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m-WEAK AMENABILITY OF $(2n)$TH DUALS OF BANACH ALGEBRAS

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Abstract. Let A be a Banach algebra such that its $(2n)$th dual for some $(n \geq 1)$ with first Arens product is $m-$weakly amenable for some $(m > 2n)$. We introduce some conditions by which if m is odd [even], then A is weakly [$2-$weakly] amenable.
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1. Introduction and Preliminaries

Let X be a normed space and $X^{'}$ be the topological dual space of X; the value of $f \in X^{'}$ at $x \in X$ is denoted by $\langle f, x\rangle$. By writing $(X^{'})^{'} = X^{''}$ we regard X as a subspace of $X^{''}$ by means of the natural mapping $i : X \to X^{''}(x \longmapsto \widehat{x})$ where $\langle \widehat{x}, f\rangle = \langle f, x\rangle (f \in X^{'})$. Also we denote the nth dual of X by $X^{(n)}$. The weak topology on X is denoted by $w = \sigma(X, X^{'})$ and weak$^*-$topology on $X^{'}$ is denoted by $w^* = \sigma(X^{'}, X)$.
Now let X, Y and Z be normed spaces and $f : X \times Y \to Z$ be a continuous bilinear map. Arens in [2] offers two extensions f^{***} and f^{t***t} of f from $X^{''} \times Y^{''}$ to $Z^{''}$ as following

$$(1) f^* : Z^{'} \times X \longrightarrow Y^{'} \quad \left(\left\langle f^*(z^{'}, x), y\right\rangle = \left\langle z^{'}, f(x, y)\right\rangle\right),$$

$$(2) f^{**} : Y^{''} \times Z^{'} \longrightarrow X^{'} \left(\left\langle f^{**}(y^{''}, z^{'}), x\right\rangle = \left\langle y^{''}, f^*(z^{'}, x)\right\rangle\right),$$

$$(3) f^{***} : X^{''} \times Y^{''} \longrightarrow Z^{''} \left(\left\langle f^{***}(x^{''}, y^{''}), z^{'}\right\rangle = \left\langle x^{''}, f^{**}(y^{''}, z^{'})\right\rangle\right).$$

The mapping f^{***} is the unique extension of f such that $x^{''} \longmapsto f^{***}(x^{''}, y^{''})$ from $X^{''}$ into $Z^{''}$ is $w^* - w^*-$continuous for every $y^{''} \in Y^{''}$. Let now $f^t : Y \times X \to Z$ be the transpose of f defined by $f^t(y, x) = f(x, y)$ for $x \in X$ and $y \in Y$. We can extend f^t as above to f^{t***} and then we have the mapping $f^{t***t} : X^{''} \times Y^{''} \longrightarrow Z^{''}$.

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If $f^{***} = f^{t***t}$ then f is called *Arens regular*. The mapping $y^{''} \longmapsto f^{t***t}(x^{''}, y^{''})$ from $Y^{''}$ into $Z^{''}$ is $w^* - w^*-$continuous for every $x^{''} \in X^{''}$. Arens regularity of f is equivalent to the following equality

$$\lim_i \lim_j \left\langle z^{'}, f(x_i, y_i) \right\rangle = \lim_j \lim_i \left\langle z^{'}, f(x_i, y_i) \right\rangle,$$

whenever both limits exist for any $z^{'} \in Z^{'}$ and all bounded nets (x_i) and (y_j) that w^*-converges to $x^{''} \in X^{''}$ and $y^{''} \in Y^{''}$, respectively.
Throughout this paper A is a Banach algebra. This algebra is called *Arens regular* if its multiplication as a bilinear map $\pi : A \times A \to A(\pi(a, b) = ab)$ is Arens regular. We shall frequently use Goldstine's theorem: for each $a^{''} \in A^{''}$, there is a net (a_i) in A such that $a^{''} = w^* - \lim\limits_i \widehat{a}_i$. Now let $a^{''} = w^* - \lim\limits_i \widehat{a}_i$ and $b^{''} = w^* - \lim\limits_j \widehat{b}_j$ be elements of $A^{''}$. The first and second *Arens products* on $A^{''}$ are denoted by symbols $\Box$ and $\Diamond$ respectively and defined by

$$a^{''} \Box b^{''} = \pi^{***}(a^{''}, b^{''}) \quad , \quad a^{''} \Diamond b^{''} = \pi^{t***t}(a^{''}, b^{''}).$$

