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# ON THE GOTTLIEB POLYNOMIALS IN SEVERAL VARIABLES

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**Abstract.** In this study, we give some new properties of the Gottlieb polynomials in several variables. The results obtained here include various families of multilinear and multilateral generating functions, integral representation and recurrence relations for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the multivariable Gottlieb polynomials and the generalized Lauricella functions. Finally, we get several results of this theorem.

**Keywords.** Gottlieb polynomials; generalized Lauricella functions; generating functions; integral representation; recurrence relation.

## 1. Introduction

The Gottlieb polynomials are defined as [6]

$$\begin{aligned}\varphi_n(x;\lambda) &= e^{-n\lambda}\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}(1-e^{\lambda})^k \\ &= e^{-n\lambda}\,{}_2F_1\left[-n,-x;1;1-e^{\lambda}\right],\end{aligned} \tag{1.1}$$

where ${}_2F_1$ denotes the hypergeometric function.

Recently, Choi [4] defined an extension of the Gottlieb polynomials $\varphi_n(x;\lambda)$ in $m$ variables by

$$\begin{aligned}\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m) &= \exp(-n\sigma_m)\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\sum_{r_3=0}^{n-r_1-r_2}\cdots\sum_{r_m=0}^{n-r_1-r_2-...-r_{m-1}} \\ &\times\frac{(-n)_{\delta_m}\prod\limits_{j=1}^{m}(-x_j)_{r_j}\prod\limits_{j=1}^{m}\left(1-e^{\lambda_j}\right)^{r_j}}{\prod\limits_{j=1}^{m}(r_j)!_{\delta_m}!}\end{aligned} \tag{1.2}$$

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where $n, m \in \mathbb{N}$ and, for convenience,

$$\sigma_m := \sum_{j=1}^{m} \lambda_j \qquad \text{and} \qquad \delta_m = \sum_{j=1}^{m} r_j. \tag{1.3}$$

Here and in the following, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

It is noted that the special case $m = 1$ of (1.2) reduces immediately to the Gottlieb polynomials in (1.1). The multivariable Gottlieb polynomials defined by (1.2) have the following two generating functions [4]:

$$\sum_{n=0}^{\infty} \varphi_n^m(x_1, ..., x_m; \lambda_1, ..., \lambda_m) t^n = (1 - te^{-\sigma_m})^{-\left(\sum_{j=1}^{m} x_j\right) - 1} \prod_{j=1}^{m} (1 - te^{\lambda_j - \sigma_m})^{x_j}, \tag{1.4}$$

where $m \in \mathbb{N}$ and $\sigma_m$ is given in (1.3).

$$\begin{aligned} \sum_{n=0}^{\infty} (\mu)_n \varphi_n^m(x_1, ..., x_m; \lambda_1, ..., \lambda_m) \frac{t^n}{n!} &= (1 - te^{-\sigma_m})^{-\mu} F_D^{(m)} \left[\mu, -x_1, ..., -x_m; 1; \right. \\ &\quad \left. \frac{t(e^{\lambda_1} - 1)e^{-\sigma_m}}{1 - te^{-\sigma_m}}, ..., \frac{t(e^{\lambda_m} - 1)e^{-\sigma_m}}{1 - te^{-\sigma_m}} \right], \end{aligned}$$

where $F_D^{(m)}[.]$ denotes one of the Lauricella series in $m$ variables ([10], p. 33, Eq. (4)) defined by

$$F_D^{(m)}[a, b_1, ..., b_m; c; x_1, ..., x_m] = \sum_{r_1, r_2, ..., r_m = 0}^{\infty} \frac{(a)_{\delta_m} (b_1)_{r_1} ... (b_m)_{r_m}}{(c)_{\delta_m}} \frac{x_1^{r_1}}{r_1!} ... \frac{x_m^{r_m}}{r_m!}$$

$$(\max\{|x_1|, ..., |x_m|\} < 1)$$

and $\sigma_m, \delta_m$ are given in (1.3).

The main object of this paper is to study different properties of multivariable Gottlieb polynomials. Various families of multilinear and multilateral generating functions are derived for these polynomials. Other miscellaneous properties of these multivariable polynomials are also discussed. Some special cases of the results presented here are also indicated.

