Miyachi's theorem, and Morgan's theorem for the Q-Fourier-Dunkl transform. Let us now be more precise and describe our results. To do so, we need to introduce some notations. Throughout this paper $\alpha > \frac{-1}{2}$. Notice that if $\alpha = \frac{-1}{2}$ then the space is the classical Lebesgue one, we can follow in this case the procedures for similar transforms, such as the Fourier transform (see for example [3, 6]).

- (1.1) $$Q(x) = \exp\left(-\int_0^x q(t)dt\right), \quad x \in \mathbb{R}$$

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where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$.

- $L^p_\alpha(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad if \quad 1 < p < \infty,$$

and $\|f\|_{\infty,\alpha} = \|f\|_\infty = esssup_{x\in\mathbb{R}}|f(x)|$.

- $L^p_Q(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,Q} = \|Qf\|_{p,\alpha} < \infty$, where Q is given by (1.1).

We consider the first singular differential-difference operator Λ defined on $\mathbb{R}$

$$\Lambda f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x} + q(x)f(x) \tag{1.2}$$

where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$. For $q = 0$ we regain the Dunkl operator Λ_α associated with reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x}.$$

1.1. Q-Fourier-Dunkl Transform

The following statements are proved in [1]

Lemma 1.1. *1. For each $\lambda \in \mathbb{C}$, the differential-difference equation*

$$\Lambda u = i\lambda u, \quad u(0) = 1$$

admits a unique $\mathcal{C}^\infty$ solution on $\mathbb{R}$, denoted by Ψ_λ, given by

$$\Psi_\lambda(x) = Q(x)e_\alpha(i\lambda x), \tag{1.3}$$

where e_α denotes the one-dimensional Dunkl kernel defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(z) \qquad (z \in \mathbb{C}),$$

and j_α being the normalized spherical Bessel function of index α given by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{z}{2})^{2n}}{n!\ \Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}). \tag{1.4}$$

2. For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $n = 0, 1, ...$ we have

$$\left| \frac{\partial^n}{\partial\lambda^n}\Psi_\lambda(x) \right| \leq Q(x)|x|^n e^{|Im\ \lambda||x|}. \tag{1.5}$$

In particular

$$| \Psi_\lambda(x) | \leq Q(x)e^{|Im\ \lambda||x|}. \tag{1.6}$$

3. *For all* $x \in \mathbb{R},\ \lambda \in \mathbb{C}$, *we have the Laplace type integral representation*

$$\Psi_\lambda(x) = a_\alpha Q(x) \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}}(1+t)e^{i\lambda xt}dt, \tag{1.7}$$

where $a_\alpha = \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$.

Definition 1.1. The Q-Fourier-Dunkl transform associated with Λ for a function in $L^1_Q(\mathbb{R})$ is defined by

$$\mathcal{F}_Q(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-\lambda}(x)x^{2\alpha+1}dx. \tag{1.8}$$

Theorem 1.1. 1. *Let* $f \in L^1_Q(\mathbb{R})$ *such that* $\mathcal{F}_Q(f) \in L^1_\alpha(\mathbb{R})$. *Then for allmost* $x \in \mathbb{R}$ *we have the inversion formula*

$$f(x)\left(Q(x)\right)^2 = m_\alpha \int_{\mathbb{R}} \mathcal{F}_Q(f)(\lambda)\Psi_\lambda(x)|\lambda|^{2\alpha+1}d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2(\alpha+1)}(\Gamma(\alpha+1))^2}.$$

2. *For every* $f \in L^2_Q(\mathbb{R})$, *we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2\left(Q(x)\right)^2|x|^{2\alpha+1}dx = m_\alpha \int_{\mathbb{R}} |\mathcal{F}_Q(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda.$$

3. *The Q-Fourier-Dunkl transform* $\mathcal{F}_Q$ *extends uniquely to an isometric isomorphism from* $L^2_Q(\mathbb{R})$ *onto* $L^2_\alpha(\mathbb{R})$.

