FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 4 (2018), 599–611
https://doi.org/10.22190/FUMI1804599P

# ON A SUBSPACE OF A SPECIAL FINSLER SPACE

**Vivek Kumar Pandey and P. N. Pandey**

**Abstract.** The present paper deals with the properties of a Finsler space $F_n^*$ whose metric is obtained from the metric of another Finsler space $F_n$ defined over the same manifold, with the help of a contravariant vector $v^i(x^j)$ satisfying the condition $LC_{jkr}v^r = \rho h_{jk}$, where $L$, $h_{jk}$ and $C_{jkr}$ are metric function, angular metric tensor and Cartan tensor of $F_n$, respectively, and $\rho$ is a scalar function of positional coordinates $x^i$. Apart from obtaining expressions for different geometric objects of $F_n^*$, a subspace of $F_n^*$ is studied. Apart from other results for the subspace of $F_n^*$, certain conditions for a subspace of $F_n^*$ to be totally geodesic and projectively flat have been obtained.

**Keywords**: Finsler space; subspace; projective change; totally geodesic subspace; projectively flat space.

## 1. Introduction

In 1952, S. Kikuchi [11] studied the theory of a subspace of a Finsler space. H. Rund [3] in 1959, H. Yasuda [4] in 1987, T. Sakaguchi [12] in 1988 and many others mathematicians contributed significantly to the theory of Finsler subspaces and obtained many important and interesting results. In 1980 during the study of conformally flat Finsler spaces, H. Izumi [2] introduced a vector $b_i$ which is $v-$covariant constant ($b_i|_j = 0$) and satisfies the condition $LC^r_{jk}b_r = \rho h_{jk}$, where $\rho$ is a scalar independent of directional arguments $y^i$. He called such vector $b_i$ as $h-$ vector. In 1990, B. N. Prasad [1] studied a Finsler space with a special metric $ds = (g_{ij}(dx)dx^i dx^j)^{1/2} + b_i(x,y)dx^i$, where $b_i$ is an $h-$vector, and obtained the Cartan connections. In 2008, M. K. Gupta and P. N. Pandey [6] worked on subspaces of a Finsler space with a special metric by taking this $h-$ vector.

Let $F_n = (M_n, L)$ be a Finsler space and $F_n^* = (M_n, L^*)$ be another Finsler space over the same manifold $M_n$, whose metric $L^*$ is obtained from the metric $L$ of $F_n$ by

$$L^*(x,y) = L(x,y) + v_i(x,y)y^i, \tag{1.1}$$

Received March 27, 2018; accepted April 10, 2018
2010 *Mathematics Subject Classification.* 53B40

where $v_i = g_{ij}v^j$, $g_{ij}$ is the metric tensor of $F_n$ and $v^i(x^j)$ is a contravariant vector satisfying

$$LC_{jkr}v^r = \rho h_{jk}, \tag{1.2}$$

where $\rho$ is a scalar function of positional coordinates $x^i$.
We call such a Finsler space $F_n^* = (M_n, L^*)$ as a special Finsler space. This special Finsler space $F_n^*$ is a generalization of the Finsler spaces considered by the authors ([1], [6]). The aim of the present paper is to obtain the Cartan connections and to study a subspace of the Finsler space $F_n^* = (M_n, L^*)$.

## 2. Preliminaries

Let the Cartan connection of an $n-$dimensional Finsler space $F_n = (M_n, L)$ is given by the triad $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$, where $G^i_j = F^i_{jk}y^k$ and $C^i_{jk}$ is the associated Cartan tensor. If $X_i(x, y)$ be a covariant vector field then its $h-$ and $v-$ covariant derivatives with respect to the Cartan connection $C\Gamma$ are given by

$$X_{i|k} = \partial_k X_i - (\dot{\partial}_r X_i)G^i_k - X_r F^r_{ik} \tag{2.1}$$

and

$$X_i|_k = \dot{\partial}_k X_i - X_r C^r_{ik} \tag{2.2}$$

respectively. Here $\partial_k$ and $\dot{\partial}_k$ denote the partial derivatives with respect to $x^k$ and $y^k$ respectively, and $\partial_k$ and $\dot{\partial}_k$ stand for $\partial/\partial x^k$ and $\partial/\partial y^k$ respectively.

