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ON THE MAPPINGS PRESERVING THE HYPERBOLIC POLYGONS OF TYPE B TOGETHER WITH THEIR HYPERBOLIC AREAS

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Abstract. In this paper, we present new characterizations of Möbius transformations and conjugate Möbius transformations by using the mappings preserving the hyperbolic polygons of type B together with their hyperbolic areas.
Keywords. Hyperbolic polygons; Möbius transformations; hyperbolic areas.

1. Introduction

A Möbius transformation $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a mapping of the form $w = \frac{az+b}{cz+d}$ satisfying $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The set of all Möbius transformations is a group under composition. Möbius transformations are conformal mappings having many useful properties. For example, a map is Möbius if and only if it preserves cross ratios. As for geometric aspect, circle-preserving is another important characterization of Möbius transformations. There are well-known elementary proofs that if f is a continuous injective map of the extended complex plane $\overline{\mathbb{C}}$ that maps circles into circles, then f is Möbius.

The Möbius invariant property is naturally related to hyperbolic geometry. For instance, see the preservation of triangular domains [6], Lambert and Saccheri quadrilaterals [10], [11], hyperbolic regular polygons [3], hyperbolic regular star polygons [4], polygons of type A [7] and others. The Möbius transformations preserving the open unit disc $B^2 = \{z \in \mathbb{C} : |z| < 1\}$ are precisely those of the form $w = e^{i\theta}\frac{a+z}{1+\overline{a}z}$, where $a, z \in B^2$ and $\theta \in \mathbb{R}$. The Poincaré disc model of hyperbolic geometry is built on B^2, more precisely the points of this model are points of B^2 and the hyperbolic lines of this model are Euclidean semicircular arcs that intersect the boundary of B^2 orthogonally including diameters of B^2. Given two distinct hyperbolic lines which intersect at a point, the measure of the angle between these hyperbolic lines is defined by the Euclidean tangents at the common point.

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Definition 1.1. [1] A Lambert quadrilateral is a hyperbolic quadrilateral with ordered interior angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$ and θ, where $0 < \theta < \frac{\pi}{2}$.

Definition 1.2. [1] A Saccheri quadrilateral is a hyperbolic quadrilateral with ordered interior angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, θ, θ, where $0 < \theta < \frac{\pi}{2}$.

Definition 1.3. [7] A hyperbolic polygon with $n-$sides is called as of type A if it has exactly two interior angles not equal to $\frac{\pi}{2}$.

Definition 1.4. [7] A hyperbolic polygon with $n-$sides is called as of type B if it has exactly a unique interior angle not equal to $\frac{\pi}{2}$.

Saccheri quadrilaterals and Lambert quadrilaterals are convex hyperbolic polygons with 4 sides having type A and type B, respectively.

The transformations defined by $f(z) = \frac{a\overline{z}+b}{c\overline{z}+d}$ from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ satisfying $ad - bc \neq 0$ are known as conjugate Möbius transformations. Clearly a conjugate Möbius transformation is a composition of the complex conjugate function with a Möbius transformation. These transformations, like Möbius transformations, have many beautiful properties. For instance they preserve angle magnitudes of angles, but notice that Möbius transformations preserve the orientation while conjugate Möbius transformations reverse it.

C. Carathéodory [2] proved that every arbitrary one to one correspondence between the points of a circular disc C and a bounded point set C' by which circles lying completely in C are transformed into circles lying in C' must always be either a Möbius transformation or a conjugate Möbius transformation. The following results are well known and they play major roles in our proofs.