The heat kernel $N(x,s)$, $x \in \mathbb{R}$, $s > 0$, associated with the Q-Fourier-Dunkl transform is given by

$$N(x,s) = m_\alpha \frac{e^{-\frac{x^2}{4s}}}{(2s)^{\alpha+\frac{1}{2}}Q(x)}. \tag{1.9}$$

Some basic properties of $N(x,s)$ are the following:

- $N(x,s)Q^2(x) = m_\alpha \int_{\mathbb{R}} e^{-sy^2}\Psi_y(x)|y|^{2\alpha+1}dy.$
- $\mathcal{F}_Q(N(.,s))(x) = e^{-sx^2}.$

we define the heat functions W_l, $l \in \mathbb{N}$ as

$$Q^2(x)W_l(x,s) = \int_{\mathbb{R}} y^l e^{-\frac{y^2}{4s}}\Psi_y(x)|y|^{2\alpha+1}dy \tag{1.10}$$

$$\mathcal{F}_Q(W_l(.,s)) = i^l y^l e^{-sy^2}. \tag{1.11}$$

The intertwining operators associated with a Q-Fourier-Dunkl transform on the real line is given by

$$X_Q(f)(x) = a_\alpha Q(x) \int_{-1}^{1} f(tx)(1-t^2)^{\alpha-\frac{1}{2}} dt,$$

its dual is given by

$$(1.12) \quad {}^t X_Q(f)(y)a = \quad a_\alpha \int_{|x|\geq|y|} f(x)Q(x)sgn(x)(x^2-y^2)^{\alpha-\frac{1}{2}}(x+y)dx$$

${}^t X_Q$ can be written as

$${}^t X_Q(f)(y) = a_\alpha \int_{\mathbb{R}} f(x)Q(x)d\nu_y(x),$$

where

$$d\nu_y(x) = a_\alpha \chi_{\{|x|\geq|y|\}} sgn(x)(x^2-y^2)^{\alpha-\frac{1}{2}}(x+y)dx$$

and $\chi_{\{|x|\geq|y|\}}$ denote the characteristic function with support in the set $\{x \in \mathbb{R} \ / \ |x| \geq |y|\}$.

Proposition 1.1. *If $f \in L^1_Q(\mathbb{R})$ then ${}^t X_Q(f) \in L^1(\mathbb{R})$ and $\|{}^t X_Q(f)\|_1 \leq \|f\|_{1,Q}$.*

For every $f \in L^1_Q(\mathbb{R})$

$$\mathcal{F}_Q = \mathcal{F} \circ^t X_Q(f), \tag{1.13}$$

where $\mathcal{F}$ is the usual Fourier transform defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x}dx.$$

2. Cowling-Price's Theorem for the Q-Fourier-Dunkl Transform

Theorem 2.1. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \frac{e^{apx^2} Q^p(x)|f(x)|^p}{(1+|x|)^k} |x|^{2\alpha+1} dx < \infty \tag{2.1}$$

and

$$\int_{\mathbb{R}} \frac{e^{bq\xi^2}|\mathcal{F}_Q(f)(\xi)|^q}{(1+|\xi|)^m} d\xi < \infty, \tag{2.2}$$

for some constants $a, b > 0$, $k > 0, m > 1$ and $1 \leq p, r \leq +\infty$.

i) If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$*, then* $f(x) = P(x)N(x,b)$ *where* P *is a polynomial with* $degP \leq \min\{\frac{k}{p} + \frac{2\alpha+1}{p'}, \frac{m-1}{r}\}$*. Especially, if*

$$k \leq 2\alpha + 2 + p\min\{\frac{k}{p} + \frac{2\alpha+1}{p'}, \frac{m-1}{r}\},$$

then $f = 0$ *almost everywhere. Furthermore, if* $m \in]1, 1+r]$ *and* $k > 2\alpha+2$*, then* f *is a constant multiple of* $N(.,b)$*.*

iii) If $ab < \frac{1}{4}$*, then for all* $\delta \in]b, \frac{1}{4}a[$ *all functions of the form* $f(x) = P(x)N(x,\delta)$ *satisfy (2.1) and (2.2).*