The components of the metric tensor $g_{ij}$ and the angular metric tensor $h_{ij}$ of the Finsler space $F_n = (M_n, L)$ are defined respectively by

$$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2 \tag{2.3}$$

and

$$h_{ij} = L\dot{\partial}_i\dot{\partial}_j L. \tag{2.4}$$

Differentiating (2.3) partially with respect to $y^k$, we obtain a tensor $C_{ijk}$ of type $(0, 3)$ defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}. \tag{2.5}$$

This tensor is called the Cartan tensor and its degree of homogeneity in $y^i$ is $-1$. The normalized supporting element $l_i = y_i/L$ satisfies $l_i = \dot{\partial}_i L$. From the equations (2.3) and (2.4), we obtain the relation

$$g_{ij} = h_{ij} + l_i l_j \tag{2.6}$$

among the metric tensor $g_{ij}$, the angular metric tensor $h_{ij}$ and the normalized supporting element $l_i$. The $h-$covariant derivatives and $v-$ covariant derivatives of $g_{ij}$, $h_{ij}$ and $l_i$ satisfy [9]

$$\begin{array}{llllll}\text{(a)} & g_{ij|k} = 0 & \text{(b)} & h_{ij|k} = 0 & \text{(c)} & L_{|i} = 0 \\ \text{(d)} & l^i_{|j} = 0 & \text{(e)} & l_i|_j = \frac{1}{L}h_{ij} & \text{(f)} & L|_i = l_i.\end{array} \tag{2.7}$$

Let $M_m(1 < m < n)$ be an $m-$dimensional subspace of the $n-$ dimensional manifold $M_n$ represented parametrically by the equations

$$x^i = x^i(u^\alpha) \qquad i = 1, 2, ...n; \quad \alpha = 1, 2, ...m, \tag{2.8}$$

where $u^\alpha$ denote the Gaussian cordinates on the subspace $M_m$.

Let $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ be the projection factors [3] and the matrix $||B^i_\alpha||$ of this projection factors be supposed to be of rank $m$. If $y^i$, the supporting element, is assumed to be tangential to the subspace $M_m$ then it can be written in terms of the projection factors as

$$y^i = B^i_\alpha(u)w^\alpha, \qquad \alpha = 1, 2, ...m. \tag{2.9}$$

Here $w = (w^\alpha)$ is assumed to be the supporting element at the point $(u^\alpha)$ of the subspace $M_m$. The metric $L(x, y)$ of the Finsler space $F_n = (M_n, L)$ induces the metric

$$\bar{L}(u, w) = L(x(u), y(u, w)) \tag{2.10}$$

on the subspace $M_m$. Thus, we obtain an $m-$dimensional Finsler subspace $F_m = (M_m, \bar{L}(u, w))$ of the space $F_n = (M_n, L)$.
Let $g_{\alpha\beta}(u, w)$ defined by

$$g_{\alpha\beta}(u, w) = \frac{1}{2}\frac{\partial^2\bar{L}^2}{\partial w^\alpha \partial w^\beta}, \tag{2.11}$$

be the metric tensor of the subspace $F_m$. Successive partial differentiations of (2.10) with respect to $w^\alpha$ and $w^\beta$ give

$$g_{\alpha\beta}(u, w) = g_{ij}(x, y)B^i_\alpha B^j_\beta. \tag{2.12}$$

A covariant vector $Y_i$ which satisfies the condition

$$Y_i B^i_\alpha(u) = 0 \tag{2.13}$$

is called normal to the subspace $F_m$. Clearly, these are $m$ equations for determinations of $n$ functions $Y_i$. So, there exist $(n - m)$ linearly independent and mutually orthogonal unit vectors $Y^i_{(a)}$, (say), satisfying the following conditions

$$g_{ij}B^i_\alpha Y^j_{(a)} = 0 \tag{2.14}$$

and

$$Y_i^{(a)} = g_{ij} Y^j_{(a)}, \tag{2.15}$$

where $(a) = m+1, m+2, ...n$. Further, (2.14) and (2.15) imply that

$$g_{ij} Y^i_{(a)} Y^j_{(b)} = \delta_{(a)(b)}, \qquad \{(a),(b) = m+1, m+2, ......n\}. \tag{2.16}$$

If $B_i^\alpha(u,w)$ is the reciprocal of the projection factors $B^i_\alpha$ defined by

$$B_i^\alpha(u,w) = g^{\alpha\beta} B^j_\beta\, g_{ij}, \tag{2.17}$$

then, in view of (2.12), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha. \tag{2.18}$$