## 2. Generating Functions

In this section, firstly, we prove a theorem which gives a generating function relation for the multivariable Gottlieb polynomials $\varphi_n^m(x_1, ..., x_m; \lambda_1, ..., \lambda_m)$. Secondly, we give a result about several families of bilinear and bilateral generating functions for these polynomials using the similar method considered in (see [1], [2], [3], [5], [8], [12]).

**Theorem 2.1.** *The following generating function holds true:*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\binom{n+k}{n}\varphi_{n+k}^{m}(x_1,...,x_m;\lambda_1,...,\lambda_m)t^n \\
= \;& \left(1-te^{-\sigma_m}\right)^{-\left(\sum\limits_{j=1}^{m}x_j\right)-k-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j} \\
&\times\varphi_k^m\left(x_1,...,x_m;\ln\left(\frac{e^{\lambda_1}-te^{\lambda_1-\sigma_m}}{1-te^{\lambda_1-\sigma_m}}\right),...,\ln\left(\frac{e^{\lambda_m}-te^{\lambda_m-\sigma_m}}{1-te^{\lambda_m-\sigma_m}}\right)\right). \qquad (2.1)
\end{aligned}$$

*Proof.* If we write $t+u$ instead of $t$ in (1.4), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m)(t+u)^n \\
= \;& (1-te^{-\sigma_m}-ue^{-\sigma_m})^{-\left(\sum\limits_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}(1-te^{\lambda_j-\sigma_m}-ue^{\lambda_j-\sigma_m})^{x_j}.
\end{aligned}$$

Here, using the binomial theorem on the left-hand side of the last equality, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m)\sum_{k=0}^{n}\binom{n}{k}t^{n-k}u^k \\
= \;& \left(1-te^{-\sigma_m}\right)^{-\left(\sum\limits_{j=1}^{m}x_j\right)-1}\left(1-\frac{ue^{-\sigma_m}}{1-te^{-\sigma_m}}\right)^{-\left(\sum\limits_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j} \\
&\times\prod_{j=1}^{m}\left(1-\frac{ue^{\lambda_j-\sigma_m}}{1-te^{\lambda_j-\sigma_m}}\right)^{x_j}.
\end{aligned}$$

Replacing $n$ by $n+k$ and using relation (1.4), then we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k}\varphi_{n+k}^m(x_1,...,x_m;\lambda_1,...,\lambda_m)t^n u^k \\
= \;& \left(1-te^{-\sigma_m}\right)^{-\left(\sum\limits_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}\sum_{k=0}^{\infty}\left(1-te^{-\sigma_m}\right)^{-k} \\
&\times\varphi_k^m\left(x_1,...,x_m;\ln\left(\frac{e^{\lambda_1}-te^{\lambda_1-\sigma_m}}{1-te^{\lambda_1-\sigma_m}}\right),...,\ln\left(\frac{e^{\lambda_m}-te^{\lambda_m-\sigma_m}}{1-te^{\lambda_m-\sigma_m}}\right)\right)u^k
\end{aligned}$$

Equating the coefficients of $u^k$ on the both sides of the last equality, we arrive at the desired result. □

**Theorem 2.2.** *Corresponding to an identically non-vanishing function*

$$\Omega_{\mu}(y_1, ..., y_r)$$

*of $r$ complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order $\mu$, let*

$$\begin{aligned} &\Lambda_{\nu,q}\left[x_1, ..., x_m; \lambda_1, ..., \lambda_m; y_1, ..., y_r; t\right] \\ : &= \sum_{n=0}^{\infty} a_n \varphi_{\nu+qn}^{m}(x_1, ..., x_m; \lambda_1, ..., \lambda_m)\Omega_{\mu+pn}(y_1, ..., y_r)t^n \end{aligned}$$

*where $a_n \neq 0$, $\mu \in \mathbb{C}$ and*

$$\theta_{n,p,q}(y_1, ..., y_r; z) := \sum_{k=0}^{[n/q]} \binom{n+\nu}{n-qk} a_k \Omega_{\mu+pk}(y_1, ..., y_r) z^k. \tag{2.2}$$

