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REMARKS ON METALLIC WARPED PRODUCT MANIFOLDS

Adara M. Blaga and Cristina E. Hreţcanu

Abstract. We characterize the metallic structure on the product of two metallic manifolds in terms of metallic maps and provide a necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic. We discuss a particular case of the product manifolds and we construct an example of the metallic warped product Riemannian manifold.
Keywords: Riemannian manifold, metallic warped product, projection mapping.

1. Introduction

Starting from a polynomial structure, which was generally defined by S. I. Goldberg, K. Yano and N. C. Petridis in ([8],[9]), we consider a polynomial structure on an m-dimensional Riemannian manifold (M, g), called by us a *metallic structure* ([6],[11],[7],[12]), determined by a $(1, 1)$-tensor field J which satisfies the equation:

$$J^2 = pJ + qI, \tag{1.1}$$

where I is the identity operator on the Lie algebra of vector fields on M identified with the set of smooth sections $\Gamma(T(M))$ (and we will simply denote $X \in T(M)$) with p and q are non-zero natural numbers). From the definition, we easily get the recurrence relation:

$$J^{n+1} = g_{n+1} \cdot J + g_n \cdot I, \tag{1.2}$$

where $(\{g_n\}_{n\in\mathbb{N}^*})$ is the generalized secondary Fibonacci sequence defined by $g_{n+1} = pg_n + qg_{n-1}$, $n \geq 1$ with $g_0 = 0$, $g_1 = 1$ and p, $q \in \mathbb{N}^*$.

If (M, g) is a Riemannian manifold endowed with a metallic structure J such that the Riemannian metric g is J-compatible (i.e. $g(JX, Y) = g(X, JY)$, for any $X, Y \in T(M)$), then (M, g, J) is called a *metallic Riemannian manifold*. In this case:

$$g(JX, JY) = pg(X, JY) + qg(X, Y), \tag{1.3}$$

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for any $X, Y \in T(M)$.

It is known ([13]) that an almost product structure F on M induces two metallic structures:

$$J_{\pm} = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I \tag{1.4}$$

and, conversely, every metallic structure J on M induces two almost product structures:

$$F_{\pm} = \pm\frac{2}{2\sigma_{p,q} - p}J - \frac{p}{2\sigma_{p,q} - p}I, \tag{1.5}$$

where $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number, which is a positive solution of the equation $x^2 - px - q = 0$, for p and q non-zero natural numbers.

In particular, if the almost product structure F is compatible with the Riemannian metric, then J_+ and J_- are metallic Riemannian structures.

On a metallic manifold (M, J) there exist two complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$ corresponding to the projection operators l and m ([13]) given by:

$$l = -\frac{1}{2\sigma_{p,q} - p}J + \frac{\sigma_{p,q}}{2\sigma_{p,q} - p}I, \quad m = \frac{1}{2\sigma_{p,q} - p}J + \frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p}I. \tag{1.6}$$

The analogue concept of a locally product manifold is considered in the context of metallic geometry. Precisely, we say that the metallic Riemannian manifold (M, g, J) is *locally metallic* if J is parallel with respect to the Levi-Civita connection associated to g.

2. Metallic warped product Riemannian manifolds

2.1. Warped product manifolds

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimensions n and m, respectively. Denote by p_1 and p_2 the projection maps from the product manifold $M_1 \times M_2$ onto M_1 and M_2 and by $\widetilde{\varphi} := \varphi \circ p_1$ the lift to $M_1 \times M_2$ of a smooth function φ on M_1. In this case, we call M_1 *the base* and M_2 *the fiber* of $M_1 \times M_2$. The unique element $\widetilde{X}$ of $T(M_1 \times M_2)$ that is p_1-related to $X \in T(M_1)$ and to the zero vector field on M_2 will be called the *horizontal lift of* X and the unique element $\widetilde{V}$ of $T(M_1 \times M_2)$ that is p_2-related to $V \in T(M_2)$ and to the zero vector field on M_1 will be called the *vertical lift of* V. Also denote by $\mathcal{L}(M_1)$ the set of all horizontal lifts of vector fields on M_1 and by $\mathcal{L}(M_2)$ the set of all vertical lifts of vector fields on M_2.

For $f > 0$ a smooth function on M_1, consider the Riemannian metric on $M_1 \times M_2$:

$$\widetilde{g} := p_1^* g_1 + (f \circ p_1)^2 p_2^* g_2. \tag{2.1}$$

Definition 2.1. ([4]) The product manifold of M_1 and M_2 together with the Riemannian metric $\widetilde{g}$ defined by (2.1) is called *the warped product* of M_1 and M_2 by the warping function f [and it is denoted by $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$].