It is easy to show that

$$a^{''} \Box b^{''} = w^* - \lim_i w^* - \lim_j \widehat{a_i b_j} \quad , \quad a^{''} \Diamond b^{''} = w^* - \lim_j w^* - \lim_i \widehat{a_i b_j}.$$

On the other hand, we can define above Arens products in stages as follows. Let $a, b \in A$, $f \in A^{'}$ and $F, G \in A^{''}$.

(1) Define $f.a$ in $A^{'}$ by $\langle f.a, b \rangle = \langle f, ab \rangle$, and $a.f$ in $A^{'}$ by $\langle a.f, b \rangle = \langle f, ba \rangle$.

(2) Define $F.f$ in $A^{'}$ by $\langle F.f, a \rangle = \langle F, f.a \rangle$, and $f.F$ in $A^{'}$ by $\langle f.F, a \rangle = \langle F, a.f \rangle$.

(3) Define $F \Box G$ in $A^{''}$ by $\langle F \Box G, f \rangle = \langle F, G.f \rangle$, and $F \Diamond G$ in $A^{''}$ by $\langle F \Diamond G, f \rangle = \langle G, f.F \rangle$.

Then $(A^{''}, \Box)$ and $(A^{''}, \Diamond)$ are Banach algebras, see [2, 7] for further details.

Now let E be a Banach $A-$module, then $E^{'}$ is a Banach $A-$module under actions

$$\langle a.f, x \rangle = \langle f, xa \rangle, \langle f.a, x \rangle = \langle f, ax \rangle \qquad (a \in A, x \in E, f \in E^{'}), \tag{1.1}$$

and $E^{''}$ is a Banach $(A^{''}, \Box)-$module under actions

$$F \bullet \Lambda = w^* - \lim_i w^* - \lim_j \widehat{a_i x_j} \quad , \quad \Lambda \bullet F = w^* - \lim_j w^* - \lim_i \widehat{x_j a_i}\ , \tag{1.2}$$

where $F = w^* - \lim\limits_i \hat{a}_i$ and $\Lambda = w^* - \lim\limits_j \hat{x}_j$ such that (a_i) and (x_j) are bounded nets in A and E, respectively.
For a Banach $A-$module E, the continuous linear map $D : A \to E$ is called a *derivation* if

$$D(ab) = aD(b) + D(a)b \qquad (a, b \in A).$$

For $x \in E$ the derivation $\delta_x : A \to E$ given by $\delta_x(a) = ax - xa$ is called *inner derivation*. The Banach algebra A is called *amenable* if every derivation $D : A \to E'$ is inner, for each Banach A-module E, [12]. If every derivation $D : A \to A'$ $[D : A \to A^{(n)}\ ,\ n \in N]$ is inner, A is called *weakly amenable [n-weakly amenable]*, see also [3, 6] for details.

