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NEW RESULTS OF ESSENTIAL APPROXIMATE AND DEFECT POINT SPECTRUM BY USING THE GAP CONVERGENCE

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Abstract. This paper focuses on exploring the relationship between the essential approximate point spectrum (and the essential defect spectrum) of a sequence of closed linear operators $(T_n)_{n\in\mathbb{N}}$ acting on a Banach space X, and the corresponding spectra of a linear operator T on X. We examine this relationship under both generalized convergence and compact convergence conditions for the sequence $(T_n)_{n\in\mathbb{N}}$ converging to T.
Keywords: essential approximate point spectrum, essential defect spectrum, convergence by the gap, convergence compactly.

1. Introduction

Let X be Banach space. By an operator T on X we mean a linear operator with domain $\mathcal{D}(T) \subset X$, and a range $\mathrm{R}(T) \subset X$. $N(T)$ denote the null space of T, the graph of T is the set defined by $G(T) := \{(x, Tx) \in X \times X, \text{for all } x \in \mathcal{D}(T)\}$. T is said to be closed if it's graph $G(T)$ is closed in the product space $X \times X$. T is said to be compact if, for every M a bounded subset of $\mathcal{D}(T)$, $T(M) \subset \mathrm{R}(T)$ is relatively compact, so that $\overline{T(M)}$ is compact. We denote by $\mathcal{C}(X)$ the set of all closed,

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densely defined linear operators on X, and let $\mathcal{L}(X)$ (respectively, $\mathcal{K}(X)$) denote the Banach algebra of all bounded linear operators (respectively, the ideal of all compact operators) on X. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $\mathrm{R}(T)$ in X. For $T \in \mathcal{C}(X)$, we let $\sigma(T)$, $\rho(T)$ respectively the spectrum, and the resolvent set of T. The reduced minimum modulus $\gamma(T)$ of T is defined by $\gamma(T) := \inf\Big\{\|Tx\| : dist(x, N(T) = 1,\ x \in D(T)\Big\}$, we set $\gamma(T) = \infty$ if $T = 0$. A useful classes of linear operators which have extensive application in spectrum theory are those of:
The set of upper semi-Fredholm operators on X is defined by

$$\Phi_+(X) := \Big\{T \in \mathcal{C}(X) :\ \alpha(T) < \infty \text{ and } \ R(T) \text{ is closed in } X\Big\}.$$

The set of lower semi-Fredholm operators on X is defined by

$$\Phi_-(X) := \Big\{T \in \mathcal{C}(X) :\ \beta(T) < \infty \text{ and } R(T) \text{ is closed in } X\Big\}.$$

The set of semi-Fredholm operators on X is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X).$$

The set of Fredholm operators on X is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

For $T \in \Phi_\pm(X)$, the number $i(T) = \alpha(T) - \beta(T)$ is called the index of T.

Let $\Phi^b(X)$, $\Phi^b_+(X)$ and $\Phi^b_-(X)$ denote the set $\Phi(X) \cap \mathcal{L}(X)$, $\Phi_+(X) \cap \mathcal{L}(X)$ and $\Phi_-(X) \cap \mathcal{L}(X)$, respectively.
Moreover, the set of Frefholm perturbations on X is defined by

$$\mathcal{F}(X) := \Big\{F \in \mathcal{L}(X) : T + F \in \Phi(X); \text{whenever } T \in \Phi(X)\Big\},$$

the set of lower semi-Fredholm perturbations on X is defined by

$$\mathcal{F}_-(X) := \Big\{F \in \mathcal{L}(X) : T + F \in \Phi_-(X); \text{whenever } T \in \Phi_-(X)\Big\},$$

and the set of upper semi-Fredholm perturbations on X is given by

$$\mathcal{F}_+(X) := \Big\{F \in \mathcal{L}(X) : T + F \in \Phi_+(X); \text{whenever } T \in \Phi_+(X)\Big\}.$$

Let $\mathcal{F}^b(X)$, $\mathcal{F}^b_+(X)$ and $\mathcal{F}^b_-(X)$ denote the set $\mathcal{F}(X) \cap \mathcal{L}(X)$, $\mathcal{F}_+(X) \cap \mathcal{L}(X)$ and $\mathcal{F}_-(X) \cap \mathcal{L}(X)$, respectively.
Let we now recall definitions and notions of concepts which we are interested throughout this study.

