FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 4 (2018), 531–538
https://doi.org/10.22190/FUMI1804531N

SOME CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH q-RUSCHEWEYH DIFFERENTIAL OPERATOR

Khalida Inayat Noor

Abstract. It is known that the q-analysis (q-calculus) has many applications in mathematics and physics. The notion of the q-derivative D_q of a function f, analytic in the open unit disc, is defined as $D_q f(z) = \frac{f(qz)-f(z)}{(q-1)z}$, $q \in (0,1)$, $(z \neq 0)$ and $D_q f(0) = f'(0)$. Using a q-analogue of the well-known Ruscheweyh differential operator D_q^n of order n, we introduce certain classes $ST_q(n)$ for $n = 0, 1, 2, ...$, and investigate a number of interesting properties such as inclusion and coefficient results. The ideas and techniques in this paper may stimulate further research in this field.

Keywords: Analytic, Starlike functions, q-derivative, Ruscheweyh operator, Subordination

1. Introduction

Let A denote the class of functions f which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m. \tag{1.1}$$

The class $S \subset A$ consists of univalent functions. A function $f \in A$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ in E if it satisfies the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in E).$$

We denote this class by $S^*(\alpha)$. For $\alpha = 0$, we have $S^*(0) = S^*$, is the well-known class of starlike functions. The class $C(\alpha)$, $(0 \leq \alpha < 1)$ consists of convex functions

Received March 21, 2018; accepted: July 07, 2018
2010 *Mathematics Subject Classification.* Primary: 30C45, 30C10; Secondary: 47B38

of order α and can be defined by the relation $f \in C(\alpha)$, if and only if, $zf' \in S^*(\alpha)$.

Let $f_1,\ f_2 \in A$. If there exists a Schwartz function $\phi(z)$ which is analytic in E with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f_1(z) = f_2(\phi(z))$, then we say that $f_1(z)$ is subordinate to $f_2(z)$ and write $f_1(z) \prec f_2(z)$, where $\prec$ denote subordination symbol.

For $f \in A$ and given by (1.1), $g : g(z) = z + \sum_{m=2}^{\infty} b_m z^m$, convolution $*$ (Hadamard product) of f and g is defined by

$$(f * g)(z) = \sum_{m=2}^{\infty} a_m b_m z^m.$$

Recently, the use of q-calculus has attracted the attention of many researchers in the field of geometric function theory. Ismail et al. [5] generalized the class S^* with the concept of q-derivative and called this class S_q^* of q-starlike functions. For recent developments, see [10, 11, 12, 13, 14, 17] and the references therein.

We first give some basic definitions and the concept of q-calculus, which we shall use in this paper. For more details, see [3, 8].

A set $B \subset \mathbb{C}$ is called q-geometric if, for $q \in (0,1)$, $qz \in B$, it contains all the sequences $\{zq^m\}_0^{\infty}$. Jackson [6, 7] defined q-derivative and q-integral of f on the set B as follows:

$$\partial_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \quad (z \neq 0, z \in (0,1)), \tag{1.2}$$

and

$$\int_0^z f(t)\partial_q t = z(1-q) \sum_{m=0}^{\infty} q^m f(zq^m), \quad q \in (0,1),$$

provided that the series converges.

It can easily be seen that, for $m = 1, 2, 3, ...,$ and $z \in E$,

$$\partial_q \left\{ \sum_{m=1}^{\infty} a_m z^m \right\} = \sum_{m=1}^{\infty} [m, q] a_m z^{m-1}, \tag{1.3}$$

where

$$[m, q] = 1 + \sum_{i=1}^{m-1} q^i = \frac{1-q^m}{1-q}, \quad [0, q] = 0. \tag{1.4}$$

For any non-negative integer m, the q-number shift factorial is defined by

$$[m,q]! = \begin{cases} 1, & m = 0 \\ [1,q][2,q][3,q]...[m,q], & m = 1,2,3,... \end{cases}$$

Also, the q-generalized Pochhamer symbol for $x > 0$ is given as

$$[m,q]_m = \begin{cases} 1, & m = 0 \\ [x,q][x+1,q]...[x+m\text{-}1,q], & m = 1,2,3,... \end{cases}$$

Throughout this paper, we shall assume $z \in E$, and $q \in (0,1)$, unless stated otherwise.