Proposition 1.1. *Let A be a Banach algebra and E be a Banach $A-$module and $D : A \to E$ is a continuous derivation, then $D'' : (A'', \Box) \to E''$ is a continuous derivation, where $E^{''}$ is considered as a Banach $A^{''}-$module in accordance to formula (1.2). ([7], theorem 2.7.17).*

Proposition 1.2. *Let A be a Banach algebra, and let $n \in \mathbb{N}$. If A is $(n+2)-$weakly amenable, then A is n-weakly amenable [6].*

It was shown in [4, 11] that the $n-$weak amenability of $A^{''}$ implies the $n-$weak amenability of A. In [13] it was shown that if the Banach algebra A is complete Arens regular and every derivation $D : A \to A'$ is weakly compact, the weak amenability of $A^{(2n)}$ for some $n \geq 1$ implies of A. The authors in [5, 10] determined the conditions that the $3-$weak amenability of A'' implies the $3-$weak amenability of A, and the $3-$weak amenability of $A^{(2n)}$ for some $(n \geq 1)$ implies the $3-$weak amenability of A.
In this paper we always use the first Arens product $\Box$ on Banach algebra $A^{(2n)}(n \geq 1)$. First, we introduce the following important notation.

If $A^{(3)}$ is considered as a dual space of $A^{''}$, we will use the symbol $A^{(3)} = (A^{''})^{'}$ and the formula (1.1) for $A^{''}-$module actions on $A^{(3)}$. On the other hand, the symbol $A^{(3)} = (A^{'})^{''}$ shows $A^{(3)}$ as the second dual of $A^{'}$, and we will use the formula (1.2) for $A^{''}-$module actions on $A^{(3)}$.

In Section 2 we investigate

- $\triangleright$ two $A^{''}-$module actions on $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$,
- $\triangleright$ two $A^{(4)}-$module actions on $A^{(5)} = ((A^{'})^{''})^{''}$ and $A^{(5)} = ((A^{''})^{''})^{'}$,

$\vdots$

- $\triangleright$ two $A^{(2n)}-$module actions on $A^{(2n+1)} = (((A^{'})^{''})^{''}\cdots)^{''}$ and $A^{(2n+1)} = ((((A^{''})^{''})\cdots)^{''})^{'}$,

and also in Section 3 we investigate

- $\triangleright$ two $A^{''}-$module actions on $A^{(4)} = ((A^{'})^{'})^{''}$ and $A^{(4)} = ((A^{''})^{'})^{'}$,
- $\triangleright$ two $A^{(4)}-$module actions on $A^{(6)} = (((A^{'})^{'})^{''})^{''}$ and $A^{(6)} = ((A^{''})^{''})^{'})^{'}$,

$\vdots$

- $\triangleright$ two $A^{(2n)}-$module actions on $A^{(2n+2)} = ((((A^{'})^{'})^{''})^{''}\cdots)^{''}$ and $A^{(2n+2)} = (((((A^{''})^{''})\cdots)^{''})^{'})^{'}$.

In these sections we shall frequently use the formulas (1.1) and (1.2), and the induction process. In each case we will find conditions to make two different actions equal. These are generalizations of the methods in [9]. In Section 4 we consider continuous derivations $D : A \to A^{'}$ and $D : A \to A^{''}$. This section is about pulling the inner-ness of $(2n)-$th duals of D down to the inner-ness of D. In our main results in Section 5 we show, using the conditions obtained from previous sections, that $m-$weak amenability of $A^{(2n)}$ for some $n \geq 1$ and $m > 2n$ implies weak or 2−weak amenability of A.

2. $A^{(2n)}-$Module actions on $A^{(2n+1)}$

First, for $n = 1$, we consider two $A''-$module actions on $A^{(3)}$ when $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$. Let $a^{(3)} = w^* - \lim_\alpha \widehat{a^{'}_\alpha}$, $a^{''} = w^* - \lim_\beta \widehat{a_\beta}$ and $b^{''} = w^* - \lim_i \widehat{b_i}$ in which $(a^{'}_\alpha)$ is a bounded net in $A^{'}$ and (a_β) and (b_i) are bounded nets in A. For the left $A''-$module action on $A^{(3)} = (A^{'})^{''}$ we can write