Proof. It follows from (2.1) that $f \in L^1_Q$ and $\mathcal{F}_Q(f)(\xi)$ exists for all $\xi \in \mathbb{R}$. Moreover, it has an entire holomorphic extension on $\mathbb{C}$ satisfying for some $s > 0$,

$$|\mathcal{F}_Q(f)(z)| \leq Ce^{\frac{Imz^2}{4a}}(1 + |Imz|)^s.$$

By (1.1) we have for all $z = \xi + i\eta \in \mathbb{C}$,

$$|\mathcal{F}_Q(f)(z)| \leq \int_{\mathbb{R}} |f(x)||\Lambda_\xi(x)||x|^{2\alpha+1}dx \tag{2.3}$$

$$\leq e^{\frac{\eta^2}{4a}} \int_{\mathbb{R}} \frac{e^{ax^2}Q(x)|f(x)|}{(1+|x|)^{\frac{k}{p}}}(1+|x|)^{\frac{k}{p}} e^{-a(x-\frac{\eta}{2a})^2}|x|^{2\alpha+1}dx. \tag{2.4}$$

By Hölder inequality we have

$$|\mathcal{F}_Q(f)(z)| \leq e^{\frac{\eta^2}{4a}} \left(\int_{\mathbb{R}} \frac{e^{pax^2}Q(x)^p|f(x)|^p}{(1+|x|)^k}|x|^{2\alpha+1}dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}}(1+|x|)^{\frac{kp'}{p}} e^{-ap'(x-\frac{\eta}{2a})^2}|x|^{2\alpha+1}dx\right)^{\frac{1}{p'}}$$

according to (2.1) we get that

$$\begin{aligned}
|\mathcal{F}_Q(f)(\xi+i\eta)| &\leq Ce^{\frac{\eta^2}{4a}} \left(\int_{\mathbb{R}}(1+|x|)^{\frac{kp'}{p}} e^{-ap'(x-\frac{\eta}{2a})^2}|x|^{2\alpha+1}dx\right)^{\frac{1}{p'}} \\
&\leq Ce^{\frac{\eta^2}{4a}} \left(\int_0^\infty (1+|x|)^{\frac{kp'}{p}+2\alpha+1} e^{-ap'(x-\frac{\eta}{2a})^2}dx\right)^{\frac{1}{p'}} \\
&\leq Ce^{\frac{\eta^2}{4a}}(1+|\eta|)^{\frac{k}{p}+\frac{2\alpha+1}{p'}}.
\end{aligned}$$

If $ab = \frac{1}{4}$, then

$$|\mathcal{F}_Q(f)(\xi+i\eta)| \leq Ce^{b\eta^2}(1+|\eta|)^{\frac{k}{p}+\frac{2\alpha+1}{p'}}.$$

We put $g(z) = e^{bz^2}\mathcal{F}_Q(f)(z)$, then

$$|g(z)| \leq Ce^{b|Rez|^2}(1+|Imz|)^{\frac{k}{p}+\frac{2\alpha+1}{p'}}.$$

It follows from (2.2) that

$$\int_{\mathbb{R}} \frac{|g(z)|^r}{(1+|\xi|)^m}d\xi < \infty.$$

Lemma 2.1. *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ce^{a|Rez|^2}(1+|Imz|)^l$$

for some $l > 0$, $a > 0$ and

$$\int_{\mathbb{R}} \frac{|h(x)|^r}{(1+|x|)^m}|P(x)|dx < \infty$$

for some $r \geq 1$, $m > 1$ and P is a polynomial with degree m. Then h is a polynomial with $degh \leq min\{l, \frac{m-M-1}{r}\}$ and if $m \leq r+M+1$, then h is a constant.