Lemma 1.1. *[1] Let $\theta_1, \theta_2, \ldots, \theta_n$ be any ordered $n-$tuple with $0 \leq \theta_j < (n-2)\pi$, $j = 1, \ldots, n$. Then there exists a hyperbolic polygon P with interior angles $\theta_1, \theta_2, \ldots, \theta_n$, occurring in this order around ∂P, if and only if $\theta_1 + \theta_2 + \ldots + \theta_n < (n-2)\pi$.*

Theorem 1.1. *(Gauss-Bonnet theorem for a hyperbolic polygon with n sides) Let P be a hyperbolic convex polygon with $n-$ sides and with interior angles $\theta_1, \theta_2, \ldots, \theta_n$. Then the hyperbolic area $\Delta(P)$ of the polygon P is*

$$\Delta(P) = (n-2)\pi - (\theta_1 + \theta_2 + \ldots + \theta_n) \tag{1.1}$$

Throughout the paper we denote by X' the image of X under f, by $[A_j, A_k]$ the geodesic segment between the points A_j and A_k, by A_jA_k the hyperbolic line passing through the points A_j and A_k, by $A_jA_kA_s$ the hyperbolic triangle with three ordered vertices A_j, A_k and A_s, by $A_1A_2 \cdots A_n$ the hyperbolic polygon with $n-$ ordered vertices $A_1, A_2, \cdots A_n$, and by $\angle A_jA_kA_s$ the angle between $[A_j, A_k]$ and $[A_s, A_k]$. We consider the hyperbolic plane $B^2 = \{z \in \mathbb{C} : |z| < 1\}$ with length differential $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$.

2. The Mappings Preserving the Hyperbolic Polygons of Type B Together With Their Hyperbolic Areas

A map $f : B^2 \to B^2$ has the property B, if it preserves $n-$sided hyperbolic polygons having type B, that is if P is a $n-$sided hyperbolic polygon of type B, then $f(P)$ is a $n-$sided hyperbolic polygon of type B, see [7]. J. Liu proved the following result in [7]:

Lemma 2.1. *[7] Let $f : B^2 \to B^2$ be a continuous bijection. If f has Property B for each $n > 3$, then f preserves the vertex where the interior angle is not right.*

Instead of using the continuity condition of functions, we try to obtain a new characterization of Möbius transformations with the condition "$n-$sided hyperbolic polygons preserving property of type B together with their hyperbolic areas " for a fixed $n > 3$. More precisely, when we say f preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas, this means that if P is a $n-$sided hyperbolic polygon of type B with hyperbolic area $\Delta(P) = \sigma$, then $f(P)$ is a $n-$sided hyperbolic polygon of type B with hyperbolic area $\Delta(f(P)) = \sigma$. Area preserving mappings are studied by V. Pambuccian in [8] and by O. Demirel in [5].

Lemma 2.2. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B for a fixed $n > 3$. Then f is injective.*

Proof. Let P and Q be two distinct points in B^2. By *Lemma 2.1*, there exists a hyperbolic polygon, say $A_1A_2 \cdots A_n$, satisfying $\angle A_nA_1A_2 = \alpha < \frac{\pi}{2}$, $\angle A_1A_2A_n = \cdots = \angle A_{n-2}A_{n-1}A_n = \angle A_{n-1}A_nA_1 = \frac{\pi}{2}$. There are three cases:

Case 1: Assume $d_H(P,Q) < d_H(A_1, A_2)$, where d_H is hyperbolic distance. $A_1A_2 \cdots A_n$ can be carried to the point Q with the help of a hyperbolic isometry g_1 such that $g_1(A_2) = Q$ and $P \in [g_1(A_1), g_1(A_2)]$. Let l be the hyperbolic line passing through P and intersects $g_1(A_{n-1})g_1(A_n)$ perpendicularly. Denote the common point of the hyperbolic lines l and $g_1(A_{n-1})g_1(A_n)$ by S. The existence of the point S is clear since $\angle Pg_1(A_n)g_1(A_{n-1}) < \frac{\pi}{2}$, $\angle Pg_1(A_{n-1})g_1(A_n) < \frac{\pi}{2}$. Hence we construct a hyperbolic polygon $PQg_1(A_3) \cdots g_1(A_{n-1})S$ which is an $n-$sided hyperbolic polygon of type B.