From (2.14), (2.15), (2.16), (2.17) and (2.18), we have

$$\begin{array}{ll} \text{(a)}\ \ B^i_\alpha Y_i^{(a)} = 0 & \text{(b)}\ \ Y^i_{(a)} B^\alpha_i = 0 \\ \text{(c)}\ \ Y^i_{(a)} Y_i^{(b)} = \delta^{(a)}_{(b)} & \text{(d)}\ \ B^i_\alpha B^\alpha_j + Y^i_{(a)} Y_j^{(a)} = \delta^i_j. \end{array} \tag{2.19}$$

If the triad $IC\Gamma = (F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$, where $G^\alpha_\beta = F^\alpha_{\beta\gamma} y^\gamma$, is the induced Cartan connection of the Finsler subspace $F_m$ then the second fundamental tensor $H^{(a)}_{\alpha\beta}$ and the normal curvature vector $H^{(a)}_\alpha$ with respect to induced Cartan connection $IC\Gamma$ can be expressed in the direction of the normal vector $Y^i_{(a)}$ by

$$H^{(a)}_{\alpha\beta} = Y_i^{(a)} (B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta) + M^{(a)}_{(b)\alpha} H^{(b)}_\beta \tag{2.20}$$

and

$$H^{(a)}_\alpha = Y_i^{(a)} (B^i_{0\alpha} + F^i_{0j} B^j_\alpha) \tag{2.21}$$

respectively, where

$$M^{(a)}_{(b)\alpha} = C^i_{jk} Y_i^{(a)} Y^j_{(b)} B^k_\alpha, \tag{2.22}$$

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \qquad B^i_{0\alpha} = v^\beta B^i_{\beta\alpha}. \tag{2.23}$$

The contraction of (2.20) by $v^\alpha$ gives us

$$H^{(a)}_{0\beta} = v^\alpha H^{(a)}_{\alpha\beta} = H^{(a)}_\beta. \tag{2.24}$$

## 3. The Finsler space $F_n^* = (M_n, L^*)$

Let $v^i = v^i(x^j)$ be a contravariant vector field in a Finsler space $F_n = (M_n, L)$ satisfying the condition (1.2).
Differentiating $v_i = g_{ir}(x,y)v^r$ partially with respect to $y^j$ and using the condition (1.2), we obtain

(3.1) $$L(\dot{\partial}_j v_i) = 2\rho h_{ij}.$$

Consider an $n-$ dimensional Finsler space $F_n^* = (M_n, L^*)$ whose metric function $L^*(x,y)$ is obtained from the metric of the space $F_n$ by the transformation (1.1). Throughout the paper, the geometric objects related to $F_n^*$ will be asterisked $*$.

Differentiating (1.1) partially with respect to $y^k$ and using (3.1), we get

(3.2) $$L_k^* = L_k + v_k,$$

where $L_k^* = \dot{\partial}_k L^*$.
The normalized supporting element $l_i^*$ of $F_n^*$ can be written as

(3.3) $$l_k^* = l_k + v_k.$$

Differentiating (3.2) partially with respect to $y^j$ and using (3.1), we obtain

(3.4) $$L_{jk}^* = L_{jk} + 2\rho h_{jk}/L,$$

where $L_{jk} = \dot{\partial}_j \dot{\partial}_k L$ and $L_{jk}^* = \dot{\partial}_j \dot{\partial}_k L^*$.
Using (2.4) in (3.4), we get

(3.5) $$L_{jk}^* = (1 + 2\rho)L_{jk}.$$

Partial Differentiation of (3.5) with respect to $y^i$ gives

(3.6) $$L_{ijk}^* = (1 + 2\rho)L_{ijk},$$

where $L_{ijk} = \dot{\partial}_k L_{ij}$ and $L_{ijk}^* = \dot{\partial}_k L_{ij}^*$.
In view of (2.4), the angular metric tensor $h_{ij}^*$ of the Finsler space $F_n^*$ is given as

(3.7) $$h_{ij}^* = \tau(1 + 2\rho)h_{ij},$$

where $\tau = \frac{L^*}{L}$.
From (3.3), (3.7) and (2.6), the fundamental metric tensor $g_{ij}^*$ of the Finsler space $F_n^*$ is given by