*Then, for $p \in \mathbb{N}$, we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} \varphi_{n+\nu}^{m}(x_1, ..., x_m; \lambda_1, ..., \lambda_m)\theta_{n,p,q}(y_1, ..., y_r; z)t^n \\ = &\left(1 - te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m} x_j\right)-\nu-1} \prod_{j=1}^{m} \left(1 - te^{\lambda_j - \sigma_m}\right)^{x_j} \\ &\times \Lambda_{\nu,q}\left(x_1, ..., x_m; \ln\left(\frac{e^{\lambda_1} - te^{\lambda_1-\sigma_m}}{1 - te^{\lambda_1-\sigma_m}}\right), ..., \ln\left(\frac{e^{\lambda_m} - te^{\lambda_m-\sigma_m}}{1 - te^{\lambda_m-\sigma_m}}\right);\right. \\ &\left. y_1, ..., y_r; z\left(\frac{t}{1 - te^{-\sigma_m}}\right)^q\right). \end{aligned} \tag{2.3}$$

*Proof.* For convenience, let $T$ denote the left side of (2.3). Using (2.2), we have

$$T = \sum_{n=0}^{\infty} \varphi_{n}^{m}(x_1, ..., x_m; \lambda_1, ..., \lambda_m) \sum_{k=0}^{[n/q]} \binom{n+\nu}{n-qk} a_k \Omega_{\mu+pk}(y_1, ..., y_r) z^k t^n.$$

Replacing $n$ by $n+qk$ and using relation (2.1), we may write

$$\begin{aligned} T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+\nu+qk}{n}\varphi_{n+\nu+qk}^{m}(x_1,...,x_m;\lambda_1,...,\lambda_m)a_k\,\Omega_{\mu+pk}(y_1,...,y_r)z^k t^{n+qk} \\ &= \left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-\nu-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}\sum_{k=0}^{\infty}\left(1-te^{-\sigma_m}\right)^{-qk}a_k\Omega_{\mu+pk}(y_1,...,y_r) \\ &\quad\times\varphi_{\nu+qk}^{m}\left(x_1,...,x_m;\ln\left(\frac{e^{\lambda_1}-te^{\lambda_1-\sigma_m}}{1-te^{\lambda_1-\sigma_m}}\right),...,\ln\left(\frac{e^{\lambda_m}-te^{\lambda_m-\sigma_m}}{1-te^{\lambda_m-\sigma_m}}\right)\right)(zt^q)^k \\ &= \left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-\nu-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j} \\ &\quad\times\Lambda_{\nu,q}\left(x_1,...,x_m;\ln\left(\frac{e^{\lambda_1}-te^{\lambda_1-\sigma_m}}{1-te^{\lambda_1-\sigma_m}}\right),...,\ln\left(\frac{e^{\lambda_m}-te^{\lambda_m-\sigma_m}}{1-te^{\lambda_m-\sigma_m}}\right);\right. \\ &\quad\left. y_1,...,y_r;z\left(\frac{t}{1-te^{-\sigma_m}}\right)^q\right). \end{aligned}$$

This completes the proof. □

It is possible to give many applications of our theorems by making appropriate choices of the multivariable functions $\Omega_{\mu+pk}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$. Furthermore, for every suitable choice of the coefficients $a_k$ $(k\in\mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r\in\mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertion of Theorem 2.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the multivariable Gottlieb polynomials.

## 3. Miscellaneous Properties

In this section, we give some properties for the multivariable Gottlieb polynomials $\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m)$ given by (1.2).

**Theorem 3.1.** *The multivariable Gottlieb polynomials have the following integral representation:*

$$\begin{aligned} \varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m) &= \frac{1}{\Gamma(x_1)...\Gamma(x_m)}\sum_{r_{m+1}=0}^{n}\prod_{j=1}^{m}\left(e^{\lambda_j-\sigma_m}\right)^{r_j} \\ &\quad\times\int_0^{\infty}...\int_0^{\infty}(-x_1...-x_m)_{n-r_{m+1}}e^{-(u_1+...+u_{m+1})}u_1^{x_1-1}...u_m^{x_m-1} \\ &\quad\times\frac{[\exp(-\sigma_m)(u_1+...+u_{m+1})]^{r_{m+1}}}{(n-r_{m+1})!r_{m+1}!}du_1...du_{m+1}, \end{aligned}$$

*where* $Re(x_j) > 0 \quad (j = 1, ..., m)$.

