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GERAGHTY EXTENSION TO k-DIMENSION

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Abstract. In this paper, we extend the Geraghty result [7] to k-dimension.

Keywords: Fixed point, Geraghty extension, k-dimension, metric space.

1. Introduction and Preliminaries

It is known that the Banach contraction principle is considered as one of the most important theorems in the classical functional analysis. There are many generalizations of this theorem. The following generalization is due to M. Geraghty [7].

Theorem 1.1. *[7] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If T satisfies the following inequality:*

$$d(Tx, Ty) \leq \beta(d(x, y))\, d(x, y)$$

for all $x, y \in X$, where $\beta : [0, \infty) \to [0, 1)$ is a function which satisfies the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \textit{implies} \quad \lim_{n\to\infty} t_n = 0.$$

Then T has a unique fixed point $u \in X$ and $\{T^n x\}$ converges to u for each $x \in X$.

The above result has been generalized by many authors. For details, see [1, 2, 3, 4, 5, 6, 8, 9].

$\mathbb{N}$ (resp. $\mathbb{N}_0$) denotes a set of positive (nonnegative) integers. We denote by $\mathcal{F}$ a set of functions β given in Theorem 1.1. The aim of this paper is to generalize and extend Theorem 1.1 to k-dimension. To be more clear, we will consider nonself mappings $T : X^k \to X$ involving a Geraghty type contraction in the class of metric spaces. Note that in the given contraction (it corresponds later to (2.1)),

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we consider two k-uplets of the form $(u_1, u_2, \ldots, u_k)$ and $(u_2, u_3, \ldots, u_{k+1})$, that is, there is a repetition of $(k-1)$-components, which are u_2, u_3, $\ldots$, u_k. This fact is different from all known multidimensional fixed point results where the two considered k-uplets are not generally dependent, i.e., of the form $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$.

2. Main results

Our main result is

Theorem 2.1. *Let (X, d) be a complete metric space and $k \in \mathbb{N}$. Let $T : X^k \to X$ be such that*

(2.1)
$$d(T(u_1, u_2, \ldots, u_k), T(u_2, u_3, \ldots, u_{k+1})) \\ \leq \beta\left(M((u_1, u_2, \ldots, u_k), (u_2, u_3, \ldots, u_{k+1}))\right) M((u_1, x_2, \ldots, u_k), (u_2, u_3, \ldots, u_{k+1})),$$

for all $u_1, u_2, \ldots, u_k, u_{k+1}$ in X, where $\beta \in \mathcal{F}$ and $M : X^k \times X^k \to [0, \infty)$ is as

$$M((u_1, u_2, \ldots, u_k), (u_2, u_3, \ldots, u_{k+1})) \\ = \max\{d(u_k, u_{k+1}), d(u_k, T(u_1, u_2, \ldots, u_k)), d(u_{k+1}, T(u_2, u_3, \ldots, u_{k+1}))\}.$$

Then there is a point u in X such that $T(u, u, \ldots, u) = u$.

Proof. We split the proof into several steps.
Step 1: Let $k \in \mathbb{N}$ be fixed. Consider as the initial point the k-uplet point $(x_1, x_2, \ldots, x_k) \in X^k$. Let

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}) \quad \text{for all } n \in \mathbb{N}.$$

In view of (2.1),

(2.2)
$$d(x_{n+k+1}, x_{n+k+2}) = d(T(x_{n+1}, x_{n+2}, \ldots, x_{n+k}), T(x_{n+2}, x_{n+3}, \ldots, x_{n+k+1})) \\ \leq \beta\left(M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1}))\right) \\ M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1})).$$

Now,

$$M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1})) \\ = \max\{d(x_{n+k}, x_{n+k+1}), d(T(x_{n+1}, x_{n+2}, \ldots, x_{n+k}), x_{n+k}), d(T(x_{n+2}, x_{n+3}, \ldots, x_{n+k+1}), x_{n+k+1})\} \\ = \max\{d(x_{n+k}, x_{n+k+1}), d(x_{n+k+2}, x_{n+k+1})\}.$$