From this Lemma, g is a polynomial, we say P_b with $degP_b \leq min\{\frac{kp'}{p} + \frac{2\alpha+1}{p'}, \frac{m-1}{r}\}$. Then $\mathcal{F}_Q(f)(x) = P_b(x)e^{-bx^2}$ then,

$$f(x) = Q_b(x)N(x,b)$$

where $degP_b = degQ_b$. Therefore, nonzero f satisfies (1.10) provided that

$$k > 2\alpha + 2 + p\min\left\{\frac{kp'}{p} + \frac{2\alpha+1}{p'}, \frac{m-1}{q}\right\}.$$

If $m < r+1$, by Lemma 1 we have g as a constant and $\mathcal{F}_Q(f)(x) = Ce^{-bx^2}$ and $f(x) = CN(x,b)$. If $m > 1$ and $k > 2\alpha+2$, these functions satisfy (2.1) and (2.2), which proves (ii).

If $ab > \frac{1}{4}$, then we can find positive constants a_1 and b_1 such that $a > a_1 = \frac{1}{4b_1} > \frac{1}{4b}$. Then f and $\mathcal{F}_Q(f)$ also satisfy (2.2) with a and b replaced by a_1 and b_1 respectively. Then $\mathcal{F}_Q(f)(x) = P_{b_1}(x)e^{-b_1x^2}$. $\mathcal{F}_Q(f)$ cannot satisfy (2.2) unless $P_{b_1} = 0$, which implies that $f = 0$, this proves (i). If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4a}[$, the functions of the form $f(x) = P(x)N(x,\delta)$, where P is a polynomial on $\mathbb{R}$, satisfy (2.1) and (2.2). This proves (iii). □

3. Mathematical Formulas

4. Miyachi's Theorem for the Q-Fourier-Dunkl Transform

Theorem 4.1. *Let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2}f \in L^p_Q(\mathbb{R}) + L^r_Q(\mathbb{R}) \tag{4.1}$$

and

$$\int_{\mathbb{R}} \log^+ \frac{|\mathcal{F}_Q(f)(\xi)e^{b\xi^2}|}{\lambda}d\xi < \infty, \tag{4.2}$$

for some constants $a, b, \lambda > 0$ and $1 \leq p, r \leq +\infty$.

(i) if $ab > \frac{1}{4}$ *then* $f = 0$ *almost everywhere.*

(ii) if $ab = \frac{1}{4}$ *then* $f = cN(.,b)$ *with* $|c| \leq \lambda$.

(iii) if $ab > \frac{1}{4}$ *then for all* $\delta \in]b, \frac{1}{4}[$, *all functions of the form* $f(x) = P(x)N(x,\delta)$, *where* P *is a polynomial on* $\mathbb{R}$ *satisfy (2.1) and (2.2).*

To prove this result, we need the following lemmas.

Lemma 4.1. [5] *Let* h *be an entire function on* $\mathbb{C}$ *such that*

$$|h(z)| \leq Ae^{B|Rez|^2},$$

and

$$\int_{\mathbb{R}} \log^+ |h(y)|dy < \infty, \tag{4.3}$$

for some constants A *and* B. *Then* h *is a constant.*

Lemma 4.2. *Let* $r \in [1, +\infty], a > 0$. *Then for* $g \in L^r_Q(\mathbb{R})$ *there exist* $c > 0$ *such that*

$$\| e^{ax^2}\ {}^tX_Q(e^{-ay^2}g) \|_r \leq c \| g \|_{r,Q}\ .$$

Proof. From the hypothesis, it follows that e^{-ay^2} belongs to $L^1_Q(\mathbb{R})$. Then by Proposition 1.1, ${}^tX_Q(e^{-ay^2}g)$ is defined almost everywhere on $\mathbb{R}$. Here we consider two cases:

i) If $r \in [1, +\infty[$ then

$$\begin{aligned}
\| e^{ax^2}\ {}^tX_Q(e^{-ay^2}g) \|_r^r &\leq \int_{\mathbb{R}} e^{arx^2} (\int_{\mathbb{R}} Q(y)e^{-ay^2}|g(y)|d\nu_x(y))^r dx, \\
&\leq \int_{\mathbb{R}} e^{arx^2} \left(\int_{\mathbb{R}} |Q(y)g(y)|^r d\nu_x(y)\right)^{\frac{r}{r}} \left(\int_{\mathbb{R}} e^{-ar'y^2} d\nu_x(y)\right)^{\frac{r}{r'}} dx
\end{aligned}$$