$$\langle a^{''} \bullet a^{(3)}, b^{''}\rangle = \lim_\beta \lim_\alpha \langle b^{''}, a_\beta . a^{'}_\alpha\rangle = \lim_\beta \lim_\alpha \lim_i \langle a^{'}_\alpha, b_i a_\beta\rangle, \tag{2.1}$$

and for the left $A''-$module action on $A^{(3)} = (A^{''})^{'}$ as dual of $A^{''}$ we can write

$$\langle a^{''}.a^{(3)}, b^{''}\rangle = \langle a^{(3)}, b^{''}\Box a^{''}\rangle = \lim_\alpha \lim_i \lim_\beta \langle a^{'}_\alpha, b_i a_\beta\rangle \tag{2.2}$$

This shows that two left $A''-$module actions on $A^{(3)} = (A^{''})^{'}$ and $A^{(3)} = (A^{'})^{''}$ are not equal. But two right $A''-$module actions $a^{(3)} \bullet a^{''}$ and $a^{(3)} \cdot a^{''}$ are equal, because they are obtained from $\pi^{*(***)}$ and $\pi^{(***)*}$, which obviously are equal.

Proposition 2.1. *Let A be a Banach algebra with the following conditions:*

(i) A is Arens regular,

(ii) the map $A \times A^{'} \to A^{'}$ $\Big((a, a^{'}) \longmapsto a.a^{'}\Big)$ is Arens regular.

Then two A''-module actions on $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$ coincide.

Proof. It is enough to prove that the left module actions in (2.1) and (2.2) coincide. We begin with the equation (2.1)

$$
\begin{aligned}
\langle a^{''} \bullet a^{(3)}, b^{''} \rangle &= \lim_{\beta} \lim_{\alpha} \langle b^{''}, a_{\beta}.a_{\alpha}^{'} \rangle \\
&= \lim_{\alpha} \lim_{\beta} \langle b^{''}, a_{\beta}.a_{\alpha}^{'} \rangle \qquad \text{(by (ii))} \\
&= \lim_{\alpha} \lim_{\beta} \lim_{i} \langle a_{\beta}.a_{\alpha}^{'}, b_i \rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{i} \langle a_{\alpha}^{'}, b_i a_{\beta} \rangle \\
&= \lim_{\alpha} \lim_{i} \lim_{\beta} \langle a_{\alpha}^{'}, b_i a_{\beta} \rangle \qquad \text{(by (i))}.
\end{aligned}
$$

This proves the equality of (2.1) and (2.2). $\square$

Now for $n = 2$ we consider two $A^{(4)}-$module actions on $A^{(5)}$ when $A^{(5)} = ((A^{'})^{''})^{''}$ and $A^{(5)} = ((A^{''})^{''})^{'}$. Let $a^{(5)} = w^* - \lim_{\alpha} \widehat{a_{\alpha}^{(3)}}$, $a^{(4)} = w^* - \lim_{\beta} \widehat{a_{\beta}^{''}}$ and $b^{(4)} = w^* - \lim_{i} \widehat{b_i^{''}}$ such that $(a_{\alpha}^{(3)})$ is a bounded net in $A^{(3)}$ and $(a_{\beta}^{''}), (b_i^{''})$ are bounded nets in $A^{''}$. For the left $A^{(4)}-$module action on $A^{(5)} = ((A^{'})^{''})^{''}$ we have

$$
\langle a^{(4)} \bullet a^{(5)}, b^{(4)} \rangle = \lim_{\beta} \lim_{\alpha} \langle b^{(4)}, a_{\beta}^{''} \bullet a_{\alpha}^{(3)} \rangle = \lim_{\beta} \lim_{\alpha} \lim_{i} \langle a_{\beta}^{''} \bullet a_{\alpha}^{(3)}, b_i^{''} \rangle, \tag{2.3}
$$

and for the left $A^{(4)}-$module action on $A^{(5)} = ((A^{''})^{''})^{'}$ we have

$$
\langle a^{(4)}.a^{(5)}, b^{(4)} \rangle = \langle a^{(5)}, b^{(4)} \Box a^{(4)} \rangle = \lim_{\alpha} \lim_{i} \lim_{\beta} \langle a_{\alpha}^{(3)}, b_i^{''} \Box a_{\beta}^{''} \rangle. \tag{2.4}
$$