Note that if f is constant (equal to 1), the warped product becomes the usual product of the Riemannian manifolds.

For $(x,y) \in \widetilde{M}$, we shall identify $X \in T(M_1)$ with $(X_x, 0_y) \in T_{(x,y)}(\widetilde{M})$ and $Y \in T(M_2)$ with $(0_x, Y_y) \in T_{(x,y)}(\widetilde{M})$ ([3]).

The projection mappings of $T(M_1 \times M_2)$ onto $T(M_1)$ and $T(M_2)$, respectively, denoted by $\pi_1 =: Tp_1$ and $\pi_2 =: Tp_2$ verify:

$$\pi_1 + \pi_2 = I, \quad \pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2, \quad \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0. \tag{2.2}$$

The Riemannian metric of the warped product manifold $\widetilde{M} = M_1 \times_f M_2$ equals to:

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2), \tag{2.3}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$ and we notice that the leaves $M_1 \times \{y\}$, for $y \in M_2$, are totally geodesic submanifolds of $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g})$.

If we denote by $\widetilde{\nabla}$, ${}^{M_1}\nabla$, ${}^{M_2}\nabla$ the Levi-Civita connections on $\widetilde{M}$, M_1 and M_2, we know that for any $X_1, Y_1 \in T(M_1)$ and $X_2, Y_2 \in T(M_2)$ ([14]):

$$\widetilde{\nabla}_{(X_1,X_2)}(Y_1, Y_2) = ({}^{M_1}\nabla_{X_1} Y_1 - \frac{1}{2} g_2(X_2, Y_2) \cdot grad(f^2),$$

$${}^{M_2}\nabla_{X_2} Y_2 + \frac{1}{2f^2} X_1(f^2) Y_2 + \frac{1}{2f^2} Y_1(f^2) X_2). \tag{2.4}$$

In particular:

$$\widetilde{\nabla}_{(X,0)}(0, Y) = \widetilde{\nabla}_{(0,Y)}(X, 0) = (0, X(\ln(f))Y).$$

Let R, R_{M_1}, R_{M_2} be the Riemannian curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{R_{M_1}}$, $\widetilde{R_{M_2}}$ the lift on $\widetilde{M}$ of R_{M_1} and R_{M_2}. Then:

Lemma 2.1. *([4]) If $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$ is the warped product of M_1 and M_2 by the warping function f and $m > 1$, then for any X, Y, $Z \in \mathcal{L}(M_1)$ and any U, V, $W \in \mathcal{L}(M_2)$, we have:*

1. *$R(X,Y)Z = \widetilde{R_{M_1}}(X,Y)Z$;*
2. *$R(U,X)Y = \frac{1}{f} H^f(X,Y)U$, where H^f is the lift on $\widetilde{M}$ of $Hess(f)$;*
3. *$R(X,Y)U = R(U,V)X = 0$;*
4. *$R(U,V)W = \widetilde{R_{M_2}}(U,V)W - \frac{|grad(f)|^2}{f^2}[g(U,W)V - g(V,W)U]$;*
5. *$R(X,U)V = \frac{1}{f} g(U,V) \widetilde{\nabla}_X grad(f)$.*

Let S, S_{M_1}, S_{M_2} be the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}. Then:

Lemma 2.2. *([4]) If $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$ is the warped product of M_1 and M_2 by the warping function f and $m > 1$, then for any X, $Y \in \mathcal{L}(M_1)$ and any V, $W \in \mathcal{L}(M_2)$, we have:*

1. $S(X,Y) = \widetilde{S_{M_1}}(X,Y) - \frac{m}{f} H^f(X,Y)$, *where H^f is the lift on $\widetilde{M}$ of $Hess(f)$;*
2. $S(X,V) = 0$;
3. $S(V,W) = \widetilde{S_{M_2}}(V,W) - \left[\frac{\Delta(f)}{f} + (m-1)\frac{|grad(f)|^2}{f^2}\right] g(V,W)$.

Remark 2.1. For the case of product Riemannian manifolds:

i) the Riemannian curvature tensors verify ([2]):

$$R(\widetilde{X},\widetilde{Y})\widetilde{Z} = (R_1(X_1,Y_1)Z_1, R_2(X_2,Y_2)Z_2), \tag{2.5}$$

for any $\widetilde{X} = (X_1,X_2), \widetilde{Y} = (Y_1,Y_2), \widetilde{Z} = (Z_1,Z_2) \in T(M_1 \times M_2)$, where R, R_1 and R_2 are respectively the Riemannian curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2);

ii) the Ricci curvature tensors verify ([2]):

$$S(\widetilde{X},\widetilde{Y}) = S_1(X_1,Y_1) + S_2(X_2,Y_2), \tag{2.6}$$

for any $\widetilde{X} = (X_1,X_2), \widetilde{Y} = (Y_1,Y_2) \in T(M_1 \times M_2)$, where S, S_1 and S_2 are respectively the Ricci curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2).