(3.8) $$g_{ij}^* = \tau(1 + 2\rho)g_{ij} + v_i v_j + l_i v_j + v_i l_j + (1 - \tau(1 + 2\rho))l_i l_j.$$

Keeping $g^{*ij} g_{jk}^* = \delta_k^i$ in view, the inverse metric tensor $g^{*ij}$ is given by

(3.9) $$g^{*ij} = \frac{1}{\tau(1 + 2\rho)} g^{ij} - \frac{1}{\tau^2(1 + 2\rho)}(l^i v^j + v^i l^j) + \frac{\tau(1 + 2\rho) + v^2 - 1}{\tau^3(1 + 2\rho)} l^i l^j$$

in the Finsler space $F_n^*$.
Differentiating (3.8) partially with respect to $y^k$, we obtain the Cartan tensor $C^*_{ijk}$ of $F_n^*$ as

$$C^*_{ijk} = \tau(1+2\rho)C_{ijk} + \frac{(1+2\rho)}{2L}(h_{jk}c_i + h_{ki}c_j + h_{ij}c_k). \tag{3.10}$$

Here $c_i = v_i - (\tau - 1)l_i$. Thus, we have

**Theorem 3.1.** *The components of the metric tensor $g^*_{ij}$, the inverse metric tensor $g^{*ij}$, the angular metric tensor $h^*_{ij}$ and the Cartan tensor $C^*_{ijk}$ of the Finsler space $F_n^*$ whose metric $L^*$ is obtained from the metric $L$ of the Finsler space $F_n$ by (1.1), are given by (3.8), (3.9), (3.7) and (3.10) respectively.*

## 4. The Cartan connection of the Finsler space $F_n^* = (M_n, L^*)$

In this section, we find the Cartan connection of the Finsler space $F_n^* = (M_n, L^*)$. Since $L_{ij}$ is $h-$covariant constant with respect to the Cartan connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$, i.e. $L_{ij|k} = 0$, (2.1) gives

$$\partial_k L_{ij} = L_{ijr}F^r_{0k} + L_{rj}F^r_{ik} + L_{ir}F^r_{jk}, \tag{4.1}$$

where $L_{ijk} = \dot{\partial}_k L_{ij}$ and $F^r_{0k} = F^r_{ik}y^i = G^r_k$.
Differentiating (3.5) covariantly with respect to $x^i$, we get

$$\partial_i L^*_{jk} = (1+2\rho)\partial_i L_{jk} + 2\rho_i L_{jk}, \tag{4.2}$$

where $\partial_i \rho = \rho_i$.
In view of (4.1), (3.5) and (3.6), (4.2) can be written as

$$(1+2\rho)\{L_{jkr}(F^{*r}_{0i} - F^r_{0i}) + L_{kr}(F^{*r}_{ji} - F^r_{ji}) + L_{jr}(F^{*r}_{ki} - F^r_{ki})\} = 2\rho_i L_{jk}. \tag{4.3}$$

Let $D^i_{jk}$ be the difference of the connections $F^{*i}_{jk}$ and $F^i_{jk}$, i.e.

$$D^i_{jk} = F^{*i}_{jk} - F^i_{jk}. \tag{4.4}$$

In view of (4.4), (4.3) reduces to

$$(1+2\rho)(L_{jkr}D^r_{0i} + L_{rk}D^r_{ji} + L_{jr}D^r_{ki}) = 2\rho_i L_{jk}. \tag{4.5}$$

Cyclic rotation of the indices $i$, $j$ and $k$ gives

$$(1+2\rho)(L_{kir}D^r_{0j} + L_{ri}D^r_{jk} + L_{kr}D^r_{ij}) = 2\rho_j L_{ki}, \tag{4.6}$$

and

$$(1+2\rho)(L_{ijr}D^r_{0k} + L_{rj}D^r_{ki} + L_{ir}D^r_{jk}) = 2\rho_k L_{ij}. \tag{4.7}$$

Using $L_{k|j} = 0$ in (2.1), we have

$$\partial_j L_k = L_{kr} F^r_{0j} + L_r F^r_{jk}. \tag{4.8}$$

Differentiating (3.2) partially with respect to $x^j$ and then using (4.8) and (2.1), we obtain

$$L^*_{kr} F^{*}_{0j} + L^* F^*_{kj} = (1+2\rho) L_{kr} F^r_{0j} + (L_r + v_r) F^r_{kj} + v_{k|j}. \tag{4.9}$$