But two right $A^{(4)}-$module actions $a^{(5)} \bullet a^{(4)}$ and $a^{(5)} \cdot a^{(4)}$ are equal. To prove the equality of the left $A^{(4)}-$module actions on $A^{(5)}$, we need the equality of two left $A^{''}-$module actions on $A^{(3)} = (A^{''})^{'}$ and $A^{(3)} = (A^{'})^{''}$ by the following lemma, whose proof is straightforward.

Lemma 2.1. *Let A be a Banach algebra with the following conditions*

(i) $A^{''}$ is Arens regular,

(ii) the map $A^{''} \times A^{'''} \to A^{'''}$ $\Big((a^{''}, a^{(3)}) \longmapsto a^{''}.a^{(3)}\Big)$ is Arens regular.

Then the conditions of the proposition 2.1 hold.

Proposition 2.2. *Let A be a Banach algebra with the conditions of Lemma 2.1, then two $A^{(4)}-$module actions on $A^{(5)} = ((A^{'})^{''})^{''}$ and $A^{(5)} = ((A^{''})^{''})^{'}$ coincide.*

Proof. By Lemma 2.1, two left $A''-$module actions on $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$ are equal. We begin with the equality (2.3)

$$
\begin{aligned}
\langle a^{(4)} \bullet a^{(5)}, b^{(4)} \rangle &= \lim_{\beta} \lim_{\alpha} \langle b^{(4)}, a_{\beta}^{''} \bullet a_{\alpha}^{(3)} \rangle \\
&= \lim_{\alpha} \lim_{\beta} \langle b^{(4)}, a_{\beta}^{''}.a_{\alpha}^{(3)} \rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{i} \langle a_{\beta}^{''}.a_{\alpha}^{(3)}, b_i^{''} \rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{i} \langle a_{\alpha}^{(3)}, b_i^{''} \Box a_{\beta}^{''} \rangle \\
&= \lim_{\alpha} \lim_{i} \lim_{\beta} \langle a_{\alpha}^{(3)}, b_i^{''} \Box a_{\beta}^{''} \rangle.
\end{aligned}
$$

This proves the equality of (2.3) and (2.4). $\square$

We can extend our results to each n, in the following proposition.

Proposition 2.3. *Let A be a Banach algebra with the following conditions for some $n \geq 1$*

(i) *A^{2n-2} is Arens regular,*

(ii) *the map $A^{(2n-2)} \times A^{(2n-1)} \to A^{(2n-1)}$ $((a, f) \longmapsto a.f)$ is Arens regular.*

Then two $A^{(2n)}-$module actions on $A^{(2n+1)} = ((((A^{''})^{''})\cdots)^{''})^{'}$ and $A^{(2n+1)} = ((((A^{'})^{''})\cdots)^{''})^{''}$ coincide.

3. $A^{(2n)}-$Module actions on $A^{(2n+2)}$

Our methods in this section are similar to those in Section 2, so we just mention our conclusions very briefly.

Proposition 3.1. *Let A be a Banach algebra with the following conditions*

(i) *$A^{''}$ is Arens regular,*

(ii) *the maps $A \times A^{'} \to A^{'}$ $((a, f) \longmapsto a.f)$ and $A^{'} \times A \to A^{'}$ $((f, a) \longmapsto f \cdot a)$ are Arens regular.*

Then two A''-module actions on $A^{(4)} = ((A^{'})^{'})^{''}$ and $A^{(4)} = ((A^{''})^{'})^{'}$ coincide.

To extend our results to $A^{(6)}$ we need the following lemma that is similar to Lemma 2.1.