Case 2: Assume $d_H(P,Q) > d_H(A_1, A_2)$. $A_1A_2 \cdots A_n$ can be carried to the point Q with the help of a hyperbolic isometry g_2 such that $g_2(A_2) = Q$ and $g_2(A_1) \in [P,Q] = [P, g_2(A_2)]$. Let k be the hyperbolic line passing through P which intersects the hyperbolic line $g_2(A_{n-1})g_2(A_n)$ perpendicularly. Denote the common point of the hyperbolic lines k and $g_2(A_{n-1})g_2(A_n)$ by R. The existence of the point R is clear since $\angle Pg_2(A_n)g_2(A_{n-1}) > \frac{\pi}{2}$. Hence we construct an $n-$sided hyperbolic polygon $RPQg_2(A_3) \cdots g_1(A_{n-2})g_1(A_{n-1})$ of type B.

Case 3: If $d_H(P,Q) = d_H(A_1, A_2)$, then $A_1A_2 \cdots A_n$ can be carried to the point Q with the help of a hyperbolic isometry g_3 such that $g_3(A_1) = P$ and $g_3(A_2) = Q$.

Hence we construct an $n-$sided hyperbolic polygon $PQg_3(A_3)\cdots g_3(A_{n-1})g_3(A_n)$ of type B.

As in the cases above, for two arbitrary points P and Q, it is possible to construct an $n-$sided hyperbolic polygon of type B by using these points. Therefore, if $PQB_1B_2\cdots B_n$ is an $n-$sided hyperbolic polygon of type B, then $P'Q'B'_1B'_2\cdots B'_n$ is also an $n-$sided hyperbolic polygon of type B. This ends the proof. □

Lemma 2.3. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B for a fixed $n > 3$. Then f preserves the collinearity and betweenness properties of the points.*

Proof. Let P and Q be two distinct points in B^2 and assume that S be an interior point of $[P,Q]$. By *Lemma 2.2*, one can easily construct an $n-$sided hyperbolic polygon of type B, say $PQA_1\cdots A_{n-2}$. Moreover, there are many more $n-$sided hyperbolic polygons of type B with common side $[P,Q]$ and all of them contain S. Hence the images of all $n-$sided hyperbolic polygons of type B with common side $[P,Q]$ under f are $n-$sided hyperbolic polygons of type B with common side $[P',Q']$ containing S'. Therefore, f preserves the collinearity and betweenness properties of the points. □

Lemma 2.4. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$. Then f preserves the vertices together with their interior angles.*

Proof. Let $A_1A_2\cdots A_n$ be an $n-$sided hyperbolic polygon of type B (directed counterclockwise) such that $\angle A_nA_1A_2 := \theta \neq \frac{\pi}{2}$. Assume $\angle A'_nA'_1A'_2 = \frac{\pi}{2}$. Clearly, $\angle A'_{n-1}A'_nA'_1 = \frac{\pi}{2}$ or $\angle A'_1A'_2A'_3 = \frac{\pi}{2}$. Without loss of generality, we may assume $\angle A'_{n-1}A'_nA'_1 = \frac{\pi}{2}$. Now draw a geodesic segment $[A_n, K]$ to the hyperbolic line A_1A_2 where the point K lies on A_1A_2 satisfying $\angle A_nKA_1 = \frac{\pi}{2}$. Notice that if $\theta < \frac{\pi}{2}$, then K lies on $[A_1,A_2]$ and if $\theta > \frac{\pi}{2}$, then A_1 lies on $[K,A_2]$. Since K lies on A_1A_2, by *Lemma 2.3*, the point K' must be lie on $A'_1A'_2$. Hence we construct a new $n-$sided hyperbolic polygon $KA_2\cdots A_n$ of type B. Therefore, $K'A'_2\cdots A'_n$ is also an $n-$sided hyperbolic polygon of type B. Since $\angle A'_{n-1}A'_nA'_1 = \frac{\pi}{2}$, we get $\angle A'_{n-1}A'_nK' \neq \frac{\pi}{2}$ which yields $\angle A'_nK'A'_2 = \angle A'_nK'A'_1 = \frac{\pi}{2}$. Obviously, this is a contradiction since the sum of the interior angles of the hyperbolic triangle $A'_nK'A'_1$ is greater then π. Thus we have $\angle A'_nA'_1A'_2 \neq \frac{\pi}{2}$. Because of the fact that f preserves the $n-$sided hyperbolic polygons of type B together with their hyperbolic areas, by *Gauss-Bonnet theorem*, we get $\angle A'_nA'_1A'_2 = \theta$, $\angle A'_{i-1}A'_iA'_{i+1} = \frac{\pi}{2}$ for all $2 \leq i \leq n-1$ and $\angle A'_{n-1}A'_nA'_1 = \frac{\pi}{2}$. □