In view of (3.2), (3.3), (3.5), and (4.4), (4.9) reduces to

$$(1+2\rho) L_{kr} D^r_{0j} + (l_r + v_r) D^r_{kj} = v_{k|j}. \tag{4.10}$$

Here subscript '0'denotes the contraction by the supporting element $y^k$.
Now, we propose

**Theorem 4.1.** *If $F_n = (M_n, L)$ and $F^*_n = (M_n, L^*)$ are two Finsler spaces over the same manifold $M_n$ and $L^*(x, y)$ is given by (1.1), then the Cartan connection of $F^*_n$ is completely determined by (4.5) and (4.10).*

To prove Theorem 4.1, first we have to prove the following lemma

**Lemma 4.1.** *The system of equations*

$$\begin{aligned} &(a)\quad (1+2\rho) L_{jk} A^k = B_j \\ &(b)\quad (l_k + v_k) A^k = B \end{aligned} \tag{4.11}$$

*has a unique solution*

$$A^k = (1+2\rho)^{-1} L B^k + \tau^{-1}(B - (1+2\rho) L B_v) l^k, \tag{4.12}$$

*where $\tau = (L^*/L)$, $B_v = B_i v^i$ and $B^i = g^{ij} B_j$ for given $B_j$ and $B$ such that* $B_j l^j = 0$.

*Proof.* From $h_{jk} = L L_{jk}$ and (2.6), (4.11(a)) can be written as

$$g_{jk} A^k = (1+2\rho)^{-1} L B_j + l_j (l_k A^k). \tag{4.13}$$

Transvecting (4.13) with $v^j$, we get

$$v_k A^k = (1+2\rho)^{-1} L B_v + (\tau - 1) l_k A^k, \tag{4.14}$$

where $B_v = B_i v^i$.
In view of (4.11(b)), (4.14) implies

$$l_k A^k = \tau^{-1}(B - (1+2\rho)^{-1} L B_v). \tag{4.15}$$

Thus, from (4.13) and (4.15), we have

$$g_{jk} A^k = (1+2\rho)^{-1} L B_j + \tau^{-1}(B - (1+2\rho)^{-1} L B_v) l_j. \tag{4.16}$$

Contraction of (4.16) by $g^{ij}$ gives the solution

$$A^i = (1+2\rho)^{-1} L B^i + \tau^{-1}(B - (1+2\rho)^{-1} L B_v) l^i \tag{4.17}$$

of the given system, where $B^i = g^{ij} B_j$ and $\tau = \frac{L^*}{L}$. □

Thus, we are in a position to prove Theorem 4.1. We complete the proof of Theorem 4.1 if we find the value of $D^i_{jk}$.
We will find the value of $D^i_{jk}$ in three steps. In the first step, we will find the value of $D^i_{00}$, in the second step we will find $D^i_{j0}$ and in the last step we will find $D^i_{jk}$.

In view of (4.10), we have

$$(1+2\rho)L_{jr}D^r_{0k}+(l_r+v_r)D^r_{jk}=v_{j|k}. \tag{4.18}$$

Simultaneously adding and subtracting (4.18) and (4.10), we get

$$(1+2\rho)(L_{jr}D^r_{0k}+L_{kr}D^r_{0j})+2(l_r+v_r)D^r_{jk}=v_{j|k}+v_{k|j} \tag{4.19}$$

and

$$(1+2\rho)(L_{jr}D^r_{0k}-L_{kr}D^r_{0j})=v_{j|k}-v_{k|j}. \tag{4.20}$$

If we take

$$(a)\quad v_{j|k}+v_{k|j}=2s_{jk} \qquad (b)\quad v_{j|k}-v_{k|j}=2t_{jk}, \tag{4.21}$$

then (4.19) and (4.20) become

$$(1+2\rho)(L_{jr}D^r_{0k}+L_{kr}D^r_{0j})+2(l_r+v_r)D^r_{jk}=2s_{jk}. \tag{4.22}$$

$$(1+2\rho)(L_{jr}D^r_{0k}-L_{kr}D^r_{0j})=2t_{jk}. \tag{4.23}$$

Subtracting (4.7) from the addition of (4.5) and (4.6), we have

$$2L_{kr}D^r_{ij}+(1+2\rho)(L_{jkr}D^r_{0i}+L_{kir}D^r_{0j}-L_{ijr}D^r_{0k})=2(\rho_iL_{jk}+\rho_jL_{ki}-\rho_kL_{ij}). \tag{4.24}$$