Using the q-derivative, we define certain new classes of analytic functions given as below.

Definition 1.1. Let $f \in A$. Then f is said to belong to the class ST_q, if

$$\left| \frac{z}{f(z)}(\partial_q f)(z) - \frac{q}{1-q^2} \right| \leq \frac{q}{1-q^2},$$

where $\partial_q f(z)$ is defined by (1.2) on q-geometric set B.

Remark 1.1. We note that, as $q \to 1^-$, the disc $|w(z) - \frac{q}{1-q^2}| \leq \frac{q}{1-q^2}$ becomes the right half plane $\mathrm{Re}\{w(z)\} > \alpha$, $\alpha \in (\frac{1}{2}, 1)$ and the class ST_q reduces to $S^*(\frac{1}{2})$.

Following the argument similar to the one used in [20], it is easily seen that $f \in ST_q$, if and only if,

$$\frac{z\partial_q f(z)}{f(z)} \prec \frac{1}{1-qz}. \tag{1.5}$$

From (1.5) it can be seen that the linear transformation $\frac{1}{1-qz}$ maps $|z| = r$ onto the circle with center $C(r) = \frac{qr^2}{1-q^2r^2}$ and the radius $\sigma(r) = \frac{qr}{1-q^2r^2}$, and we can write

$$\frac{1-qr+qr^2}{(1-qr)(1+qr)} \leq \left\{ \mathrm{Re}\, \frac{z\partial_q f(z)}{f(z)} \right\} \leq \frac{1+qr+qr^2}{(1-qr)(1+qr)}. \tag{1.6}$$

Now, with $\partial_q(\log f(z)) = \frac{\partial_q f(z)}{f(z)}$, $\mathrm{Re}\, \frac{\partial_q f(z)}{f(z)} = r\frac{\partial_q \log|f(z)|}{dr}$ and some computation, we have from (1.6)

$$\frac{1}{r} + \frac{q}{1+qr} \leq \frac{\partial_q}{dr} \log|f(z)| \leq \frac{1}{r} + \frac{q}{1-qr}. \tag{1.7}$$

Taking the q-integral on both sides of (1.7) and simplifying, we get

$$\frac{1}{(1+qr)^{qq_1}} \leq \left| \frac{f(z)}{z} \right| \leq \frac{1}{(1-qr)^{qq_1}}, \quad q_1 = \frac{1-q}{\log q^{-1}}. \tag{1.8}$$

Since $\lim_{q\to 1^-} \frac{1-q}{\log q^{-1}} = 1$, (1.8) gives us a distortion result for $f \in S^*(\frac{1}{2})$ as

$$\frac{r}{(1+r)} \leq |f(z)| \leq \frac{r}{(1-r)}, \quad \text{see [4]}$$

For $n \in \mathbb{N}_\circ = \{0, 1, 2, 3, ...\}$, the Ruscheweyh derivative D^n of order n, is defined as

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \quad f \in A.$$

We now proceed to discuss the q-analogue of the Ruscheweyh derivative.

Let the function $F_{q,n+1}$ be defined as

$$F_{q,n+1}(z) = z + \sum_{m=2}^{\infty} \frac{[m+n-1,q]!}{[n,q]![m-1,q]!} z^m, \tag{1.9}$$

where the series converges absolutely in E.

Using (1.9), the q-Ruscheweyh differential operator of order n, $D_q^n : A \to A$ is defined for $f(z)$ given by (1.1) as

$$\begin{aligned} D_q^n f(z) &= F_{q,n+1}(z) * f(z) \\ &= z + \sum_{m=2}^{\infty} \frac{[m+n-1,q]!}{[n,q]![m-1,q]!} a^m z^m, \quad \text{see [9].} \end{aligned} \tag{1.10}$$

We note that

$$D_q^0 f(z) = f(z) \quad \text{and} \quad D_q' f(z) = z\partial_q f(z).$$

Also (1.10) can be written as

$$D_q^n f(z) = \frac{z\partial_q^n(z^{n-1}f(z))}{[n,q]!}, \quad n \in \mathbb{N}.$$

As $q \to 1^{-1}$, $\lim_{q\to 1^-} F_{q,n+1}(z) = \frac{z}{(1-z)^{n+1}}$, and $\lim_{q\to 1^-} D_q^n f(z) = D^n f(z)$, that is, the q-Ruscheweyh derivative reduces to the Ruscheweyh derivative as $q \to 1^-$. See [18].