*Proof.* *If we denote the left-hand side of (1.4) by* $S$ *and use the identity*

$$a^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-at} t^{v-1} dt \qquad (Re(v) > 0),$$

*we obtain*

$$S = (1 - te^{-\sigma_m})^{-x_1 - \dots - x_m - 1} (1 - te^{\lambda_1 - \sigma_m})^{x_1} \dots (1 - te^{\lambda_m - \sigma_m})^{x_m}$$

$$= \frac{1}{\Gamma(x_1)} \int_0^\infty e^{-(1-te^{-\sigma_m})u_1} u_1^{x_1-1} du_1 \dots \frac{1}{\Gamma(x_m)} \int_0^\infty e^{-(1-te^{-\sigma_m})u_m} u_m^{x_m-1} du_m \frac{1}{\Gamma(1)}$$

$$\times \int_0^\infty e^{-(1-te^{-\sigma_m})u_{m+1}} du_{m+1} \sum_{r_1=0}^\infty \frac{(-x_1)_{r_1}}{r_1!} \left(te^{\lambda_1 - \sigma_m}\right)^{r_1} \dots \sum_{r_m=0}^\infty \frac{(-x_m)_{r_m}}{r_m!} \left(te^{\lambda_m - \sigma_m}\right)^{r_m}$$

$$= \frac{1}{\Gamma(x_1)\dots\Gamma(x_m)} \int_0^\infty \dots \int_0^\infty \sum_{r_1,\dots,r_m=0}^\infty \frac{(-x_1)_{r_1}\dots(-x_m)_{r_m}}{r_1!\dots r_m!} t^{r_1+\dots+r_m} \prod_{j=1}^m \left(e^{\lambda_j - \sigma_m}\right)^{r_j}$$

$$\times e^{-(u_1+\dots+u_m+u_{m+1})} e^{te^{-\sigma_m}(u_1+\dots+u_m+u_{m+1})} u_1^{x_1-1} \dots u_m^{x_m-1} du_1 \dots du_m du_{m+1}$$

$$= \frac{1}{\Gamma(x_1)\dots\Gamma(x_m)} \int_0^\infty \dots \int_0^\infty \sum_{r_1,\dots,r_m=0}^\infty \frac{(-x_1)_{r_1}\dots(-x_m)_{r_m}}{r_1!\dots r_m!} t^{r_1+\dots+r_m} \prod_{j=1}^m \left(e^{\lambda_j - \sigma_m}\right)^{r_j}$$

$$\times e^{-(u_1+\dots+u_{m+1})} \sum_{r_{m+1}=0}^\infty \frac{\left[t \exp(-\sigma_m)(u_1 + \dots + u_{m+1})\right]^{r_{m+1}}}{r_{m+1}!} u_1^{x_1-1} \dots u_m^{x_m-1} du_1 \dots du_{m+1}.$$

*If we use the identity [11]*

$$\sum_{r_1,\dots,r_m=0}^\infty f(r_1 + \dots. + r_m)(\nu_1)_{r_1} \dots (\nu_m)_{r_m} \frac{x^{r_1+\dots+r_m}}{r_1!\dots r_m!} = \sum_{n=0}^\infty f(n)(\nu_1 + \dots + \nu_m)_n \frac{x^n}{n!}$$

*and replace* $n$ *by* $n - k$, *we have*

$$\begin{aligned} S \quad &= \quad \frac{1}{\Gamma(x_1)\dots\Gamma(x_m)} \sum_{n=0}^\infty \Bigg[ \sum_{r_{m+1}=0}^n \prod_{j=1}^m \left(e^{\lambda_j - \sigma_m}\right)^{r_j} \\ &\times \int_0^\infty \dots \int_0^\infty \frac{(-x_1 \dots - x_m)_{n - r_{m+1}} \left[\exp(-\sigma_m)(u_1 + \dots + u_{m+1})\right]^{r_{m+1}}}{(n - r_{m+1})! r_{m+1}!} \\ &\times e^{-(u_1+\dots+u_{m+1})} u_1^{x_1-1} \dots u_m^{x_m-1} du_1 \dots du_{m+1} \Bigg] t^n. \end{aligned}$$

*From the coefficients of* $t^n$ *on the both sides of the last equality, one can get the desired result.* □

We now discuss some miscellaneous recurrence relations of the multivariable Gottlieb polynomials. By differentiating each member of the generating function relation (1.4) with respect to $x_j$ $(j = 1, 2, ..., m)$ and after some calculations, we arrive at the following (differential) recurrence relation for the multivariable Gottlieb polynomials:

$$\frac{\partial}{\partial x_j}\varphi_n^m = \sum_{r=0}^{n-1} \frac{1}{r+1}\left[\left(e^{-\sigma_m}\right)^{r+1} - \left(e^{\lambda_j-\sigma_m}\right)^{r+1}\right]\varphi_{n-r-1}^m.$$