Definition 1.1. Let X be Banach space. For each $T \in \mathcal{C}(X)$. We define

(i) The Weyl essential spectrum of the operator T by

$$\sigma_w(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma(A+K),$$

(ii) The essential approximate point spectrum of the operator T by

$$\sigma_{eap}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap}(T+K)$$

where, $\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \inf_{\|x\|=1, x \in \mathcal{D}(T)} \|\lambda - Tx\| = 0)\}$,

(iii) The essential defect spectrum of the operator T by

$$\sigma_{e\delta}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma_{\delta}(T+K)$$

where, $\sigma_{\delta}(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{is not surjective}\}$.

It's clear that, for $T \in \mathcal{C}(X)$, it holds $\sigma_w(T) := \sigma_{eap}(T) \cup \sigma_{e\delta}(T)$. The essential approximate point spectrum was introduced by V. Rakočević in [10], and the essential defect spectrum was introduced by C. Schmoeger in [14].

A characterization of the essential approximate point spectrum, and the essential defect spectrum by means of upper and lower semi-Fredholm operators is given by the following proposition

Proposition 1.1. [4, Proposition 3.1] *Let* $T \in \mathcal{C}(X)$, *then*

(i) $\lambda \notin \sigma_{eap}(T)$ *if, and only if,* $\lambda - T \in \Phi_+(X)$ *and* $i(\lambda - T) \leq 0$.

(ii) $\lambda \notin \sigma_{e\delta}(T)$ *if, and only if,* $\lambda - T \in \Phi_-(X)$ *and* $i(\lambda - T) \geq 0$.

We organize our paper as follows: The next section is a preliminaries, where we review some of the concepts and properties that concern us in our study. In section 3 we present our main results to investigate the essential approximate point spectrum and the essential defect spectrum of a sequence of linear operators T_n on Banach space X, where T_n converges in the generalized sense, and the same when T_n converges compactly.

2. Preliminaries

In this section we gather some notations and results of each of convergence in generalized sense and the convergence compactly, that we need to prove our results later.

2.1. The convergence in the generalized sense

While the distance between two bounded linear operators can be defined as the norm of their difference, the distance between to unbounded linear operators has to be measured in a different ways. One possibility is to use the gap between their graphs, which leads to the notion of convergence in the generalized sense, which essentially represents the convergence between their graphs. This concept of convergence can be found in the literature (see [5])

Definition 2.1. The gap between two linear subspaces M and N of a normed space X is defined by

$$\widehat{\delta}(M,N) := \max\Big\{\delta(M,N),\delta(N,M)\Big\}$$

where

$$\delta(M,N) := \begin{cases} \sup_{\|x\|=1} dist(x,N), & \text{if } M\neq\{0\} \\ 0, & \text{otherwise} \end{cases}.$$

From the Definition 2.1, it follows that $\delta(M,N)=0$ if, and only if, $\overline{M}\subset\overline{N}$. The set of all closed linear subspaces of X equipped with the distance $\widehat{\delta}(\cdot,\cdot)$ forms a metric space, and a sequence of closed linear subspaces M_n converges to M if $\widehat{\delta}(M_n,M)\to 0$. The gap between two closed subspaces was introduced in Hilbert space by M.G. Krein and M. A. Krasnoselkii in [6]. This notion was later extended to arbitrary Banach spaces in a paper by M. G. Krein, M. A. Krasnoselski, and D. P. Milman in [7].
If T, S $\in\mathcal{C}(X)$, their graphs $G(T),G(S)$ are closed linear subspaces in the product space X × X. Thus the distance between T and S can be measured by the "gap" between the closed linear subspaces $G(T),G(S)$.

Definition 2.2. Let X be a Banach space, and let T, S be two closed linear operators on X. Let us define

$$\delta(T,S)=\delta\big(G(T),G(S)\big) \text{ and } \widehat{\delta}(T,S)=\widehat{\delta}\big(G(T),G(S)\big).$$

$\widehat{\delta}\big(T,S\big)$ is called the gap between S and T.

The next theorem contains some basic properties of the gap between two closed linear operators

Theorem 2.1. [5, Chapter IV Section 2] *Let T and S be two closed densely defined linear operators. Then, we have:*

(i) If S and T are one-to-one, then $\delta(S,T)=\delta(S^{-1},T^{-1})$ and $\widehat{\delta}(S,T)=\widehat{\delta}(S^{-1},T^{-1})$.