where r' is the conjugate exponent for r. Since

$$\int_{\mathbb{R}} e^{-ry^2} d\nu_x(y) = Ce^{-rx^2}, \tag{4.4}$$

for $r > 0$ it follows from (4.4) that

$$\begin{aligned}
\| e^{ax^2}\ {}^tX_Q(e^{-ay^2}g) \|_r^r &\leq C \int_{\mathbb{R}} {}^tX_Q(|g|^r)(x)dx, \\
&= C \int_{\mathbb{R}} |g(x)|^r |x|^{2\alpha+1} dx < \infty.
\end{aligned}$$

ii) If $r = \infty$ then it follows from (4.4) that

$$\begin{aligned}
\| e^{ax^2}\ {}^tX_Q(e^{-ay^2}g) \|_r &\leq e^{ax^2}\ {}^tX_Q(e^{-ay^2})(x)\|g\|_{Q,\infty} \\
&= C\|g\|_{Q,\infty}.
\end{aligned}$$

□

Lemma 4.3. *Let $r, p \in [1, +\infty]$ and let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2} f \in L^p_Q(\mathbb{R}) + L^r_Q(\mathbb{R}) \tag{4.5}$$

for some $a > 0$. Then for all $z \in \mathbb{C}$, the integral

$$\mathcal{F}_Q(f)(z) = \int_{\mathbb{R}} f(x)\Lambda_{z,Q}(-x)|x|^{2\alpha+1}dx$$

is well defined. $\mathcal{F}_Q(f)(z)$ is entire and there exists $C > 0$ such that for all $\xi,\ \eta \in \mathbb{R}$,

$$|\mathcal{F}_Q(f)(\xi + i\eta)| \leq Ce^{\frac{\eta^2}{4a}}. \tag{4.6}$$

Proof. From (5) and Hölder's inequality we have the first assertion. For (4.6) using (4.5) we have $f \in L^1_Q(\mathbb{R})$ and ${}^tX_Q(f) \in L^1(\mathbb{R})$. for all $\xi,\ \eta \in \mathbb{R}$,

$$\mathcal{F}_Q(f)(\xi + i\eta) = \int_{\mathbb{R}} {}^tX_Q(f)(x)e^{-ix(\xi+i\eta)}dx$$

$$\begin{aligned} |\mathcal{F}_Q(f)(\xi + i\eta)| &\leq e^{\frac{\eta^2}{4a}} \int_{\mathbb{R}} e^{ax^2} \left|{}^tX_Q(f)(x)\right| e^{-ax^2+x\eta-\frac{\eta^2}{4a}}dx \\ &\leq e^{\frac{\eta^2}{4a}} \int_{\mathbb{R}} e^{ax^2} \left|{}^tX_Q(f)(x)\right| e^{-a\left(x-\frac{\eta}{2a}\right)^2}dx. \end{aligned}$$

From (4.5) we can deduce that there exists $u \in L^p_Q(\mathbb{R})$ and $v \in L^r_Q(\mathbb{R})$ such that

$$f(x) = e^{-ax^2}u(x) + e^{-ax^2}v(x),$$

by Lemma 4 we have

$$\int_{\mathbb{R}} e^{ax^2} \left|{}^tX_Q(f)(x)\right| e^{-a\left(x-\frac{\eta}{2a}\right)^2}dx \leq C\left(\|u\|_{p,Q} + \|v\|_{r,Q}\right) < \infty,$$

which proves the Lemma. □

Proof of Theorem

- If $ab > \frac{1}{4}$. Let h be a function on $\mathbb{C}$ defined by

$$h(z) = e^{\frac{z^2}{4a}}\mathcal{F}_Q(f)(z).$$

h is entire function on $\mathbb{C}$, it follows from (4.6) that

$$\forall \xi \in \mathbb{R},\ \forall \eta \in \mathbb{R}\ \ |h(\xi + i\eta)| \leq Ce^{\frac{\xi^2}{4a}}. \tag{4.7}$$

