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SOME GEOMETRIC PROPERTIES OF WEIGHTED LEBESGUE SPACES $L^p_w(G)$

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Abstract. In this paper, we deal with some geometric properties of weighted Lebesgue spaces $L^p_w(G)$, where G is locally compact Abelian group and w is a Beurling weight. Also, we study the uniformly convexity of the space $L^p(G) \cap L^r(G)$ with $1 < p, r < \infty$.
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1. Introduction

Throughout this paper, G is a locally compact Abelian group and dx is a Haar measure on G. If $1 \leq p < \infty$, then $L^p(G)$ will denote the space of functions f such that $|f|^p$ is integrable [2]. A Beurling weight on G is a measurable, locally bounded function w satisfying for each $x, y \in G$ the following two properties: $w(x) \geq 1$ and $w(x+y) \leq w(x).w(y)$. By the definition of $w$it is deduced easily that wdx is a positive measure on G. We denote by $L^p_w(G),\ 1 \leq p < \infty$, the Banach spaces of equivalence classes of real valued measurable functions on G with the system of following norm

$$\|f\|_{p,w} = \left(\int\limits_G |f(x)|^p \, w(x) dx \right)^{\frac{1}{p}} < \infty.$$

The conjugate space of $L^p_w(G)$ is the $L^{p'}_{w'}(G)$, where $w^{'} = w^{1-p^{'}}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. It can be easily seen that $L^p_w(G)$ is a reflexive Banach space [3], [4], [8], [9].

A Banach space X is said to be *strictly convex* if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$, then $|(1-\lambda)x + \lambda y| < 1$ for all $\lambda \in (0,1)$.

A Banach space X is said to be *uniformly convex if for all* $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that the conditions

$$\|x\| \leq 1,\ \ \|y\| \leq 1 \text{ and } \|x-y\| \geq \varepsilon \text{ imply } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

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for all $x, y \in X$.

The number

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1, \ \ \|x-y\| \geq \varepsilon \right\}$$

is called the *modulus of convexity*. Note that if $\varepsilon_1 < \varepsilon_2$, then $\delta(\varepsilon_1) < \delta(\varepsilon_2)$ and $\delta(0) = 0$ since $x = y$ if $\varepsilon = 0$ [1],[7].

We will need some auxiliary lemmas to prove that the spaces $L_w^p(G)$ are uniformly convex whenever $1 < p < \infty$.

Let us first remind that the Minkowski inequality for the space $L_w^p(G), \ p \geq 1$; If $f, g \in L_w^p(G)$, then

$$\left(\int_G |f(x) + g(x)|^p \, w(x) dx \right)^{\frac{1}{p}} \leq \left(\int_G |f(x)|^p \, w(x) dx \right)^{\frac{1}{p}} + \left(\int_G |g(x)|^p \, w(x) dx \right)^{\frac{1}{p}}.$$

Lemma 1.1. *Let $0 < p < 1$, we have $(a+b)^p \leq a^p + b^p$ for all $a \geq 0$ and $b \geq 0$.*

Lemma 1.2. *If $p \geq 1$, then $(a+b)^p \leq 2^{p-1} (a^p + b^p)$ for all positive numbers a and b.*

2. Main Results

Theorem 2.1. *The space $L_w^p(G)$ is convex whenever $0 < p < \infty$.*

Proof. Let $f, g \in L_w^p(G)$. We need to show that $tf + (1-t)g \in L_w^p(G)$ for $0 \leq t \leq 1$. Let us consider this in two cases; $p \geq 1$ and $0 < p < 1$.

Case $p \geq 1$. By lemma 2 and the Minkowski inequality, we have

$$\begin{aligned}
\int_G |tf(x) + (1-t)g(x)|^p \, w(x)dx &= \int_G \left| (tf(x) + (1-t)g(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx \\
&= \left[\left(\int_G \left| (tf(x) + (1-t)g(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx \right)^{\frac{1}{p}} \right]^p \\
&\leq \left[\left(\int_G \left| (tf(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx \right)^{\frac{1}{p}} \right. \\
&\quad \left. + \left(\int_G \left| ((1-t)g(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx \right)^{\frac{1}{p}} \right]^p \\
&\leq 2^{p-1} \left[\int_G \left| (tf(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx \right. \\
&\quad \left. + \int_G \left| ((1-t)g(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx \right] \\
&= 2^{p-1} \left[|t|^p \int_G |f(x)|^p \, w(x)dx \right. \\
&\quad \left. + |1-t|^p \int_G |g(x)|^p \, w(x)dx \right] \\
&= 2^{p-1} \left(|t|^p \, \|f\|_{p,w}^p + |1-t|^p \, \|g\|_{p,w}^p \right) \\
&< \infty
\end{aligned}$$

which shows that $tf + (1-t)g \in L_w^p(G)$ for $p \geq 1$.