(ii) Let $A\in\mathcal{L}(X)$. Then $\widehat{\delta}(A+S,A+T)\le 2(1+\|A\|^2)\widehat{\delta}(S,T)$.

(iii) Let T be Fredholm operator (respectively semi-Fredholm operator). If $\widehat{\delta}(T,S) < \gamma(T)(1+[\gamma(T)]^2)^{\frac{-1}{2}}$, *then S is Fredholm operator (respectively semi-Fredholm operator),* $\alpha(S) \leq \alpha(T)$ *and* $\beta(S) \leq \beta(T)$. *Furthermore, there exists $b > 0$ such that* $\widehat{\delta}(T,S) < b$, *which implies* $i(S) = i(T)$.

(iv) Let $T \in \mathcal{L}(X)$. *If* $S \in \mathcal{C}(X)$ *and* $\widehat{\delta}(T,S) \leq \left[1 + \|T\|^2\right]^{-\frac{1}{2}}$, *then S is bounded operator (so that $\mathcal{D}(S)$ is closed).*

A complete discussion of all the above definitions and properties may be found in T. Kato [5]. For the case of closable linear operators, the authors A. Ammar, and A. Jeribi have introduced in [1] the following definitions and results

Definition 2.3. Let S and T be two closable operators. We define the gap between T and S by $\delta(T,S) = \delta(\overline{T},\overline{S})$ and $\widehat{\delta}(T,S) = \widehat{\delta}(\overline{T},\overline{S})$.

Definition 2.4. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closable linear operators on X and let T be a closable linear operator on X. $(T_n)_{n\in\mathbb{N}}$ is said to converge in the generalized sense to T, written $T_n \stackrel{g}{\longrightarrow} T$, if $\widehat{\delta}(T_n,T)$ converges to 0 when $n \to \infty$.

It should be remarked that the notion of generalized convergence introduced above for closed and closable operators can be thought as a generalization of convergence in norm for linear operators that may be unbounded. Moreover, an important passageway between these two notions is developed in the following theorem

Theorem 2.2. [1, Theorem 2.3] *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closable linear operators on X and let T be a closable linear operator on X.*

(i) $T_n \stackrel{g}{\longrightarrow} T$, *if, and only if,* $T_n + S \stackrel{g}{\longrightarrow} T + S$, *for all* $S \in \mathcal{L}(X)$.

(ii) Let $T \in \mathcal{L}(X)$. $T_n \stackrel{g}{\longrightarrow} T$ *if, and only if,* $T_n \in \mathcal{L}(X)$ *for sufficiently larger n and T_n converges to T.*

(iii) Let $T_n \stackrel{g}{\longrightarrow} T$. *Then,* T^{-1} *exists and* $T^{-1} \in \mathcal{L}(Y)$, *if, and only if,* T_n^{-1} *exists and* $T_n^{-1} \in \mathcal{L}(X)$ *for sufficiently large n and* T_n^{-1} *converges to* T^{-1}.

.

2.2. The convergence compactly

Definition 2.5. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators mapping on X, $(T_n)_{n\in\mathbb{N}}$ is said to be converge to zero compactly, written $T_n \stackrel{c}{\longrightarrow} 0$, if for all $x \in X$, $T_n x \to 0$ and $(T_n x_n)_n$ is relatively compact for every bounded sequence $(x_n)_n \subset X$.

Theorem 2.3. [2, Theorem 4] *Let K_n be a sequence of bounded linear operators such that $K_n \xrightarrow{c} 0$, and let T be a closed linear operator. If T is a semi-Fredholm operator, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

(i) $(T + K_n)$ *is semi-Fredholm,*
(ii) $\alpha(T + K_n) < \alpha(T)$,
(iii) $\beta(T + K_n) < \beta(T)$, *and*
(iv) $i(T + K_n) = i(T)$.

Definition 2.6. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators mapping on X and let $T \in \mathcal{L}(X)$, $(T_n)_{n\in\mathbb{N}}$ is said to be converge to T compactly, written $T_n \xrightarrow{c} T$ if, and only if, $T_n - T$ converges to zero compactly.

Proposition 2.1. [1, Proposition 3.1] *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators which converges compactly to a bounded operator T. Then,*

(i) if $T_n \in \mathcal{F}^b(X)$, then $T \in \mathcal{F}^b(X)$,

(ii) if $T_n \in \mathcal{F}_+^b(X)$, then $T \in \mathcal{F}_+^b(X)$, and

(iii) if $T_n \in \mathcal{F}_-^b(X)$, then $T \in \mathcal{F}_-^b(X)$.