On the other hand, we have

$$\begin{aligned}\int_{\mathbb{R}} \log^{+} |h(y)|\, dy &= \int_{\mathbb{R}} \log^{+} \left| e^{\frac{y^2}{4a}} \mathcal{F}_Q(f)(y) \right| dy \\ &= \int_{\mathbb{R}} \log^{+} \frac{\left| e^{by^2} \mathcal{F}_Q(f)(y) \right|}{\lambda} \lambda e^{\left(\frac{1}{4a}-b\right)y^2} dy \\ &\leq \int_{\mathbb{R}} \log^{+} \frac{\left| e^{by^2} \mathcal{F}_Q(f)(y) \right|}{\lambda} dy + \int_{\mathbb{R}} \lambda e^{\left(\frac{1}{4a}-b\right)y^2} dy\end{aligned}$$

because $\log_{+}(cd) \leq \log_{+}(c) + d$ for all c, $d > 0$. Since $ab > \frac{1}{4}$, (2.2) implies that

$$\int_{\mathbb{R}} \log^{+} |h(y)| dy < \infty. \tag{4.8}$$

A combination of (4.7), (4.8) and Lemma 3 shows that h is a constant and

$$\mathcal{F}_Q(f)(y) = Ce^{-\frac{1}{4a}y^2}.$$

Since $ab > \frac{1}{4}$, (2.2) holds whenever $C = 0$ and the injectivity of $\mathcal{F}_Q$ implies that $f = 0$ almost everywhere.

- If $ab = \frac{1}{4}$. We deduce from previous case that $\mathcal{F}_Q(f) = Ce^{-\frac{\xi^2}{4a}}$. Then (2.2) holds whenever $|C| \leq \lambda$. Hence $f = CN(.,b)$ with $|C| \leq \lambda$.
- If $ab < \frac{1}{4}$. If f is a given form, then $\mathcal{F}_Q(f)(y) = Q(y)e^{-\delta y^2}$ for some Q.

In the contintion, we will give an analogue of Hardy's theorem [?] for the Q-Fourier-Dunkl transform.

Theorem 4.2. Hardy *Let $N \in \mathbb{N}$. Assume that $f \in L^2_Q(\mathbb{R})$ is such that*

$$|f(x)| \leq Me^{-\frac{1}{4a}x^2} \ a.e \ , \ \forall y \in \mathbb{R}, |\mathcal{F}_Q(f)(y)| \leq M(1+|y|)^N e^{-by^2}, \tag{4.9}$$

for some constants $a > 0$, $b > 0$ and $M > 0$. Then,
i) If $ab > \frac{1}{4}$, then $f = 0$ a.e.
ii) If $ab = \frac{1}{4}$, then the function f is of the form

$$f(x) = \sum_{|s| \leq N} a_s W_s\left(\frac{1}{4a}, x\right) \ a.e. \ , \ a_s \in \mathbb{C}.$$

iii) If $ab < \frac{1}{4}$, there are infinitely many nonzero functions of f satisfying the conditions (4.9).

Proof. The first condition of (4.9) implies that $f \in L^1_Q(\mathbb{R})$. So by Proposition 1.1, the function ${}^t X_Q(f)$ is defined almost everywhere. By using the relation (1.13), we deduce that for all $x \in \mathbb{R}$,

$$|{}^t X_Q(f)(x)| \leq M_0 e^{-ax^2},$$

where M_0 is a positive constant. So

$$|{}^tX_Q(f)(x)| \leq M_0(1+|x|)^N e^{-ax^2}, \tag{4.10}$$

On the other hand from (1.13) and (4.9) we have for all $x \in \mathbb{R}$,

$$|\mathcal{F}({}^tX_Q)(f)(y)| \leq M(1+|y|)^N e^{-b|y|^2}, \tag{4.11}$$

The relations (4.10) and (4.11) show that the conditions of Proposition 3.4 of [2], p.36, are satisfied by the function ${}^tX_Q(f)$. Thus we get:
i) If $ab > \frac{1}{4}$, ${}^tX_Q(f) = 0$ a.e. Using (1.13) we deduce

$$\forall y \in \mathbb{R}, \mathcal{F}_Q(f)(y) = \mathcal{F} \circ ({}^tX_Q)(f)(y) = 0.$$