The case that $M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1})) = d(x_{n+k+2}, x_{n+k+1})$ for some n, is impossible. Indeed, by (2.2) and the fact that $\beta \in \mathcal{F}$,

$$d(x_{n+k+1}, x_{n+k+2}) \leq \beta\left(d(x_{n+k+2}, x_{n+k+1})\right) d(x_{n+k+2}, x_{n+k+1}) < d(x_{n+k+2}, x_{n+k+1}),$$

which is a contradiction. Hence $M((x_{n+1}, x_{n+2}, \dots, x_{n+k}), (x_{n+2}, x_{n+3}, \dots, x_{n+k+1})) = d(x_{n+k}, x_{n+k+1})$ for all $n \geq 0$. Again by (2.2),

(2.3)
$$d(x_{n+k+1}, x_{n+k+2}) \leq \beta\left(d(x_{n+k}, x_{n+k+1})\right) d(x_{n+k}, x_{n+k+1}) < d(x_{n+k}, x_{n+k+1})$$
for all $n \in \mathbb{N}_0$.

So the sequence $\{d(x_{n+k}, x_{n+k+1})\}$ is non-negative and non-increasing. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) = r$. We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. So for a large n, $d(x_{n+k}, x_{n+k+1}) > 0$. (2.3) implies that
$$\frac{d(x_{n+k+1}, x_{n+k+2})}{d(x_{n+k}, x_{n+k+1})} \leq \beta(d(x_{n+k}, x_{n+k+1})) < 1.$$
Taking the limit as $n \to \infty$, we get that
$$\lim_{n\to\infty} \beta\left(d(x_{n+k}, x_{n+k+1})\right) = 1.$$
Since $\beta \in \mathcal{F}$,
$$\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) = 0. \tag{2.4}$$

Step 2: We shall prove that $\{x_{n+k}\}$ is a Cauchy sequence. We argue by contradiction. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(p)+k}\}$ and $\{x_{n(p)+k}\}$ of $\{x_{n+k}\}$ with $m(p) > n(p) > p$ such that for every p
$$d(x_{m(p)+k}, x_{n(p)+k}) \geq \varepsilon. \tag{2.5}$$
Moreover, corresponding to $n(p)$ we can choose $m(p)$ in such a way that it is the smallest integer with $m(p) > n(p)$ and satisfying (2.5). Then
$$d(x_{m(p)+k-1}, x_{n(p)+k}) < \varepsilon. \tag{2.6}$$
By the triangle inequality, (2.5) and (2.6), we get
$$\begin{aligned} d(x_{n(p)+k-1}, x_{m(p)+k-1}) &\leq d(x_{n(p)+k-1}, x_{n(p)+k}) + d(x_{n(p)+k}, x_{m(p)+k-1}) \\ &< \varepsilon + d(x_{n(p)+k-1}, x_{n(p)+k}), \end{aligned} \tag{2.7}$$
and

(2.8)
$$\begin{aligned} \varepsilon &\leq d(x_{n(p)+k}, x_{m(p)+k}) \\ &\leq d(x_{n(p)+k}, x_{n(p)+k-1}) + d(x_{n(p)+k-1}, x_{m(p)+k-1}) + d(x_{m(p)+k-1}, x_{m(p)+k}). \end{aligned}$$
Using (2.4) in (2.7) and (2.8), we obtain
$$\lim_{p\to\infty} d(x_{n(p)+k-1}, x_{m(p)+k-1}) = \varepsilon. \tag{2.9}$$

On the other hand,

$$
\begin{aligned}
&M((x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), (x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1}))\\
&= \max\{d(x_{n(p)+k-1}, x_{m(p)+k-1}), d(T(x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), x_{n(p)+k-1}),\\
&d(T(x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1}), x_{m(p)+k-1})\}\\
&= \max\{d(x_{n(p)+k-1}, x_{m(p)+k-1}), d(x_{n(p)+k}, x_{n(p)+k-1}), d(x_{m(p)+k}, x_{m(p)+k-1})\}.
\end{aligned}
$$