Transvection of (4.22), (4.23) and (4.24) with $y^k$ and utilization of $L_{ij}y^j=0$ give us

$$(1+2\rho)L_{jr}D^r_{00}+2(l_r+v_r)D^r_{0j}=2s_{j0}, \tag{4.25}$$

$$(1+2\rho)L_{jr}D^r_{00}=2t_{j0}, \tag{4.26}$$

$$(1+2\rho)(L_{jr}D^r_{0i}+L_{ir}D^r_{0j}+L_{ijr}D^r_{00})=2\rho_0L_{ij}. \tag{4.27}$$

Transvecting (4.25) with $y^j$, we find

$$(l_r+v_r)D^r_{00}=s_{00}. \tag{4.28}$$

Applying Lemma 4.1 in (4.26) and (4.28), we get

$$D^r_{00}=\frac{L}{L^*(1+2\rho)}\{2L^*t^r_0+l^r((1+2\rho)s_{00}-2Lt_{v0})\}. \tag{4.29}$$

Here $t^r_0 = g^{ir}t_{i0}$ and $t_{v0} = t_{i0}v^i$.
Putting $k$ in palace of $i$ in (4.27) and then adding with (4.23), we find

$$L_{jr}D^r_{0k} = \frac{1}{2(1+2\rho)}(2t_{jk} + 2\rho_0 L_{jk} - \frac{1}{2}(1+2\rho)L_{jkr}D^r_{00}). \tag{4.30}$$

If we take

$$\frac{1}{2(1+2\rho)}(2t_{jk} + 2\rho_0 L_{jk} - \frac{1}{2}(1+2\rho)L_{jkr}D^r_{00}) = A_{jk}, \tag{4.31}$$

then (4.30) reduces to

$$L_{jr}D^r_{0k} = A_{jk}. \tag{4.32}$$

From (4.29) and (4.31), we have

$$A_{jk} = \frac{1}{2L^*(1+2\rho)}\{2L^*(t_{jk} - LL_{jkr}t^r_0) + L_{jk}((1+2\rho)s_{00} - 2Lt_{v0} + 2L^*\rho_0)\}. \tag{4.33}$$

This shows that $A_{jk}$ is known.
If we write

$$s_{k0} - \frac{1}{2}(1+2\rho)L_{kr}D^r_{00} = A_k, \tag{4.34}$$

the equation (4.25) assumes the form

$$(l_r + v_r)D^r_{0k} = A_k. \tag{4.35}$$

Putting the value of $D^r_{00}$ from (4.29) in (4.34), we get

$$A_k = s_{k0} - LL_{kr}t^r_o. \tag{4.36}$$

In view of Lemma 4.1, the system of equations (4.32) and (4.35) give

$$D^r_{0k} = \frac{L}{L^*}(L^*A^r_k + l^r(A_k - LA_{vk})), \tag{4.37}$$

where $A_{vk} = A_{jk}v^j$ and $A^r_k = g^{ri}A_{ik}$.
Now we can express (4.22) in the form

$$(l_r + v_r)D^r_{jk} = B_{jk}, \tag{4.38}$$

where

$$B_{jk} = s_{jk} - \frac{1}{2}(1+2\rho)(L_{jr}D^r_{0k} + L_{kr}D^r_{0j}). \tag{4.39}$$

The equation (4.24) may be written as

$$L_{ir}D^r_{jk} = B_{ijk}, \tag{4.40}$$

where

(4.41) $B_{ijk} = (\rho_j L_{ki} + \rho_k L_{ij} - \rho_i L_{jk}) - \frac{1}{2}(1+2\rho)(L_{kir} D^r_{0j} + L_{ijr} D^r_{0k} - L_{jkr} D^r_{0i}).$

Putting the value of $D^r_{0i}$ from (4.37), we see that $B_{jk}$ and $B_{ijk}$ are known quantities. Applying the Lemma 4.1, for the system of equations (4.38) and (4.40), we obtain

(4.42) $$D^r_{jk} = \frac{L}{L^*}\{L^* B^r_{jk} + l^r(B_{jk} - L B_{vjk})\},$$

where $B^r_{jk} = g^{ir} B_{ijk}$ and $B_{vjk} = B_{ijk} v^i$. The quantities $B_{jk}$ and $B_{ijk}$ are given by respectively (4.39) and (4.41) together with (4.37).
Thus, the proof is completed.