The following identity can easily be derived from (1.10).

$$z\partial_q(D_q^n f(z)) = \left(1 + \frac{[n,q]}{q^n}\right) D_q^{n+1} f(z) - \frac{[n,q]}{q^n} D_q^n f(z). \tag{1.11}$$

When $q \to 1^-$, (1.11) reduces to the well-known identity for the Ruscheweyh derivative as

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z).$$

Using the q-operator D_q^n, we define the following.

Definition 1.2. Let $f \in A$ and let the operator $D_q^n : A \to A$ be defined by (1.10). Then $f \in ST_q(n)$, if and only if, $D_q^n f \in ST_q$ in E.

In other words $\frac{z\partial_q(D_q^n f(z))}{D_q^n f(z)} \prec \frac{1}{1-qz}$ implies $f \in ST_q(n)$. We note that, if

$$p(z) \prec \frac{1}{1-qz}, \quad \text{then} \quad \operatorname{Re} p(z) > \frac{1}{1+q}, \quad z \in E.$$

2. Main Results

Theorem 2.1. *For $n \in \mathbb{N}_\circ$, $ST_q(n+1) \subset ST_q(n)$.*

Proof. Let $f \in ST_q(n+1)$. Set

$$\frac{z\partial_q(D_q^n f(z))}{D_q^n f(z)} = p(z). \tag{2.1}$$

We note that $p(z)$ is analytic in E and $p(0) = 1$. We shall show that $p(z) \prec \frac{1}{1-qz}$.

The q-logarithmic differentiation of (2.1) and the use of identity (1.11) yields

$$\frac{z\partial_q(D_q^{n+1} f(z))}{D_q^{n+1} f(z)} = p(z) + \frac{z\partial_q p(z)}{p(z) + N_q}, \quad N_q = \frac{[n,q]}{q^n}. \tag{2.2}$$

Let

$$p(z) = \frac{1}{1 - q\phi(z)}. \tag{2.3}$$

$\phi(z)$ is analytic in E and $\phi(0) = 0$. We shall show that $|\phi(z)| < 1$, for all $z \in E$.

Suppose, on the contrary, that there exists a $z_0 \in E$ such that $|\phi(z_0)| = 1$. Since $f \in ST_q(n+1)$, it follows from (2.2) that

$$\operatorname{Re}\left\{p(z) + \frac{z\partial_q p(z)}{p(z) + N_q}\right\} > \frac{1}{1+q}, \quad N_q = \frac{[n,q]}{q^n}.$$

Using (2.3), we have

$$\operatorname{Re}\left[p(z) + \frac{z\partial_q p(z)}{p(z) + N_q}\right] = \operatorname{Re}\left[\frac{1}{1-q\phi(z_0)} + \frac{qz_0\partial_q\phi(z_0)}{(1-q\phi(z_0))[(1+N_q) - qN_q\phi(z_0)]}\right]. \tag{2.4}$$

Let $\phi(z_0) = e^{i\theta}$. Then

$$\operatorname{Re}\frac{1}{1 - q\phi(z_0)} = \frac{1 - q\cos\theta}{1 - 2q\cos\theta + q^2}. \tag{2.5}$$

Also

$$z_0\partial_q\phi(z_0) = k\phi(z_0), \quad k \geq 1, \tag{2.6}$$

by using q-Jacks's Lemma given in [1].
Using (2.5), (2.6), $\phi(z_0) = e^{i\theta}$ in (2.4), we have

$$\text{Re}\left\{\frac{z\partial_q(D_q^{n+1}f(z_0))}{D_q^{n+1}f(z_0)}\right\} = \text{Re}\left\{\frac{1}{1-qe^{i\theta}} + \frac{qke^{i\theta}}{(1-qe^{i\theta})(N_q+1-qN_qe^{i\theta})}\right\}. \tag{2.7}$$