Lemma 2.5. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$. Then f preserves hyperbolic distance.*

Proof. Let X, Y and Z be three distinct points in B^2 such that XYZ is a hyperbolic triangle (directed counterclockwise) with $\angle ZXY := \alpha_1, \angle XYZ := \alpha_2$

and $\angle YZX := \alpha_3$. Now, by *Lemma 2.1*, there exists a hyperbolic polygon of type B, say $A_1A_2\dots A_n$ (directed counterclockwise), such that $\angle A_nA_1A_2 = \alpha_1$. The angle $\angle A_nA_1A_2$ of the hyperbolic polygon $A_1A_2\dots A_n$ can be moved to the vertex X of the hyperbolic triangle XYZ by an appropriate Möbius transformation g such that the points $g(A_2)$ and $g(A_n)$ lie on the hyperbolic lines XY and XZ, respectively. By the properties of f and g, we immediately get that $g(A_1)'g(A_2)'\dots g(A_n)'$, that is $X'g(A_2)'\dots g(A_n)'$, is an $n-$sided hyperbolic polygon of type B. By *Lemma 2.4*, we have $\angle ZXY = \angle A_nA_1A_2 = \angle g(A_n)Xg(A_2) = \angle g(A_n)'X'g(A_2)' = \angle A_n'A_1'A_2' = \angle Z'X'Y' = \alpha_1$. Hence f preserves the interior angle $\angle ZXY$ of the hyperbolic triangle XYZ. Following the same way, one can easily prove that $\angle XYZ = \angle X'Y'Z'$ and $\angle YZX = \angle Y'Z'X'$ hold true. It is well known that, in hyperbolic plane, the lengths of a hyperbolic triangle are determined by its interior angles, see [9]. Therefore, we get that $d_H(X,Y) = d_H(X',Y')$, $d_H(X,Z) = d_H(X',Z')$ and $d_H(Y,Z) = d_H(Y',Z')$. □

Lemma 2.6. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$. Then f is surjective.*

Proof. To prove that f is surjective, we will show that for any point Y in B^2, there exists a point X in B^2 such that $f(X) = Y$. Let A, B, C be three three distinct points in B^2, each of which is different from Y. Now construct three hyperbolic circles with radius $r_1 = d_H(A',Y)$, $r_2 = d_H(B',Y)$ and $r_3 = d_H(C',Y)$ centered at A', B', C', respectively. These circles meet together only at Y. Because of the fact that f is a distance preserving mapping by *Lemma 2.5*, the pre-images of circles meet together only at a point, say X. Hence, $X' = Y$. □

Theorem 2.1. *The mapping $f : B^2 \to B^2$ is Möbius or conjugate Möbius if, and only if, f preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$.*

Proof. Because of the fact that f is an isometry, the "only if" part is clear. Conversely, we may assume that f preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$. Without loss of generality we may assume $f(O) = O$ by composing an isometry if necessary. Let x and y be two different points in B^2. Since f preserves the hyperbolic distance by *Lemma 2.5*, one can easily get $d_H(0,x) = d_H(0,x')$ and $d_H(0,y) = d_H(0,y')$, namely $|x| = |x'|$ and $|y| = |y'|$, where $|\cdot|$ is the Euclidean norm. Hence we have $|x-y| = |x'-y'|$, since f preserves the angles by *Lemma 2.4*. Finally, we get

$$2\langle x,y\rangle = |x|^2 + |y|^2 - |x-y|^2 = |x'|^2 + |y'|^2 - |x'-y'|^2 = 2\langle x',y'\rangle. \tag{2.1}$$

Therefore, f preserves the inner product and then is the restriction on B^2 of an orthogonal transformation, that is, f is Möbius transformation or conjugate Möbius transformation by *Carathédory's theorem.* If the orientation of the angles preserved under f, then f is a Möbius transformation, otherwise; f is a conjugate Möbius transformation. □