Besides, by differentiating each member of the generating function relation (1.4) with respect to $t$, we have the following another recurrence relation:

$$(n+1)\varphi_{n+1}^m = (1+\sum_{j=1}^m x_j)\sum_{p=0}^n \left(e^{-\sigma_m}\right)^{p+1}\varphi_{n-p}^m - \sum_{r=0}^n\sum_{j=1}^m x_j\left(e^{\lambda_j-\sigma_m}\right)^{r+1}\varphi_{n-r}^m.$$

## 4. Bilateral Generating Functions for the Multivariable Gottlieb Polynomials and the Generalized Lauricella Functions

In this section, we derive various families of bilateral generating functions for the multivariable Gottlieb polynomials and the Srivastava-Daoust (or generalized Lauricella) functions.

The Srivastava-Daoust (or generalized Lauricella) function is defined by [9]

$$F_{C:D^{(1)};...;D^{(n)}}^{A:B^{(1)};...;B^{(n)}}\left(\begin{array}{ll}\left[(a):\theta^{(1)},...,\theta^{(n)}\right]: & \left[(b^{(1)}):\phi^{(1)}\right];\quad ...; \\ \left[(c):\psi^{(1)},...,\psi^{(n)}\right]: & \left[(d^{(1)}):\delta^{(1)}\right];\quad ...;\end{array}\right.$$
$$\left.\begin{array}{l}\left[(b^{(n)}):\phi^{(n)}\right]; \\ \left[(d^{(n)}):\delta^{(n)}\right];\end{array} z_1,...,z_n\right)$$
$$= \sum_{m_1,...,m_n=0}^{\infty}\Omega(m_1,...,m_n)\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},$$

where for convenience

$$\Omega(m_1,...,m_n) := \frac{\prod_{j=1}^A (a_j)_{m_1\theta_j^{(1)}+...+m_n\theta_j^{(n)}}}{\prod_{j=1}^C (c_j)_{m_1\psi_j^{(1)}+...+m_n\psi_j^{(n)}}}\frac{\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}}...\frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}},$$

the coefficients

$$\theta_j^{(k)}\ (j=1,...,A;\ k=1,...,n),\ \text{and}\ \phi_j^{(k)}\ (j=1,...,B^{(k)};\ k=1,...,n),$$

$$\psi_j^{(k)}\ (j=1,...,C;\ k=1,...,n),\quad \text{and } \delta_j^{(k)}\ (j=1,...,D^{(k)};\ k=1,...,n)$$

are real constants and $\left(b_{B^{(k)}}^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$ $(j = 1,...,B^{(k)};\ k=1,...,n)$ with similar interpretations for other sets of parameters [7].

For a suitably bounded nonvanishing multiple sequence $\{\Omega(m_1,...,m_r)\}_{m_1,...,m_r\in\mathbb{N}_0}$ of real or complex parameters, we define a function $\phi_n(u_1;u_2,...,u_r)$ of $r$ (real or complex) variables $u_1,u_2,...,u_r$ by [7]

$$\begin{aligned}\phi_n(u_1;u_2,...,u_r) \ :&= \sum_{m_1=0}^{n}\sum_{m_2,...,m_r=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}}\\ &\times\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right)\frac{u_1^{m_1}}{m_1!}...\frac{u_r^{m_r}}{m_r!}\end{aligned}$$

where, for convenience

$$((b))_{m_1\phi}=\prod_{j=1}^{B}(b_j)_{m_1\phi_j}\quad \text{and}\quad ((d))_{m_1\delta}=\prod_{j=1}^{D}(d_j)_{m_1\delta_j}.$$

**Theorem 4.1.** *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m)\phi_n(u_1;u_2,...,u_r)t^n\\ =\ &\left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}\\ &\times\sum_{r_1,...,r_m,m_1,...,m_r=0}^{\infty}\frac{((b))_{(m_1+r_1+...+r_m)\phi}}{((d))_{(m_1+r_1+...+r_m)\delta}}(r_1+...+r_m+1)_{m_1}\prod_{j=1}^{m}(-x_j)_{r_j}\\ &\times\Omega\left(f(m_1+r_1+...+r_m,m_2,...,m_r),m_2,...,m_r\right)\frac{\left(\frac{-u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-te^{\lambda_j-\sigma_m}}{e^{\lambda_j}-te^{\lambda_j-\sigma_m}}\right)^{m_1}}{m_1!}\\ &\times\frac{\left(\frac{u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-e^{\lambda_1}}{1-te^{\lambda_1-\sigma_m}}\right)^{r_1}}{r_{1!}}...\frac{\left(\frac{u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-e^{\lambda_m}}{1-te^{\lambda_m-\sigma_m}}\right)^{r_m}}{r_m!}\frac{u_2^{m_2}}{m_2!}...\frac{u_r^{m_r}}{m_r!}.\end{aligned}$$