In (2.7), we take $\theta = \pi$, and this gives us

$$\text{Re}\left\{\frac{z\partial_q(D_q^{n+1}f(z))}{D_q^{n+1}f(z)} - \frac{1}{1+q}\right\} < 0, \quad z \in E,$$

which is a contradiction. Thus, $|\phi(z)| < 1$ for all $z \in E$ and this proves $p(z) \prec \frac{1}{1-qz}$. Consequently, $f \in ST_q(n)$ in E. □

Using the identity (1.11) and the definition, the proof of the following result is straightforward.

Theorem 2.2. *Let $f \in ST_q(n)$ and let $I_nf : A \to A$ be defined as*

$$I_nf(z) = \frac{[n+1,q]}{q^nz^n}\int_0^z t^{n-1}f(t)\mathrm{d}_qt, \quad n \in \mathbb{N}_\circ.$$

Then $I_nf(z) \in ST_q(n+1)$.

This operator was introduced by Bernardi [2] for $q \to 1^-$. For $n = 1$, $I_1f(z)$ is the q-analogue of the Libera integral operator, see [15, 16].

In [19], it has been proved that $\cap_{0<q<1}S_q^*(\alpha) = S^*(\alpha)$, $0 \leq \alpha < 1$. From this we can easily deduce that
(i). $\cap_{0<q<1}ST_q = S^*(\frac{1}{2})$.
(ii). $f \in [\cap_{0<q<1}ST_q(n)]$ implies $D^nf \in S^*(\frac{1}{2})$.

We have the following.

Theorem 2.3. *$\cap_{n=0}^{\infty}ST_q(n) = \{id\}$ where id is the identity function.*

Proof. Let $f(z) = z$. Then it follows trivially that $z \in ST_q(n)$, for $n \in \mathbb{N}_\circ$.
On the contrary, assume $f \in \cap_{n=0}^{\infty}ST_q(n)$ with $f(z)$ given by (1.1).
From (1.5) and (1.10), we deduce that $f(z) = z$. □

Theorem 2.4. *Let $f \in ST_q(n)$ and be given by (1.1). Then*

$$a_m = O(1)\frac{([n,q]!)([m-1,q]!)}{[m,q]([m+n-1,q]!)}m^{(qq_1+\frac{1}{2})}, \quad q_1 = \frac{1-q}{\log q^{-1}},$$

where $O(1)$ is a constant depending on q.

Proof. Let

$$D_q^n f(z) = z + \sum_{m=2}^{\infty} A_m(n) z^m.$$

Then, from (1.10), we have

$$A_m(n) = \frac{[m+n-1,q]!}{[n,q]![m-1,q]!} a_m.$$

Since $f \in ST_q(n)$, $\quad D_q^n f \in ST_q$, and we can write

$$z\partial_q(D_q^n f(z)) = (D_q^n f(z))(p(z)), \quad p(z) \prec \frac{1}{1-qz}.$$

The Cauchy Theorem, (1.8) and the Schwartz inequality gives us

$$[m,q]|A_m(n)| \leq c_1(q) \frac{1}{(1-r)^{qq_1+\frac{1}{2}}}, \quad q_1 = \frac{1-q}{\log q^{-1}},$$

where $c_1(q)$ is a constant. Taking $r = 1 - \frac{1}{m}$, $(m \to \infty)$, we obtain the desired result. □

As a special case for $n = 0$, $D_q^0 f \in ST_q$ and $a_m = O(1) \cdot m^{(qq_1 - \frac{1}{2})}$, $m \to \infty$.
We observe here that, $\lim_{q \to 1^-} ST_q = S^*(\frac{1}{2})$ and $f(z) \prec \frac{z}{1-z}$. Using the Schwartz inequality and subordination, we get

$$a_m = O(1) \cdot \frac{n!(m-1)!}{(m+n-1)!}.$$

Acknowledgements

The author would like to thank the Rector of COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and academic environment.

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Khalida Inayat Noor
Department of Mathematics
COMSATS University Islamabad
Islamabad, Pakistan
khalidan@gmail.com