Case $0 < p < 1$. Let $f, g \in L_w^p(G)$ and $t \in [0,1]$. By lemma 1, we get

$$\begin{aligned}
\int_G |tf(x) + (1-t)g(x)|^p \, w(x)dx &= \int_G \left| (tf(x) + (1-t)g(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx \\
&\leq \int_G \left| (tf(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx + \int_G \left| ((1-t)g(x)) \, (w(x))^{\frac{1}{p}} \right|^p dx \\
&= |t|^p \, \|f\|_{p,w}^p + |1-t|^p \, \|g\|_{p,w}^p \\
&< \infty.
\end{aligned}$$

This completes the proof. $\square$

Theorem 2.2. *The space $L_w^p(G)$, $1<p<\infty$, is strictly convex.*

Proof. Let $f,g\in L_w^p(G)$ with $f\neq g$, $\|f\|_{p,w}=1, \|g\|_{p,w}=1$ and $0<t<1$. Then, by strictly convexity of $L^p(G)$ we have

$$\begin{aligned}\|(1-t)f+tg\|_{p,w} &= \left(\int\limits_G \left|((1-t)f(x)+tg(x))\,(w(x))^{\frac{1}{p}}\right|^p dx\right)^{\frac{1}{p}}\\ &= \left\|((1-t)f+tg)\,w^{\frac{1}{p}}\right\|_p\\ &< 1.\end{aligned}$$

□

Lemma 2.1. *Let $2\le p<\infty$ and $a,b\in\mathbb{R}$, then we have*

$$|a+b|^p+|a-b|^p\le 2^{p-1}\left(|a|^p+|b|^p\right).$$

[5].

Lemma 2.2. *Let $2\le p<\infty$. For any $f,g\in L_p$, we have*

$$\|f+g\|_p^p+\|f-g\|_p^p\le 2^{p-1}\left(\|f\|_p^p+\|g\|_p^p\right)$$

[6].

We will also need the following inequality.

Lemma 2.3. *For $2\le p<\infty$ and any $f,g\in L_w^p(G)$, we have*

$$\|f+g\|_{p,w}^p+\|f-g\|_{p,w}^p\le 2^{p-1}\left(\|f\|_{p,w}^p+\|g\|_{p,w}^p\right).$$

Proof. Let $f,g\in L_w^p(G)$. Then $fw^{\frac{1}{p}}, gw^{\frac{1}{p}}\in L_p$. By lemma 4, we get

$$\begin{aligned}\|f+g\|_{p,w}^p+\|f-g\|_{p,w}^p &= \left\|fw^{\frac{1}{p}}+gw^{\frac{1}{p}}\right\|_p^p+\left\|fw^{\frac{1}{p}}-gw^{\frac{1}{p}}\right\|_p^p\\ &\le 2^{p-1}\left(\left\|fw^{\frac{1}{p}}\right\|_p^p+\left\|gw^{\frac{1}{p}}\right\|_p^p\right)\\ &= 2^{p-1}\left(\|f\|_{p,w}^p+\|g\|_{p,w}^p\right).\end{aligned}$$

□

Theorem 2.3. *$L_w^p(G)$ is uniformly convex for $2\le p<\infty$.*

Proof. Let $f, g \in L^p_w(G)$ with $\|f\|_{p,w} \leq 1,\ \|g\|_{p,w} \leq 1$ and $\|f-g\|_{p,w} \geq \varepsilon$. Then, we have

$$\|f+g\|_{p,w}^p \leq 2^{p-1}\left(\|f\|_{p,w}^p + \|g\|_{p,w}^p\right) - \|f-g\|_{p,w}^p$$

which implies that

$$\begin{aligned} \|f+g\|_{p,w}^p &\leq 2^{p-1}.2 - \varepsilon^p \\ &= 2^p\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right). \end{aligned}$$