Lemma 3.1. *Let A be a Banach algebra with the following conditions*

(i) *$A^{(4)}$ is Arens regular,*

(ii) *the maps $A^{''} \times A^{'''} \to A^{'''}$ $((F, \Lambda) \longmapsto F.\Lambda)$ and $A^{'''} \times A^{''} \to A^{'''}$ $((\Lambda, F) \longmapsto \Lambda.F)$ are Arens regular.*

Then the conditions of the proposition 3.1 hold.

Proposition 3.2. *Let A be a Banach algebra with the conditions of Lemma 3.1, then two $A^{(4)}$-module actions on $A^{(6)} = (((A^{'})^{'})^{''})^{''}$ and $A^{(6)} = (((A^{''})^{''})^{'})^{'}$ coincide.*

Similar to the proposition 2.3 we have the following extension.

Proposition 3.3. *Let A be a Banach algebra with the following conditions for some $n \geq 1$*

(i) $A^{(2n)}$ *is Arens regular,*

(ii) the maps $\big(A^{(2n-2)} \times A^{(2n-1)} \to A^{(2n-1)}(f,\Lambda) \longmapsto f.\Lambda\big)$ *and* $\big(A^{(2n-1)} \times A^{(2n-2)} \to A^{(2n-1)}(\Lambda, f) \longmapsto \Lambda.f\big)$ *are Arens regular.*

Then two $A^{(2n)}-$*module actions on* $A^{(2n+2)} = (((((A^{'})^{'})^{''}\cdots)^{''})^{''}$ *and* $A^{(2n+2)} = (((((A^{''})^{''})\cdots)^{''})^{'})^{'}$ *coincide.*

Remark 3.1. There are many other module actions in sections 2 and 3 that we do not need to mention. We just introduce the module actions that we will apply in the next sections.

4. Duals of derivations $D : A \to A^{'}$ and $D : A \to A^{''}$

We consider the following duals of the continuous derivation $D : A \to A^{'}$ as in the proposition 1.1

$$
\begin{array}{rcl}
D^{''} & : & A^{''} \longrightarrow A^{(3)} = (A^{'})^{''} \\
D^{(4)} & : & A^{(4)} = (A^{''})^{''} \longrightarrow A^{(5)} = ((A^{'})^{''})^{''} \\
\vdots & & \\
D^{(2n)} & : & A^{(2n)} = ((A^{''})^{''}\cdots)^{''} \longrightarrow A^{(2n+1)} = (((A^{'})^{''})^{''}\cdots)^{''},
\end{array}
$$

and the following duals of the continuous derivation $D : A \to A^{''} = (A^{'})^{'}$

$$
\begin{array}{rcl}
D^{''} & : & A^{''} \longrightarrow A^{(4)} = ((A^{'})^{'})^{''} \\
D^{(4)} & : & A^{(4)} = (A^{''})^{''} \longrightarrow A^{(6)} = (((A^{'})^{'})^{''})^{''} \\
\vdots & & \\
D^{(2n)} & : & A^{(2n)} = ((A^{''})^{''}\cdots)^{''} \longrightarrow A^{(2n+2)} = (((A^{'})^{'})^{''}\cdots)^{''}.
\end{array}
$$

We recall that the above $D^{''}, D^{(4)}, \cdots, D^{(2n)}$ are also continuous derivations.