.

3. Main results

The first main result is embodied in the following theorem, when we discuss and study the essential approximate point spectrum and the essential defect spectrum of a sequence of closed linear operators perturbed by a bounded operator, and converges in the generalized sense to a closed linear operator in a Banach space

Theorem 3.1. *Let X be Banach space, Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closed linear operators converges in the generalized sense in $\mathcal{C}(X)$ to a closed linear operator T, and let B a bounded linear operator mapping on X such that $\rho(T + B) \neq \emptyset$. If $\lambda_0 \in \rho(T + B)$, then*

(i) There exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, we have

$$\sigma_{eap}(T_n + B - \lambda_0) \subseteq \sigma_{eap}(T + B - \lambda_0) + \mathcal{U}, \tag{3.1}$$

and

$$\sigma_{e\delta}(T_n + B - \lambda_0) \subseteq \sigma_{e\delta}(T + B - \lambda_0) + \mathcal{U}, \tag{3.2}$$

when $\mathcal{U} \subset \mathbb{C}$ an open containing 0. In particular, for all $n \geq n_0$

(3.3)
$$\delta\Big(\sigma_{eap}(T_n+B-\lambda_0),\sigma_{eap}(T+B-\lambda_0)\Big)=\delta\Big(\sigma_{e\delta}(T_n+B-\lambda_0),\sigma_{e\delta}(T+B-\lambda_0)\Big)=0.$$

(ii) There exists $\varepsilon>0$, and $n_0\in\mathbb{N}$, such that, for all $S\in B(X)$, and $\|S\|<\varepsilon$, we have
$$\sigma_{eap}(T_n+B+S-\lambda_0)\subseteq\sigma_{eap}(T+B-\lambda_0)+\mathcal{U},\ \textit{for all } n\geq n_0,$$

and
$$\sigma_{e\delta}(T_n+B+S-\lambda_0)\subseteq\sigma_{e\delta}(T+B-\lambda_0)+\mathcal{U},\ \textit{for all } n\geq n_0.$$

In particular, for all $n\geq n_0$

(3.4)
$$\delta\Big(\sigma_{eap}(T_n+B+S-\lambda_0),\sigma_{eap}(T+B-\lambda_0)\Big)=\delta\Big(\sigma_{e\delta}(T_n+B+S-\lambda_0),\sigma_{e\delta}(T+B-\lambda_0)\Big)=0.$$

Proof. For (i), before proof, we make some preliminary observations. Since $T_n\xrightarrow{g}T$, then by Theorem 2.2 (i), $(T_n+B-\lambda_0)\xrightarrow{g}(T+B-\lambda_0)$, furthermore, we have $(T+B-\lambda_0)^{-1}\in\mathcal{L}(X)$, which implies according to Theorem 2.2 (iii), that $\lambda_0\in\rho(T_n+B)$ for a sufficiently large n and $(T_n+B-\lambda_0)^{-1}$ converges to $(T+B-\lambda_0)^{-1}$. We recall that the essential approximate point spectrum of a bounded linear operator is compact, but this property is not valid for the case of unbounded operators, for this reason, using the compactness of $\sigma_{eap}(T+B-\lambda_0)^{-1}$ because $(T+B-\lambda_0)^{-1}$ is bounded, as a first step, we will prove the existence of $n_0\in\mathbb{N}$, such that for all $n\geq n_0$, we have

$$\sigma_{eap}(T_n+B-\lambda_0)^{-1}\subseteq\sigma_{eap}(T+B-\lambda_0)^{-1}+\mathcal{U}. \tag{3.5}$$

The proof by contradiction. Suppose that (3.5) does not hold. Then, by studying a subsequence (if necessary) we may assume that, for each n there exists $\lambda_n\in\sigma_{eap}(T_n+B-\lambda_0)^{-1}$ such that $\lambda_n\notin\sigma_{eap}(T+B-\lambda_0)^{-1}+\mathcal{U}$. Since (λ_n) is bounded, we may assume that $\lim\limits_{n\to+\infty}\lambda_n=\lambda$, which implies that $\lambda\notin\sigma_{eap}(T+B-\lambda_0)^{-1}+\mathcal{U}$. Since $0\in\mathcal{U}$ then we have $\lambda\notin\sigma_{eap}(T+B-\lambda_0)^{-1}$. Therefore $\lambda-(T+B-\lambda_0)^{-1}\in\Phi_+^b(X)$ and $i\big(\lambda-(T+B-\lambda_0)^{-1}\big)\leq 0$. As $\big(\lambda_n-(T_n+B-\lambda_0)^{-1}\big)$ converges to $\big(\lambda-(T+B-\lambda_0)^{-1}\big)$, we deduce that

$$\widehat{\delta}\big(\lambda_n-(T_n+B-\lambda_0)^{-1},\lambda-(T+B-\lambda_0)^{-1}\big)\to 0 \text{ when } n\to\infty.$$