Therefore, we get

$$\left\|\frac{f+g}{2}\right\|_{p,w}^p \leq 1 - \left(\frac{\varepsilon}{2}\right)^p.$$

That is, $\delta(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}$ and this is known to be exact. □

Lemma 2.4. *(The Minkowski inequality for $p \in (0,1)$) Let $0 < p < 1$ and let f and g be positive functions in $L_p(G)$, then $f+g \in L_p(G)$ and*

$$\|f+g\|_p \geq \|f\|_p + \|g\|_p.$$

Lemma 2.5. *If $1 < p < 2$ and $q = \frac{p}{p-1}$, then*

$$|a+b|^q + |a-b|^q \leq 2\left(|a|^p + |b|^p\right)^{q-1}$$

for all real numbers a and b [6].

Lemma 2.6. *Let $1 < p \leq 2$ and $q = \frac{p}{p-1}$. For any $f,\ g \in L^p(G)$, we have*

$$\|f+g\|_p^q + \|f-g\|_p^q \leq 2\left(\|f\|_p^p + \|g\|_p^p\right)^{q-1}.$$

Theorem 2.4. *Let $1 < p \leq 2$ and let $q = \frac{p}{p-1}$. For any $f,\ g \in L^p_w(G)$, we have*

$$\|f+g\|_{p,w}^q + \|f-g\|_{p,w}^q \leq 2\left(\|f\|_{p,w}^p + \|g\|_{p,w}^p\right)^{q-1}.$$

Proof. First notice that

$$\begin{aligned} \|f\|_{p,w}^q &= \left(\left(\int_G |f(x)|^p\, w(x)dx\right)^{\frac{1}{p}}\right)^q \\ &= \left(\int_G |f(x)|^p\, w(x)dx\right)^{\frac{1}{p-1}} \\ &= \left(\int_G |f(x)|^{q(p-1)}\, w(x)dx\right)^{\frac{1}{p-1}} \\ &= \left\||f|^q\right\|_{p-1,w}. \end{aligned}$$

Let $f,g\in L^p_w(G)$. By the Minkowski inequality for $0<r<1$, we have

$$\left(\int_G |F(x)+G(x)|^r\,dx\right)^{\frac{1}{r}} \geq \left(\int_G |F(x)|^r\,dx\right)^{\frac{1}{r}} + \left(\int_G |G(x)|^r\,dx\right)^{\frac{1}{r}}. \tag{1}$$

Since $1<p<2$, we have $0<\frac{p}{q}<1$. Let us define $F(x)=\left|(f(x)+g(x))\,w(x)^{\frac{1}{p}}\right|^q$ and $G(x)=\left|(f(x)-g(x))\,w(x)^{\frac{1}{p}}\right|^q$. By lemma 7, we get

$$\begin{aligned}
&\left(\int_G \left|(f(x)+g(x))\,w(x)^{\frac{1}{p}}\right|^p dx\right)^{\frac{q}{p}} + \left(\int_G \left|(f(x)-g(x))\,w(x)^{\frac{1}{p}}\right|^p dx\right)^{\frac{q}{p}}\\
\leq\ & \left[\int_G \left|\left|(f(x)+g(x))\,w(x)^{\frac{1}{p}}\right|^q + \left|(f(x)-g(x))\,w(x)^{\frac{1}{p}}\right|^q\right|^{\frac{p}{q}} dx\right]^{\frac{q}{p}}\\
=\ & \left[\int_G \left|\left|f(x)w(x)^{\frac{1}{p}}+g(x)w(x)^{\frac{1}{p}}\right|^q + \left|f(x)w(x)^{\frac{1}{p}}-g(x)w(x)^{\frac{1}{p}}\right|^q\right|^{\frac{p}{q}} dx\right]^{\frac{q}{p}}\\
\leq\ & \left[\int_G \left(2\left(\left|f(x)w(x)^{\frac{1}{p}}\right|^p+\left|g(x)w(x)^{\frac{1}{p}}\right|^p\right)^{q-1}\right)^{\frac{p}{q}} dx\right]^{\frac{q}{p}}\\
=\ & 2\left[\int_G \left(\left|f(x)w(x)^{\frac{1}{p}}\right|^p+\left|g(x)w(x)^{\frac{1}{p}}\right|^p\right) dx\right]^{\frac{q}{p}}\\
=\ & 2\left[\int_G \left(|f(x)|^p\,w(x)+|g(x)|^p\,w(x)\right) dx\right]^{\frac{q}{p}}
\end{aligned}$$