Lemma 4.1. *Let A be a Banach algebra with the hypothesis of the proposition 2.1. If the second dual $D^{''}$ of the continuous derivation $D : A \to A^{'}$ is inner, then D is inner.*

Proof. Since $D^{''} : A^{''} \longrightarrow A^{(3)} = (A^{'})^{''}$ is inner, there is $a^{(3)} \in A^{(3)}$ such that for every $a^{''} \in A^{''}$ we have

$$D^{''}(a^{''}) = a^{''} \bullet a^{(3)} - a^{(3)} \bullet a^{''},$$

Now let $a^{'} =: i^*(a^{(3)})$, where $i : A \longrightarrow A^{''}$ is the natural map and so $i^* : (A^{''})^{'} = A^{(3)} \longrightarrow A^{'}$. Then for each $a, b \in A$ we can write

$$\begin{aligned}
\langle D(a), b\rangle &= \langle D^{''}(\widehat{a}), \widehat{b}\rangle \\
&= \langle \widehat{a} \bullet a^{(3)} - a^{(3)} \bullet \widehat{a}, \widehat{b}\rangle \\
&= \langle a^{(3)}, \widehat{b}\Box\widehat{a} - \widehat{a}\Box\widehat{b}\rangle \qquad \text{(by proposition 2.1)} \\
&= \langle a^{(3)}, \widehat{ba - ab}\rangle \\
&= \langle i^*(a^{(3)}), ba - ab\rangle \\
&= \langle a^{'}, ba - ab\rangle \\
&= \langle a.a^{'} - a^{'}.a, b\rangle,
\end{aligned}$$

hence $D(a) = a.a^{'} - a^{'}.a$. $\square$

Using the reasoning similar to that in the proof of the previous lemma we have the next lemmas.

Lemma 4.2. *Let A be a Banach algebra with hypothesis of the proposition 2.3. If $(2n)-th$ dual $D^{(2n)}$ of the continuous derivation $D : A \to A^{'}$ is inner for some $n \geq 1$, then $D^{(2n-2)}, \cdots, D^{''}$ and D are inner.*

Lemma 4.3. *Let A be a Banach algebra with the hypothesis of the proposition 3.3. If $(2n)-th$ dual $D^{(2n)}$ of the continuous derivation $D : A \to A^{''}$ is inner for some $n \geq 1$, then $D^{(2n-2)}, \cdots, D^{''}$ and D are inner.*

5. Main results

The results of this section are immediate consequences of the previous sections, and so the proofs will be very short.

Proposition 5.1. *Let A be a Banach algebra with the hypothesis of the proposition 2.1. If $A^{''}$ is weakly amenable, then A is weakly amenable.*

Proof. Suppose that $D : A \to A^{'}$ is a continuous derivation. Then $D^{''} : A^{''} \longrightarrow A^{(3)} = (A^{'})^{''}$ is a continuous derivation by the proposition 1.1. But two $A^{''}-$module actions on $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$ are equal by the proposition 2.1, hence $D^{''} : A^{''} \longrightarrow A^{(3)} = (A^{''})^{'}$ is also a continuous derivation in which $A^{(3)} = (A^{''})^{'}$ is considered a dual of $A^{''}$. Since $A^{''}$ is weakly amenable, then $D^{''}$ is inner. Therefore D is inner by Lemma 4.1. This completes the proof. $\square$

Using the same reasoning as in the proof of the previous proposition we have the next results.

Proposition 5.2. *Let A be a Banach algebra with the conditions in the proposition 2.3 for some $n \geq 1$. If $A^{(2n)}$ is weakly amenable, then A is weakly amenable.*

Proof. This is a consequence of Lemma 4.2. $\square$

Proposition 5.3. *Let A be a Banach algebra with the conditions of the proposition 3.3 for some $n \geq 1$. If $A^{(2n)}$ is 2-weakly amenable, then A is 2-weakly amenable.*

Proof. This is a consequence of Lemma 4.3. $\square$

Finally we obtain the following general results.