*Proof.* For convenience, let $S$ denote the first member of the assertion in Theorem 4.1 and use definition of $\phi_n$. Then we have

$$S=\sum_{n=0}^{\infty}\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m)\sum_{m_1=0}^{n}\sum_{m_2,...,m_r=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}}$$

$$\times\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right)\frac{u_1^{m_1}}{m_1!}...\frac{u_r^{m_r}}{m_r!}t^n$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \left( \sum_{n=0}^{\infty} \binom{n+m_1}{n} \varphi_{n+m_1}^{m}(x_1,...,x_m;\lambda_1,...,\lambda_m)t^n \right) \frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}$$

$$\times\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right)(-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_r^{m_r}}{m_r!}.$$

By using the relationship (2.1), we get

$$\begin{aligned} S &= \left(1-te^{-\sigma_m}\right)^{-\left(\sum\limits_{j=1}^{m} x_j\right)-1} \prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j} \\ &\times \sum_{m_1,,\ldots,m_r=0}^{\infty} \varphi_{m_1}^{m}\left(x_1,...,x_m;\ln\left(\frac{e^{\lambda_1}-te^{\lambda_1-\sigma_m}}{1-te^{\lambda_1-\sigma_m}}\right),...,\ln\left(\frac{e^{\lambda_m}-te^{\lambda_m-\sigma_m}}{1-te^{\lambda_m-\sigma_m}}\right)\right) \end{aligned}$$

$$\times\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right)\left(\frac{-u_1t}{1-te^{-\sigma_m}}\right)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_r^{m_r}}{m_r!}.$$

Now, if we use the definition (1.2) and after some calculation, we arrive at the desired result. □

By appropriately choosing the multiple sequence $\Omega(m_1,...,m_r)$ in Theorem 4.1, we obtain several interesting results including, for example, the following bilateral generating functions.

**I**. By letting

$$\begin{aligned} &\Omega\left(f(m_1,...,m_r);m_2,...,m_r\right) \\ &= \frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_r\theta_j^{(r)}}}{\prod\limits_{j=1}^{E}(c_j)_{m_1\psi_j^{(1)}+...+m_r\psi_j^{(r)}}} \frac{\prod\limits_{j=1}^{B^{(2)}}(b_j^{(2)})_{m_2\phi_j^{(2)}}}{\prod\limits_{j=1}^{D^{(2)}}(d_j^{(2)})_{m_2\delta_j^{(2)}}} \cdots \frac{\prod\limits_{j=1}^{B^{(r)}}(b_j^{(r)})_{m_r\phi_j^{(r)}}}{\prod\limits_{j=1}^{D^{(r)}}(d_j^{(r)})_{m_r\delta_j^{(r)}}} \end{aligned}$$

in Theorem 4.1,we obtain the following result:

**Corollary 4.1.** *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty}\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m)$$
$$\times F^{A:B+1;B^{(2)};...;B^{(r)}}_{E:D;D^{(2)};...;D^{(r)}}\left(\begin{array}{lll}\left[(a):\theta^{(1)},...,\theta^{(r)}\right]: & [-n:1], & [(b):\phi];\\ \left[(c):\psi^{(1)},...,\psi^{(r)}\right]: & & [(d):\delta];\end{array}\right.$$
$$\left.\begin{array}{ll}[(b^{(2)}):\phi^{(2)}]; \quad ...; \quad [(b^{(r)}):\phi^{(r)}]; & \\ & u_1,u_2,...,u_r\\ [(d^{(2)}):\delta^{(2)}]; \quad ...; \quad [(d^{(r)}):\delta^{(r)}]; & \end{array}\right)t^n$$
$$= \left(1-te^{-\sigma_m}\right)^{-\left(\sum\limits_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}F^{A+B:1;1;B^{(2)};...;B^{(r)}}_{E+D:0;0;D^{(2)};...;D^{(r)}}$$
$$\left(\begin{array}{lllll}\left[(e):\varphi^{(1)},...,\varphi^{(m+r)}\right]: & [r_1+...+r_m+1:1]; & [-x_1:1]; & ...;[-x_m:1], \\ \left[(f):\xi^{(1)},...,\xi^{(m+r)}\right]: & -; & -; & ...; \quad -;\end{array}\right.$$
$$\begin{array}{ll}[(b^{(2)}):\phi^{(2)}]; \quad ...; \quad [(b^{(r)}):\phi^{(r)}]; & \\ & \left(\frac{-u_1t}{1-te^{-\sigma_m}}\prod\limits_{j=1}^{m}\frac{1-te^{\lambda_j-\sigma_m}}{e^{\lambda_j}-te^{\lambda_j-\sigma_m}}\right),\\ [(d^{(2)}):\delta^{(2)}]; \quad ...; \quad [(d^{(r)}):\delta^{(r)}]; & \end{array}$$
$$\left.\left(\frac{u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-e^{\lambda_1}}{1-te^{\lambda_1-\sigma_m}}\right),...,\left(\frac{u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-e^{\lambda_m}}{1-te^{\lambda_m-\sigma_m}}\right),u_2,...,u_r\right)$$

*where the coefficients* $e_j, f_j, \varphi_j^{(k)}$ *and* $\xi_j^{(k)}$ *are given by*

$$e_j=\left\{\begin{array}{ll}a_j, & 1\le j\le A\\ b_{j-A}, & A<j\le A+B\end{array}\right. , \qquad f_j=\left\{\begin{array}{ll}c_j, & 1\le j\le E\\ d_{j-E}, & E<j\le E+D\end{array}\right.$$

$$\varphi_j^{(s)}=\left\{\begin{array}{ll}\theta_j^{(1)}, & 1\le j\le A;\ 1\le s\le m+1\\ \theta_j^{(s-k)}, & 1\le j\le A;\ m+1<s\le m+r\\ \phi_{j-A}, & A<j\le A+B;\ 1\le s\le m+1\\ 0, & A<j\le A+B;\ m+1<s\le m+r\end{array}\right.$$

*and*

$$\xi_j^{(s)}=\left\{\begin{array}{ll}\psi_j^{(1)}, & 1\le j\le E;\ 1\le s\le m+1\\ \psi_j^{(s-k)}, & 1\le j\le E;\ m+1<s\le m+r\\ \delta_{j-E}, & E<j\le E+D;\ 1\le s\le m+1\\ 0, & E<j\le E+D;\ m+1<s\le m+r\end{array}\right.$$

*respectively.*

**II.**Upon setting

$$\Omega\left(f(m_1,...,m_r);m_2,...,m_r\right)=\frac{(a)_{m_1+...+m_r}(b_2)_{m_2}...(b_r)_{m_r}}{(c_1)_{m_1}...(c_r)_{m_r}}$$

and

$$\phi=\delta=0 \ \ \text{(that is, } \phi_1=...=\phi_B=\delta_1=...=\delta_D=0)$$

in Theorem 4.1 we obtain the following result.

**Corollary 4.2.** *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty}\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m)F_A^{(r)}\left[a,-n,b_2,...,b_r;c_1,...,c_r;u_1,u_2,...,u_r\right]t^n$$

$$= \left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}F^{1:1;1;...;1;1;...;1}_{1:0;0;...;0;1;...;1}$$

$$\left(\begin{array}{l}[(a):1,...,1]:\quad [r_1+...+r_m+1:1];\quad [-x_1:1];\quad ...;[-x_m:1];\\ \left[(c_1):\psi^{(1)},...,\psi^{(m+r)}\right]:\quad -;\quad -;\quad ...;\quad -;\end{array}\right.$$

$$\begin{array}{l}[b_2:1];\quad ...;\quad [b_r:1];\\ \qquad\qquad\left(\frac{-u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-te^{\lambda_j-\sigma_m}}{e^{\lambda_j}-te^{\lambda_j-\sigma_m}}\right),\\ [c_2:1];\quad ...;\quad [c_r:1];\end{array}$$

$$\left.\left(\frac{u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-e^{\lambda_1}}{1-te^{\lambda_1-\sigma_m}}\right),...,\left(\frac{u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-e^{\lambda_m}}{1-te^{\lambda_m-\sigma_m}}\right),u_2,...,u_r\right),$$