Then by the injectivity of $\mathcal{F}_Q$ we have $f = 0$ a.e.
ii) If $ab = \frac{1}{4}$, then ${}^tX_Q(f)(x) = P(x)e^{-ax^2}$, where P is a polynomial of degree lower than N. Using this relation and (1.13), we deduce that

$$\forall x \in \mathbb{R},\ \mathcal{F}_Q(f)(y) = \mathcal{F} \circ^t X_Q(f)(y) = \mathcal{F}(P(x)e^{-\delta x^2})(y).$$

but

$$\forall x \in \mathbb{R},\ \mathcal{F}(P(x)e^{-\delta x^2})(y) = S(y)e^{\frac{-y^2}{4\delta}},$$

with S a polynomial of degree lower than N.
Thus from (1.11), we obtain

$$\forall x \in \mathbb{R},\ \mathcal{F}_Q(f)(y) = \mathcal{F}_Q\left(\sum_{|s|<\frac{N-1}{2}} a_s W_s(\frac{1}{4\delta},.)\right)(y).$$

The injectivity of the transform $\mathcal{F}_Q$ implies

$$f(x) = \sum_{|s|\leq N} a_s W_s(\frac{1}{4a}, x)\ a.e.$$

iii) If $ab < \frac{1}{4}$, let $t \in]a, \frac{1}{4b}[$ and $f(x) = C\dfrac{e^{-tx^2}}{Q(x)}$ for some real constant C, these functions satisfy the conditions (4.9).
□In the next part, we will give an analogue of Morgan's theorem [7] for the Q-Fourier-Dunkl transform.

Theorem 4.3. Morgan *Let $1 < p < 2$ and r be the conjugate exponent of p. Assume that $f \in L^2_Q(\mathbb{R})$ satisfies*

$$\int_{\mathbb{R}} e^{\frac{a^p}{p}|x|^p}|f(x)||x|^{2\alpha+1}dx < +\infty,\ \textit{and}\ \int_{\mathbb{R}} e^{\frac{b^r}{r}|y|^r}|\mathcal{F}_Q(f)(y)|dy < +\infty, \tag{4.12}$$

for some constants $a > 0$, $b > 0$.
Then if $ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}}$, we have $f = 0$ a.e.

Proof. The first condition of (4.12) implies that $f \in L^1_Q(\mathbb{R})$. So by Proposition 1.1, the function ${}^tX_Q(f)$ is defined almost everywhere. By using the relation (4.12) and Proposition 1.1, we deduce that:

$$\int_{\mathbb{R}} |{}^tX_Q(f)(x)| e^{\frac{a^p}{p}|x|^p} dx \leq \int_{\mathbb{R}} e^{\frac{a^p}{p}|x|^p} |f(x)||x|^{2\alpha+1} dx < +\infty.$$

So

$$\int_{\mathbb{R}} |{}^tX_Q(f)(x)| e^{\frac{a^p}{p}|x|^p} dx < +\infty \tag{4.13}$$

On the other hand, from (1.13) and (4.12) we have:

$$\int_{\mathbb{R}} e^{\frac{b^q}{q}|y|^q} |\mathcal{F}_Q(f)(y)| dy = \int_{\mathbb{R}} e^{\frac{b^q}{q}|y|^q} |\mathcal{F}({}^tX_Q)(f)(y)| dy < +\infty. \tag{4.14}$$

The relations (4.13) and (4.14) are the conditions of Theorem 1.4, p.26 of [2], which are satisfied by the function ${}^tX_Q(f)$. Thus we deduce that if $ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}}$ we have ${}^tX_Q(f) = 0$ a.e. Using the same proof as in the end of Theorem 4, we have obtained $f(y) = 0$. a.e. $y \in \mathbb{R}$. □

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