Corollary 5.1. *Let $n \geq 1, m > 2n$ and suppose that A is a Banach algebra such that the conditions of the preposition 2.3 hold for n. If $A^{(2n)}$ is $m-$weakly amenable and m is odd, then A is weakly amenable.*

Proof. $A^{(2n)}$ is weakly amenable by the proposition 1.2, and hence A is weakly amenable by the proposition 5.2. $\square$

Corollary 5.2. *Let $n \geq 1, m > 2n$ and suppose that A be a Banach algebra such that the conditions of the preposition 3.3 hold for n. If $A^{(2n)}$ is $m-$weakly amenable and m is even, then A is 2-weakly amenable.*

Proof. $A^{(2n)}$ is $2-$weakly amenable by the proposition 1.2, and hence A is 2-weakly amenable by the proposition 5.3. $\square$

Example 5.1. Take a non-reflexive complex Banach space A and a bounded linear map $\varphi : A \longrightarrow \mathbb{C}$. One can define a multiplication on A by

$$ab =: \langle \varphi, b\rangle a \ , (a, b \in A).$$

This makes A a Banach algebra which is called *ideally factored algebra associated to φ* and sometimes it is denoted by A_φ, [1]. One can write for $a, b \in A$

$$\varphi(ab) = \varphi(\langle \varphi, b\rangle a) = \langle \varphi, a\rangle\langle \varphi, b\rangle = \varphi(ba),$$

this shows that φ is multiplicative. It is easy to conclude the following equations

$$\begin{array}{rcl} a^{'}.a & = & \langle a^{'}, a\rangle\varphi \\ a.a^{'} & = & \langle \varphi, a\rangle a^{'} \\ a^{''}\square b^{''} & = & a^{''}\Diamond b^{''} = \langle b^{''}, \varphi\rangle a^{''} \\ a^{'''}.a^{''} & = & \langle a^{'''}, a^{''}\rangle\widehat{\varphi} \\ a^{''}.a^{'''} & = & \langle a^{''}, \varphi\rangle a^{'''}, \end{array}$$

whenever $a \in A, a^{'} \in A^{'}, a^{''}, b^{''} \in A^{''}$ and $a^{'''} \in A^{'''}$. Now for bounded nets (a_i) and $(a_j^{'})$ in A and $A^{'}$, respectively, we have

$$\begin{aligned} w^* - \lim_j w^* - \lim_i \widehat{a_i a_j^{'}} &= w^* - \lim_j w^* - \lim_i \langle \varphi, a_i\rangle \widehat{a_j^{'}} \\ &= \lim_i \langle \varphi, a_i\rangle w^* - \lim_j \widehat{a_j^{'}}. \end{aligned}$$

This proves Arens regularity of the map $A \times A^{'} \to A^{'} \Big((a, a^{'}) \longmapsto a.a^{'}\Big)$. Since A is not reflexive, there exist bounded nets (a_i) and $(a^{'}_j)$ in A and $A^{'}$, respectively such that $\lim_i \lim_j \langle a^{'}_j, a_i\rangle \neq \lim_j \lim_i \langle a^{'}_j, a_i\rangle$, and hence the map $A^{'} \times A \to A^{'} \Big((a^{'}, a) \longmapsto a^{'}.a\Big)$ is not Arens regular, because

$$\begin{aligned} w^* - \lim_j w^* - \lim_i \widehat{a^{'}_j a_i} &= w^* - \lim_j w^* - \lim_i \langle a^{'}_j, a_i\rangle \varphi \\ &\neq w^* - \lim_i w^* - \lim_j \langle a^{'}_j, a_i\rangle \varphi. \end{aligned}$$

By using a similar reasoning we conclude that the map $A^{''} \times A^{'''} \to A^{'''} \Big((a^{''}, a^{'''}) \longmapsto a^{''}.a^{'''}\Big)$ is Arens regular, but the map $A^{'''} \times A^{''} \to A^{'''} \Big((a^{'''}, a^{''}) \longmapsto a^{'''}.a^{''}\Big)$ is not Arens regular. It is obvious that the algebras A and $A^{(2n)}$ for all $n \geq 1$ are Arens regular. In fact we have $(A_\varphi)^{''} = (A^{''})_\varphi$. Finally, all the conditions of propositions in section 2 hold , but the conditions of section 3 hold in commutative case.

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