Note that the Riemannian curvature tensor of a locally metallic Riemannian manifold has the following properties:

Proposition 2.1. *If (M, g, J) is a locally metallic Riemannian manifold, then for any $X, Y, Z \in T(M)$:*

$$R(X,Y)JZ = J(R(X,Y)Z), \tag{2.7}$$

$$R(JX,Y) = R(X,JY), \tag{2.8}$$

$$R(JX,JY) = qR(JX,Y) + pR(X,Y), \tag{2.9}$$

$$R(J^{n+1}X,Y) = g_{n+1} \cdot R(JX,Y) + g_n \cdot R(X,Y), \tag{2.10}$$

where $(\{g_n\}_{n \in \mathbb{N}^*})$ is the generalized secondary Fibonacci sequence defined by $g_{n+1} = pg_n + qg_{n-1}$, $n \geq 1$ with $g_0 = 0$, $g_1 = 1$ and p, $q \in \mathbb{N}^*$.

Proof. The locally metallic condition $\nabla J = 0$ is equivalent to $\nabla_X JY = J(\nabla_X Y)$, for any $X, Y \in T(M)$ and (2.7) follows from the definition of R. The relations (2.8), (2.9) and (2.10) follow from the symmetries of R and from the recurrence relation $J^{n+1} = g_{n+1} \cdot J + g_n \cdot I$. $\square$

Theorem 2.1. *If* $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *is a locally metallic Riemannian warped product manifold, then* M_2 *is* $\widetilde{J}$*-invariant submanifold of* $\widetilde{M}$.

Proof. Applying (2.8) from Proposition 2.1 and Lemma 2.1, we obtain $H^f(X,Y)\widetilde{J}U = H^f(\widetilde{J}X,Y)U$, for any X, $Y \in \mathcal{L}(M_1)$ and any $U \in \mathcal{L}(M_2)$, where H^f is the lift on $\widetilde{M}$ of $Hess(f)$. □

2.2. Metallic warped product Riemannian manifolds

2.2.1. Metallic Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by the projection operators

The endomorphism

$$F := \pi_1 - \pi_2 \tag{2.11}$$

verifies $F^2 = I$ and $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$, thus F is an almost product structure on $M_1 \times M_2$.

By using relations (1.4) we can construct on $M_1 \times M_2$ two metallic structures, given by:

$$\widetilde{J}_\pm = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I. \tag{2.12}$$

Also from $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$ follows $\widetilde{g}(\widetilde{J}_\pm\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}_\pm\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.2. *There exist two metallic Riemannian structures* $\widetilde{J}_\pm$ *on* $(\widetilde{M}, \widetilde{g})$ *given by:*

$$\widetilde{J}_\pm = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I, \tag{2.13}$$

where $\widetilde{M} = M_1 \times_f M_2$ *and* $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$, *for any* $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$.

Note that for $\widetilde{J}_+ = \frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I$, the projection operators are $\pi_1 = m$, $\pi_2 = l$ and for $\widetilde{J}_- = -\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I$ we have $\pi_1 = l$, $\pi_2 = m$, where m and l are given by (1.6).

Remark 2.2. If we denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\widetilde{M}$ with respect to $\widetilde{g}$, we obtain that $\widetilde{\nabla}F = 0$ [hence $\widetilde{\nabla}\widetilde{J}_\pm = 0$ and so $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J}_\pm)$ is a locally metallic Riemannian manifold].

For the case of a product Riemannian manifold $(\widetilde{M} = M_1 \times M_2, \widetilde{g})$ with $\widetilde{g}$ given by (2.1) for $f = 1$ and $\widetilde{J}_\pm$ defined by (2.13), we deduce that the Riemann curvature of $\widetilde{\nabla}$ verifies (2.7), (2.8), (2.9), (2.10).