In view of (2.4) and (2.9),

(2.10)
$$\lim_{p\to\infty} M((x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), (x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1})) = \varepsilon.$$

By (2.1) and (2.5),

(2.11)
$$
\begin{aligned}
&\varepsilon \le d(x_{n(p)+k}, x_{m(p)+k})\\
&= d(T(x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), T(x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1}))\\
&\le \beta\left(M((x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), (x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1}))\right)\\
&M((x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), (x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1}))\\
&< M((x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), (x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1}))\\
&= \max\{d(x_{n(p)+k-1}, x_{m(p)+k-1}), d(x_{n(p)+k}, x_{n(p)+k-1}), d(x_{m(p)+k}, x_{m(p)+k-1})\}.
\end{aligned}
$$

Using (2.10), we deduce from (2.11)

$$\lim_{p\to\infty} \beta\left(M((x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), (x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1}))\right) = 1.$$

Since $\beta \in \mathcal{F}$, we have

$$\lim_{p\to\infty} M((x_{n(p)}, x_{n(p)+1}, \dots, x_{n(p)+k-1}), (x_{m(p)}, x_{m(p)+1}, \dots, x_{m(p)+k-1})) = 0,$$

which is a contradiction with respect to (2.10). Thus $\{x_{n+k}\}$ is Cauchy in (X, d).
Step 3: Now, by using the completeness property of X, there exists a point u in X such that

$$\lim_{n\to\infty} x_{n+k} = u. \tag{2.12}$$

Assume that $u \neq T(u, u, \dots, u)$. We have

$$
\begin{aligned}
&M((x_n, x_{n+1}, \dots, x_{n+k-1}), (u, u, \dots, u))\\
&\max\{d(x_{n+k-1}, u), d(x_{n+k-1}, T(x_n, x_{n+1}, \dots, x_{n+k-1}), d(u, T(u, u, \dots, u))\}\\
&= d(u, x_{n+k}) + \max\{d(x_{n+k-1}, u), d(x_{n+k-1}, x_{n+k}), d(u, T(u, u, \dots, u))\}
\end{aligned}
$$

From (2.4) and (2.12),

$$\lim_{n\to\infty} M((x_n, x_{n+1}, \dots, x_{n+k-1}), (u, u, \dots, u)) = d(u, T(u, u, \dots, u)). \tag{2.13}$$

On the other hand, by (2.1)

$$\begin{aligned}
& d(u, T(u,u,\dots,u)) \leq d(u, x_{n+k}) + d(x_{n+k}, T(u,u,\dots,u)) \\
& = d(u, x_{n+k}) + d(T(x_n, x_{n+1}, \dots, x_{n+k-1}), T(u,u,\dots,u)) \\
& \leq d(u, x_{n+k}) + \beta\left(M((x_n, x_{n+1}, \dots, x_{n+k-1}), (u,u,\dots,u))\right) \\
& .M((x_n, x_{n+1}, \dots, x_{n+k-1}), (u,u,\dots,u)) \\
& < d(u, x_{n+k}) + M((x_n, x_{n+1}, \dots, x_{n+k-1}), (u,u,\dots,u)).
\end{aligned} \tag{2.14}$$

Using (2.13) in (2.14), we obtain

$$\lim_{n\to\infty} \beta\left(M((x_n, x_{n+1}, \dots, x_{n+k-1}), (u,u,\dots,u))\right) = 1,$$

that is,

$$\lim_{n\to\infty} M((x_n, x_{n+1}, \dots, x_{n+k-1}), (u,u,\dots,u)) = 0.$$

It is a contradiction with respect to (2.13). Thus, $d(u, T(u,u,\dots,u)) = 0$. This completes the proof. $\square$

Remark 2.1. Taking $k = 1$ in Theorem 2.1, we get a generalization of Theorem 1.1. Our main result is then a generalization and an extension of the Geraghty theorem to k-dimension.

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