## 5. Subspace of the Finsler space $F^*_n = (M_n, L^*)$

Suppose $F_m$ and $F^*_m$ are the subspaces of the Finsler spaces $F_n$ and $F^*_n$ respectively.
Contracting (2.14) by $w^\alpha$ and using (2.9) and $y^i g_{ij} = y_j$, we obtain

(5.1) $$y_j Y^j_{(a)} = 0.$$

Again contracting (3.8) with $Y^i_{(a)} Y^j_{(b)}$ and using (2.16), (5.1) and $\tau = (L^*/L)$, we have

(5.2) $$g^*_{ij} Y^i_{(a)} Y^j_{(b)} = \frac{L^*}{L}(1+2\rho)\delta_{(a)(b)} + v_i Y^i_{(a)} v_k Y^k_{(b)}.$$

Fixing the index $(a)$ and taking $(a) = (b)$ in (5.2), we get

(5.3) $$g^*_{ij} Y^i_{(a)} Y^j_{(a)} = \frac{L^*}{L}(1+2\rho) + (v_r Y^r_{(a)})^2.$$

Hence

(5.4) $$g^*_{ij}\left(\frac{Y^i_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho) + (v_r Y^r_{(a)})^2}}\right)\left(\frac{Y^j_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho) + (v_r Y^r_{(a)})^2}}\right) = 1.$$

From (5.4), it is clear that $\left(\frac{Y^i_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho) + (v_r Y^r_{(a)})^2}}\right)$ is a unit vector.
Contracting (3.8) by $B^i_\alpha Y^j_{(a)}$ and using (2.14) & (5.1), we obtain

(5.5) $$g^*_{ij} B^i_\alpha Y^j_{(a)} = (v_j Y^j_{(a)})(v_i + l_i) B^i_\alpha.$$

From (5.5), we can say that $Y^j_{(a)}$ is normal to the subspace $F^*_m$ if and only if the condition

(5.6) $$(v_j Y^j_{(a)})(v_i + l_i) B^i_\alpha = 0$$

holds. This implies at least one of the conditions $v_j Y^j_{(a)} = 0$ and $(v_i + l_i)B^i_\alpha = 0$. Suppose $(v_i + l_i)B^i_\alpha = 0$. Contracting this condition by $w^\alpha$ and using $B^i_\alpha w^\alpha = y^i$, we get $L + v_i y^i = L^* = 0$ which is not possible. Hence, we have the first condition, i.e.

$$v_j Y^j_{(a)} = 0. \tag{5.7}$$

Thus, the vector $Y^j_{(a)}$ is normal to the subspace $F^*_m$ if and only if the vector $v_j$ is tangent to the subspace $F_m$. From (5.4), (5.5) and (5.7), we find that $\left(\frac{Y^j_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho)}}\right)$ is a unit normal vector of the subspace $F^*_m$. In view of (2.14), (2.15) and (2.16), we obtain

$$Y^{*j}_{(a)} = \frac{Y^j_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho)}}. \tag{5.8}$$

Contracting (3.8) by $Y^{*i}_{(a)}$ and using (2.15), we obtain

$$Y^{*(a)}_j = \sqrt{\frac{L^*}{L}(1+2\rho)}Y^{(a)}_i. \tag{5.9}$$

Thus, we have

**Theorem 5.1.** *Let $F^*_n = (M_n, L^*)$ be a Finsler space whose metric function $L^*$ is obtained from the metric function $L$ of the Finsler space $F_n = (M_n, L)$ by the transformation (1.1). If $F^*_m$ and $F_m$ are $m-$ dimensional subspaces of $F^*_n$ and $F_n$ respectively, then the vector $v_i(x, y)$ satisfying the condition (1.2) is tangent to $F_m$ if and only if any vector $Y^i_{(a)}$ normal to $F_m$ is also normal to $F^*_m$.*