Let $\delta=\gamma\big(\lambda-(T+B-\lambda_0)^{-1}\big)>0$, then there exists $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$ we have $\widehat{\delta}\big(\lambda_n-(T_n+B-\lambda_0)^{-1},\lambda-(T+B-\lambda_0)^{-1}\big)\leq\frac{\delta}{\sqrt{1+\delta^2}}$. By using

Theorem 2.1 (iv) we infer that $\lambda_n - (T_n + B - \lambda_0)^{-1} \in \Phi_+(X)$. Furthermore, there exists $b > 0$ such that

$$\widehat{\delta}\big(\lambda_n - (T_n + B - \lambda_0)^{-1}, \lambda - (T + B - \lambda_0)^{-1}\big) < b,$$

which implies $i\big(\lambda_n - (T_n + B - \lambda_0)^{-1}\big) = i\big(\lambda - (T + B - \lambda_0)^{-1}\big) \leq 0$. Then we obtain $\lambda_n \notin \sigma_{eap}\big((T_n + B - \lambda_0)^{-1}\big)$, which is a contradiction. Hence (3.5) holds. Now, we assume that $\lambda \in \sigma_{eap}(T_n + B - \lambda_0)$ then $\dfrac{1}{\lambda} \in \sigma_{eap}\big((T_n + B - \lambda_0)^{-1}\big)$. By using the inclusion (3.5), we have $\dfrac{1}{\lambda} \in \sigma_{eap}\big((T+B-\lambda_0)^{-1}\big)+\mathcal{U}$. which implies that $\dfrac{1}{\lambda} \in \sigma_{eap}\big((T+B-\lambda_0)^{-1}\big)$ because $0 \in \mathcal{U}$, now we claim that $\lambda \in \sigma_{eap}(T+B-\lambda_0)+\mathcal{U}$. In fact, let us assume that $\lambda \notin \sigma_{eap}(T + B - \lambda_0) + \mathcal{U}$. The fact that $0 \in \mathcal{U}$ implies that $\lambda \notin \sigma_{eap}(T + B - \lambda_0)$ and so, $\dfrac{1}{\lambda} \notin \sigma_{eap}\big((T_n + B - \lambda_0)^{-1}\big)$ which is a contradiction. So $\lambda \in \sigma_{eap}(T + B - \lambda_0) + \mathcal{U}$. This implies that (3.1) holds. Since $\mathcal{U}$ is an arbitrary neighborhood of 0 and by using (3.1) we get $\sigma_{eap}(T_n + B - \lambda_0) \subseteq \sigma_{eap}(T + B - \lambda_0)$, for all $n \geq n_0$, hence $\delta\Big(\sigma_{eap}(T_n + B - \lambda_0), \sigma_{eap}(T + B - \lambda_0)\Big) = \delta\Big(\overline{\sigma_{eap}(T_n + B - \lambda_0)}, \overline{\sigma_{eap}(T + B - \lambda_0)}\Big) = 0$, for all $n \geq n_0$. With the same procession as we do for (3.1), and using Proposition 1.1 (ii), the inclusion (3.2) yields. Therefore, (i) holds.