Thus, we obtain

$$\|f+g\|^q_{p,w}+\|f-g\|^q_{p,w}\leq 2\left(\|f\|^p_{p,w}+\|g\|^p_{p,w}\right)^{q-1}.$$

□

Theorem 2.5. *The space $L^p_w(G)$ is uniformly convex for $1<p<2$.*

Proof. Let $f,g\in L^p_w(G)$, $1<p<2$, with $\|f\|_{p,w}\leq 1$, $\|g\|_{p,w}\leq 1$ and $\|f-g\|_{p,w}\geq \varepsilon$. Then, by the theorem 4, we have

$$\begin{aligned}
\|f+g\|^q_{p,w} &\leq 2\left(\|f\|^p_{p,w}+\|g\|^p_{p,w}\right)^{q-1}-\|f-g\|^q_{p,w}\\
&\leq 2.2^{q-1}-\varepsilon^q\\
&= 2^q\left(1-\left(\frac{\varepsilon}{2}\right)^q\right).
\end{aligned}$$

Hence, we get

$$\begin{aligned}\left\|\frac{f+g}{2}\right\|_{p,w} &\leq \left(1-\left(\frac{\varepsilon}{2}\right)^{q}\right)^{\frac{1}{q}} \\ &\leq 1-\delta\end{aligned}$$

where $\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^{q}\right)^{\frac{1}{q}}$. $\square$

Let us define $B_{p,r}=L^{p}(G)\cap L^{r}(G)$ with $1<p,r<\infty$. It is known that $B_{p,r}$ is a normed space with the norm

$$\|f\|_{p,r}=\max\left\{\|f\|_{p},\|f\|_{r}\right\}$$

[7].

Theorem 2.6. *The space $B_{p,r}$ is uniformly convex space for $1<p,r<\infty$.*

Proof. Let $f,g\in B_{p,r}$ with $\|f\|_{p,r}\leq 1$, $\|g\|_{p,r}\leq 1$ and $\|f-g\|_{p,r}\geq\varepsilon$. By definition of the space $B_{p,r}$, we have $f,g\in L^{p}(G)$ and $f,g\in L^{r}(G)$. Assume that

$$\|f+g\|_{p,r}=\max\left\{\|f+g\|_{p},\|f+g\|_{r}\right\}=\|f+g\|_{p}\,.$$

By assumption, we have $\|f+g\|_{r}\leq\|f+g\|_{p}$.

Let $1<p,r<2$. By lemma 8, we have

$$\|f+g\|_{p}^{q}\leq 2\left(\|f\|_{p}^{p}+\|g\|_{p}^{p}\right)^{q-1}-\|f-g\|_{p}^{q}$$

where $q=\frac{p}{p-1}$. Then, we get

$$\begin{aligned}\|f+g\|_{p,r}^{q} &= \|f+g\|_{p}^{q}\leq 2\left(\|f\|_{p}^{p}+\|g\|_{p}^{p}\right)^{q-1}-\|f-g\|_{p}^{q} \\ &\leq 2.2^{q-1}-\varepsilon^{q} \\ &= 2^{q}\left(1-\left(\frac{\varepsilon}{2}\right)^{q}\right)\end{aligned}$$

which gives $\left\|\frac{f+g}{2}\right\|_{p,r}\leq\left(1-\left(\frac{\varepsilon}{2}\right)^{q}\right)^{\frac{1}{q}}$. By choosing

$\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^{q}\right)^{\frac{1}{q}}$, the proof is completed for $1<p,r<2$.

If $2\leq p,r<\infty$, then we have, by lemma 4,

$$\begin{aligned}\|f+g\|_{p,r}^{p} &= \|f+g\|_{p}^{p}\leq 2^{p-1}\left(\|f\|_{p}^{p}+\|g\|_{p}^{p}\right)-\|f-g\|_{p}^{p} \\ &\leq 2^{p}\left(1-\left(\frac{\varepsilon}{2}\right)^{p}\right)\end{aligned}$$

and we get $\left\|\frac{f+g}{2}\right\|_{p,r}\leq\left(1-\left(\frac{\varepsilon}{2}\right)^{p}\right)^{\frac{1}{p}}$. If we choose $\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^{p}\right)^{\frac{1}{p}}$, the proof is completed. $\square$

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