*where the coefficients* $\psi^{(\eta)}$ *are given by*

$$\psi^{(\eta)}=\begin{cases}1, & 1\le\eta\le m+1\\ 0, & m+1<\eta\le m+r\end{cases}$$

*and* $F_A^{(r)}[.]$ *is the Lauricella function.*

**III.** If we put

$$\Omega\left(f(m_1,...,m_r);m_2,...,m_r\right)=\frac{(a_1^{(1)})_{m_2}...(a_1^{(r-1)})_{m_r}(a_2^{(1)})_{m_2}...(a_2^{(r-1)})_{m_r}}{(c)_{m_1+\cdots+m_r}}$$

and $B=1,b_1=b,\phi_1=1$ and $\delta=0$ in Theorem 4.1,we obtain the following result:

**Corollary 4.3.** *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty}\varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m)$$
$$\times F_B^{(r)}\left[-n,a_1^{(1)},...,a_1^{(r-1)},b,a_2^{(1)},...,a_2^{(r-1)};c;u_1,u_2,...,u_r\right]t^n$$
$$= \left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}F_{1:0;0;...;0;0;...;0}^{1:1;1;...;1;2;...;2}$$
$$\left(\begin{array}{l}\left[(b):\theta^{(1)},...,\theta^{(k+r)}\right]: \quad [r_1+...+r_m+1:1]; \quad [-x_1:1]; \quad ...;[-x_m:1]; \\ \quad [(c):1,...,1]: \qquad -; \qquad -; \qquad -; \\ \quad \left[a^{(1)}:1\right]; \quad ...; \quad \left[a^{(r-1)}:1\right]; \\ \qquad \left(\frac{-u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-te^{\lambda_j-\sigma_m}}{e^{\lambda_j}-te^{\lambda_j-\sigma_m}}\right), \\ \quad -; \quad ...; \quad -; \\ \quad \left(\frac{u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-e^{\lambda_1}}{1-te^{\lambda_1-\sigma_m}}\right),...,\left(\frac{u_1t}{1-te^{-\sigma_m}}\prod_{j=1}^{m}\frac{1-e^{\lambda_m}}{1-te^{\lambda_m-\sigma_m}}\right),u_2,...,u_r\end{array}\right),$$

*where the coefficients* $\theta^{(\eta)}$ *are given by*

$$\theta^{(\eta)}=\begin{cases}1, & 1\leq\eta\leq m+1\\ 0, & m+1<\eta\leq m+r\end{cases}.$$

*and* $F_B^{(r)}[.]$ *is the Lauricella function.*

**IV**. By letting

$$\Omega\left(f(m_1,m_2,...,m_r);m_2,...,m_r\right)=\frac{(a)_{m_1+...+m_r}(b_2)_{m_2}...(b_r)_{m_r}}{(c)_{m_1+...+m_r}}$$

and $\phi=\delta=0$ in Theorem 4.1,we obtain the following result:

**Corollary 4.4.** *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} \varphi_n^m(x_1,...,x_m;\lambda_1,...,\lambda_m) F_D^{(r)}\left[a,-n,b_2,...,b_r;c;u_1,u_2,...,u_r\right] t^n$$

$$= \left(1-te^{-\sigma_m}\right)^{-\left(\sum\limits_{j=1}^{m} x_j\right)-1} \prod_{j=1}^{m} \left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}$$

$$F_D^{(m+r)}\left[a, r_1+...+r_m+1, -x_1,...,-x_m, b_2,...,b_r; c; \left(\frac{-u_1 t}{1-te^{-\sigma_m}} \prod_{j=1}^{m} \frac{1-te^{\lambda_j-\sigma_m}}{e^{\lambda_j}-te^{\lambda_j-\sigma_m}}\right),\right.$$

$$\left.\left(\frac{u_1 t}{1-te^{-\sigma_m}} \prod_{j=1}^{m} \frac{1-e^{\lambda_1}}{1-te^{\lambda_1-\sigma_m}}\right),...,\left(\frac{u_1 t}{1-te^{-\sigma_m}} \prod_{j=1}^{m} \frac{1-e^{\lambda_m}}{1-te^{\lambda_m-\sigma_m}}\right), u_2,...,u_r\right],$$

*where* $F_D^{(r)}\left[.\right]$ *is the Lauricella function.*

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