(ii) Since S is bounded, it's clear by using Theorem 2.2 (i), that the operators sequence $A_n = (T_n + B + S - \lambda_0)$ converges in the generalized sense to the operator $A = (T + B + S - \lambda_0)$, then we need, for applying (i), to prove that $\rho(T + B) \subset \rho(T+B+S)$. Let $\lambda_0 \in \rho(T+B)$, for $S \in \mathcal{L}(X)$ such that $\|S\| < \dfrac{1}{\|(T + B - \lambda_0)^{-1}\|} = \varepsilon_1$, we have $\|S(T + B - \lambda_0)^{-1}\| < 1$, which gives that $\big(I + S(T + B - \lambda_0)^{-1}\big)^{-1}$ exists and bounded, when the existence is given by the convergence of the Neumann serie $\sum_{k=0}^{\infty} \big(- S(T + B - \lambda_0)^{-1}\big)^k$, and the boundedness is immediately from the inequality

$$\|\big(I + S(T + B - \lambda_0)^{-1}\big)^{-1}\| < \frac{1}{1 - \|S\|\|(T + B - \lambda_0)^{-1}\|},$$

which implies that the operator

$$\big((T + B - \lambda_0) + S\big)^{-1} = (T + B - \lambda_0)^{-1}\big(I + S(T + B - \lambda_0)^{-1}\big)^{-1}$$

exists and bounded, then $0 \in \rho(T + B + S - \lambda_0)$. Now applying (i) to A_n and A, we deduce that there exists $n_0 \in \mathbb{N}$, such that $\sigma_{eap}(T_n + B + S - \lambda_0) \subseteq \sigma_{eap}(T + B + S - \lambda_0) + \mathcal{U}$, for all $n \geq n_0$, and $\mathcal{U} \subset \mathbb{C}$ is an open containing 0, we will prove that $\sigma_{eap}(T + B + S - \lambda_0) \subseteq \sigma_{eap}(T + B - \lambda_0)$, by contradiction. Let $\lambda \notin \sigma_{eap}(T + B - \lambda_0)$, then $\big(\lambda - (T + B - \lambda_0)\big) \in \Phi_+(X)$ and $i\big(\lambda - (T + B - \lambda_0)\big) \leq 0$. From [5, Chapter IV. Theoreme 5.22, p 236], we deduce that there exists $\varepsilon_2 > 0$ such that for $\|S\| < \varepsilon_2$, one has $\big(\lambda - (T+B+S-\lambda_0)\big) \in \phi_+(X)$ and $i\big(\lambda - (T+B+S-\lambda_0)\big) = i\big(\lambda - (T+B-\lambda_0)\big) \le 0$. This implies that $\lambda \notin \sigma_{eap}(T+B+S-\lambda_0)$. Then by transitivity

$$\sigma_{eap}(T_n+B+S-\lambda_0) \subseteq \sigma_{eap}(T+B-\lambda_0)$$

From what has been mentioned and if we take $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, then for all $\|S\| < \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that $\sigma_{eap}(T_n+B+S-\lambda_0) \subseteq \sigma_{eap}(T+B-\lambda_0) + \mathcal{U}$, for all $n \ge n_0$. With the same procedure of the previous prove, the inclusion concerned in $\sigma_{e\delta}$ yields. Since $\mathcal{U}$ is an arbitrary neighborhood of the origin then we have $\delta\Big(\sigma_{eap}(T_n+S+B-\lambda_0), \sigma_{eap}(T+B+S-\lambda_0)\Big) = 0$. Therefore, (ii) holds. □

A particular case is obtained From Theorem 3.1, if we replace B by 0, and λ_0 by 0, which requires that $0 \in \rho(T)$, then we have the following corollary

Corollary 3.1. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closed linear operators and T a closed operator such that $(T_n) \xrightarrow{g} T$, we suppose that $0 \in \rho(T)$ then,*
(i) If $\mathcal{U} \subset \mathbb{C}$ is open and $0 \in \mathcal{U}$, then there exists $n_0 \in \mathbb{N}$ such that, for every $n \ge n_0$, we have

$$\sigma_{eap}(T_n) \subseteq \sigma_{eap}(T) + \mathcal{U}. \tag{3.6}$$

and

$$\sigma_{e\delta}(T_n) \subseteq \sigma_{e\delta}(T) + \mathcal{U}. \tag{3.7}$$

In particular, for all $n \ge n_0$

$$\delta\Big(\sigma_{eap}(T_n), \sigma_{eap}(T)\Big) = \delta\Big(\sigma_{e\delta}(T_n), \sigma_{e\delta}(T)\Big) = 0, \tag{3.8}$$

In the next theorem we discuss the essential approximate point spectrum and the essential defect spectrum of a sequence of linear operators converges compactly

Theorem 3.2. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(X)$ and let T be a bounded linear operator on X.*