Let us assume that the transformation given by (1.1) is projective. Then, we have

$$G^{*i}_j = G^i_j + p_j y^i + p\delta^i_j, \tag{5.10}$$

where $p$ is a function of directional argument and of homogeneity one in $y^i$ and $\dot{\partial}_j p = p_j$.
Contracting (4.4) by $y^k$ and using $F^i_{jk} y^k = G^i_j$, we get

$$G^{*i}_j = G^i_j + D^i_{0j}. \tag{5.11}$$

Thus, from (5.10) and (5.11), we have

$$D^i_{0j} = p_j y^i + p\delta^i_j. \tag{5.12}$$

Contracting (5.12) by $B^j_\alpha Y^{(a)}_i$, using (2.15) and (2.19(a)), we obtain

$$Y^{(a)}_i D^i_{0j} B^j_\alpha = 0. \tag{5.13}$$

If every geodesic in the subspace $F_m$ with respect to the induced metric is also a geodesic in the enveloping space $F_n$, the subspace $F_m$ is called totally geodesic subspace and this type of space is characterized by

$$H^{(a)}_\alpha = 0, \tag{5.14}$$

i.e. its normal curvature vector vanishes identically.
In view of (2.21), the normal curvature vector $H^{*(a)}_\alpha$ of the subspace $F^*_m$ in the direction $Y^{(a)}_i$ is given by

$$H^{*(a)}_\alpha = Y^{*(a)}_i (B^i_{0\alpha} + G^{*i}_j B^j_\alpha). \tag{5.15}$$

Using (5.9) and (5.11) in (5.15), we get

$$H^{*(a)}_\alpha = \sqrt{\frac{L^*}{L}(1+2\rho)}\; H^{(a)}_\alpha + \sqrt{\frac{L^*}{L}(1+2\rho)}\; Y^{(a)}_i D^i_{0j} B^j_\alpha = 0. \tag{5.16}$$

In view of (5.13), (5.16) reduces to

$$H^{*(a)}_\alpha = \sqrt{\frac{L^*}{L}(1+2\rho)}\; H^{(a)}_\alpha. \tag{5.17}$$

$\sqrt{\frac{L^*}{L}(1+2\rho)} \neq 0$ for $\sqrt{\frac{L^*}{L}(1+2\rho)} = 0$ implies $\rho = -(1/2)$, a contradiction to the fact that $\rho$ is a function of $x^i$. Hence, we conclude from (5.17) that $H^{(a)}_\alpha$ vanishes if and only if $H^{*(a)}_\alpha$ vanishes as $\sqrt{\frac{L^*}{L}(1+2\rho)} \neq 0$. Therefore, we have

**Theorem 5.2.** *If a contravariant vector field $v^i$ satisfying the condition (1.2) is tangent to a subspace $F_m$ of the space $F_n$ then $F_m$ is totally geodesic if and only if the subspace $F^*_m$ of $F^*_n$ is totally geodesic.*

If there exists a projective change between the Finsler spaces $F_n = (M_n, L)$ and $F^*_n = (M_n, L^*)$ over the underlying manifold $M_n$ such that the later space is locally Minkowskian then the space $F_n$ is said to be projectively flat.

In 2005, M. Kitayama [7] showed that a totally geodesic subspace of a projectively flat Finsler space is also projective.

Makato Matsumoto [8] proved that a Finsler space $F^n (n > 3)$ is projectively flat if the Weyl torsion tensor $W^i_{jk}$ and the Douglas tensor $D^i_{jkh}$ vanish, i.e.

$$(a) \quad W^i_{jk} = 0, \qquad (b) \quad D^i_{jkh} = 0 \tag{5.18}$$

and the converse part is also true.
Under a projective change $W^{*i}_{jk} = W^i_{jk}$ and $D^{*i}_{jkh} = D^i_{jkh}$, i.e. both tensors are invariant [10]. Thus, we conclude following theorem, from Theorem 5.2 in view of (5.18)

**Theorem 5.3.** *Let the metric function $L^*$ of a Finsler space $F^*_n = (M_n, L^*)$ be obtained from the metric function $L$ of a projective flat Finsler space $F_n = (M_n, L), n > 3$, by the transformation (1.1). If a subspace $F_m$ of $F_n$ is totally geodesic and the vector field $v^i$ satisfying (1.2) is tangential to it, then the corresponding subspace $F^*_m$ of $F^*_n$ is projectively flat.*

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Vivek Kumar Pandey
Department of Mathematics
University of Allahabad
Allahabad-211002, India
vivekpandey9415@gmail.com

P. N. Pandey
Department of Mathematics
University of Allahabad
Allahabad-211002, India
pnpiaps@gmail.com