Corollary 2.1. *The mapping $f : B^2 \to B^2$ is Möbius or conjugate Möbius if, and only if, f preserves the Lambert quadrilaterals together with their hyperbolic areas.*

Naturally, one may wonder whether *Corollary 2.1* is valid for Saccheri quadrilaterals. Now we give the affirmative answer as follows:

Corollary 2.2. *The mapping $f : B^2 \to B^2$ is Möbius or conjugate Möbius if, and only if, f preserves all Saccheri quadrilaterals together with their hyperbolic areas.*

Proof. Because of the fact that f is an isometry, the "only if" part is clear. Conversely, we may assume that f preserves all Saccheri quadrilaterals together with their hyperbolic areas. The injectivity, collinearity and the betweenness properties of f can be easily proved following the ways in the proofs of *Lemma 2.2*, *Lemma 2.3*.

Step 1: We claim that f preserves the right angles of Saccheri quadrilaterals. Let $ABCD$ be a Saccheri quadrilateral with $\angle DAB = \angle ABC = \frac{\pi}{2}$ and $\angle BCD = \angle CDA := \theta < \frac{\pi}{2}$. For each point $X_i \in [A, D]$, there exists a point $Y_i \in [C, B]$ such that X_iABY_i is a Saccheri quadrilateral. Notice that $d_H(A, X_i) = d_H(B, Y_i)$. Assume $\angle Y_iX_iA = \angle BY_iX_i := \theta_i$ for all $i \in I \subset \mathbb{R}$. Since f preserves the Saccheri quadrilaterals together with their hyperbolic areas, we immediately get that $X_i'A'B'Y_i'$ are Saccheri quadrilaterals with $\Delta(X_i'A'B'Y_i') = \Delta(X_iABY_i)$ for all $i \in I$. Notice that, by injectivity property of f, the sets $\{X_i' : i \in I\}$ and $\{Y_i' : i \in I\}$ are consist of collinear points, that is $X_i' \in [A', D']$ and $Y_i' \in [B', C']$ hold true for all $i \in I$. Because of the fact that all the Saccheri quadrilaterals $X_i'A'B'Y_i'$ have common two interior angles $\frac{\pi}{2}, \frac{\pi}{2}$ and have common two vertices A' and B', this implies that $\angle X_i'A'B' = \angle A'B'Y_i' = \frac{\pi}{2}$. Thus f preserves right angles of Saccheri quadrilaterals.

Step 2: By *Step 1*, f preserves the other interior angles of Saccheri quadrilaterals which are not right angles.

Step 3: Let $ABCD$ be a Lambert quadrilateral with $\angle CDA := \theta < \frac{\pi}{2}$ and $\angle DAB = \angle ABC = \angle BCD = \frac{\pi}{2}$. By reflecting $ABCD$ with respect to geodesic BC, we get a Saccheri quadrilateral $AEFD$, where the points E and F are the reflections of the points A and D, respectively. Thus, the quadrilateral $A'E'F'D'$ must be a Saccheri quadrilateral with $\Delta(A'E'F'D') = \Delta(AEFD)$. Since $B \in [A, E]$ and $C \in [D, F]$, we have $B' \in [A', E']$ and $C' \in [D', F']$. Therefore, $A'E'F'D'$ contains two quadrilaterals $A'B'C'D'$ and $B'E'F'C'$. By *Step 1* and *Step 2*, we get $\angle D'A'B' = \angle B'E'F' = \frac{\pi}{2}$ and $\angle C'D'A' = \angle E'F'C' = \theta$. By reflecting $ABCD$ in the geodesic AB, one can easily see that $\angle D'C'B' = \frac{\pi}{2}$ holds true. This implies that C' is the midpoint of D' and F' which implies that $\angle A'B'C' = \frac{\pi}{2}$. Hence the quadrilateral $A'B'C'D'$ must be a Lambert quadrilateral with $\Delta(A'B'C'D') = \Delta(ABCD)$ and this implies that f is a Möbius transformation or a conjugate Möbius transformation.

□

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