(i) If T_n converges to T compactly, $\mathcal{U} \subseteq \mathbb{C}$ is open and $0 \in \mathcal{U}$, then there exists $n_0 \in \mathbb{N}$ such that, for every $n \ge n_0$

$$\sigma_{eap}(T_n) \subseteq \sigma_{eap}(T) + \mathcal{U}, \tag{3.9}$$

and

$$\sigma_{e\delta}(T_n) \subseteq \sigma_{e\delta}(T) + \mathcal{U}. \tag{3.10}$$

(ii) If T_n converges to zero compactly then there exists $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$

$$\sigma_{eap}(T+T_n) \subseteq \sigma_{eap}(T), \tag{3.11}$$

and

$$\sigma_{e\delta}(T+T_n) \subseteq \sigma_{e\delta}(T). \tag{3.12}$$

Proof. (i) The proof by contradiction. Assume that the inclusion is fails. Then by passing to a subsequence (if necessary) it may be assumed that, for each n, there exists $\lambda_n \in \sigma_{eap}(T_n)$ such that $\lambda_n \notin \sigma_{eap}(T_n) + \mathcal{U}$, since λ_n is bounded, we suppose (if necessary pass to a subsequence) that $\lim\limits_{n\to+\infty} \lambda_n = \lambda$, which implies that $\lambda \notin \sigma_{eap}(T) + \mathcal{U}$. Using the fact that $0 \in \mathcal{U}$, we have $\lambda \notin \sigma_{eap}(T)$. Therefore $(\lambda - T) \in \Phi_+(X)$, and $i(\lambda - T) \leq 0$. As $(\lambda_n - T_n) - (\lambda - T) \xrightarrow{c} 0$, which implies by Theorem 2.3 (i) and (iv) that $(\lambda_n - T_n) \in \Phi_+(X)$, and $i(\lambda_n - T_n) = i(\lambda - T) \leq 0$, then $\lambda_n \notin \sigma_{eap}(T_n)$, which is a contradiction, hence the inclusion (3.9) holds. The statement for the essential defect spectrum can be proved similarly.

(ii) We have T is bounded, and $T_n \xrightarrow{c} 0$, then for $\lambda \notin \sigma_{eap}(T)$, $\lambda - T \in \Phi_+^b(X)$, and $i(\lambda - T) \leq 0$. Since $T_n \xrightarrow{c} 0$, then if we apply Theorem 2.3 (i) and (iv) we obtain that there exists $n_0 \in \mathbb{N}$, such that for all $n > n_0$, $(\lambda - T) - T_n = \big(\lambda - (T + T_n)\big) \in \Phi_+(X)$, and $i(\lambda - T) = i\big(\lambda - (T + T_n)\big) \leq 0$, which implies that $\lambda \notin \sigma_{eap}(T_n)$, then the inclusion (3.11) is valid. For the inclusion (3.12) the proof is similarly. $\square$

From the above result we deduce the following consequence concerned on a sequence of closed linear operators

Corollary 3.2. *Let T be a closed linear operator and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closed linear operators on X, such that $\rho(T_n) \cap \rho(T) \neq \phi$, let $\eta \in \rho(T_n) \cap \rho(T)$, if $(T_n - \eta)^{-1} - (T - \eta)^{-1}$ converges to zero compactly then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$*

$$\sigma_{eap}(T_n - \eta) \subseteq \sigma_{eap}(T - \eta), \tag{3.13}$$

and

$$\sigma_{e\delta}(T_n - \eta) \subseteq \sigma_{e\delta}(T - \eta). \tag{3.14}$$

Proof. If we put $K_n = (T_n - \eta)^{-1} - (T - \eta)^{-1}$, we have $K_n \xrightarrow{c} 0$, and $(T_n - \eta)^{-1} = (T - \eta)^{-1} + K_n$, by using the inclusions (3.11) and (3.12) respectively, we infer that $\sigma_{eap}\big((T_n - \eta)^{-1}\big) = \sigma_{eap}\big((T - \eta)^{-1} + K_n\big) \subseteq \sigma_{eap}\big((T - \eta)^{-1}\big)$, and $\sigma_{e\delta}\big((T_n - \eta)^{-1}\big) = \sigma_{e\delta}\big((T - \eta)^{-1} + K_n\big) \subseteq \sigma_{e\delta}\big((T - \eta)^{-1}\big)$. Then $\sigma_{eap}(T_n - \eta) \subseteq \sigma_{eap}(T - \eta)$, and $\sigma_{e\delta}(T_n - \eta) \subseteq \sigma_{e\delta}(T - \eta)$. $\square$

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