2.2.2. Metallic Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by two metallic structures on M_1 and M_2

For any vector field $\widetilde{X} = (X, Y) \in T(M_1 \times M_2)$ we define a linear map $\widetilde{J}$ of tangent space $T(M_1 \times M_2)$ into itself by:

$$\widetilde{J}\widetilde{X} = (J_1 X, J_2 Y), \tag{2.14}$$

where J_1 and J_2 are two metallic structures defined on M_1 and M_2, respectively, with $J_i^2 = pJ_i + qI$, $i \in \{1, 2\}$ and p, q non zero natural numbers. It follows that:

$$\widetilde{J}^2\widetilde{X} = \widetilde{J}(J_1 X, J_2 Y) = (J_1^2 X, J_2^2 Y) = p(J_1 X, J_2 Y) + q(X, Y). \tag{2.15}$$

Also from $g_i(J_i X_i, Y_i) = g_i(X_i, J_i Y_i)$, $i \in \{1, 2\}$, we get $\widetilde{g}(\widetilde{J}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.3. *If (M_1, g_1, J_1) and (M_2, g_2, J_2) are metallic Riemannian manifolds with $J_i^2 = pJ_i + qI$, $i \in \{1, 2\}$ and p, q non-zero natural numbers, then there exists a metallic Riemannian structure $\widetilde{J}$ on $(\widetilde{M}, \widetilde{g})$ given by:*

$$\widetilde{J}\widetilde{X} = (J_1 X, J_2 Y), \tag{2.16}$$

for any $\widetilde{X} = (X, Y) \in T(\widetilde{M})$, where $\widetilde{M} = M_1 \times_f M_2$ and $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$, for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$.

For the case of a product Riemannian manifold $(\widetilde{M} = M_1 \times M_2, \widetilde{g})$ with $\widetilde{g}$ given by (2.1) for $f = 1$ and $\widetilde{J}_\pm$ defined by (2.13), we deduce that the Riemann curvature of $\widetilde{\nabla}$ verifies (2.7), (2.8), (2.9), (2.10).

Now we shall obtain a characterization of the metallic structure on the product of two metallic manifolds (M_1, J_1) and (M_2, J_2) in terms of *metallic maps*, that are smooth maps $\Phi : M_1 \to M_2$ satisfying:

$$T\Phi \circ J_1 = J_2 \circ T\Phi.$$

In a way similar to the case of Golden manifolds ([5]), we have:

Proposition 2.2. *The metallic structure $\widetilde{J} := (J_1, J_2)$ given by (2.16) is the only metallic structure on the product manifold $\widetilde{M} = M_1 \times M_2$ such that the projections p_1 and p_2 on the two factors M_1 and M_2 are metallic maps.*

A necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic will be further provided:

Theorem 2.4. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by (2.1) and* $\widetilde{J}$ *given by (2.16)) be the warped product of the locally metallic Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. Then* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *is locally metallic if and only if:*

$$\begin{cases} (df^2 \circ J_1) \otimes I = df^2 \otimes J_2 \\ g_2(J_1 \cdot, \cdot) \cdot grad(f^2) = g_2(\cdot, \cdot) \cdot J_1(grad(f^2)) \end{cases}.$$

Proof. Replacing the expression of $\widetilde{\nabla}$ from (2.4), under the assumptions ${}^{M_1}\nabla J_1 = 0$ and ${}^{M_2}\nabla J_2 = 0$ we obtain the conclusion. □

Theorem 2.5. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by (2.1) and* $\widetilde{J}$ *(2.16)) be the warped product of the metallic Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. If* M_1 *and* M_2 *have* J_1*- and* J_2*-invariant Ricci tensors, respectively (i.e.* $Q_{M_i} \circ J_i = J_i \circ Q_{M_i}$*,* $i \in \{1, 2\}$*), then* $\widetilde{M}$ *has* $\widetilde{J}$*-invariant Ricci tensor if and only if*

$$Hess(f)(J_1 \cdot, \cdot) - Hess(f)(\cdot, J_1 \cdot) \in \{0\} \times T(M_2).$$

Proof. If we denote by S, S_{M_1}, S_{M_2} the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}, by using Lemma 2.2, for any X, $Y \in \mathcal{L}(M_1)$, we have:

$$S(\widetilde{J}X, Y) = \widetilde{S_{M_1}}(\widetilde{J}X, Y) - \frac{m}{f} H^f(\widetilde{J}X, Y) = \widetilde{S_{M_1}}(X, \widetilde{J}Y) - \frac{m}{f} H^f(\widetilde{J}X, Y) =$$

$$= S(X, \widetilde{J}Y) + \frac{m}{f} H^f(X, \widetilde{J}Y) - \frac{m}{f} H^f(\widetilde{J}X, Y),$$

where H^f is the lift on $\widetilde{M}$ of $Hess(f)$. Also, for any V, $W \in \mathcal{L}(M_2)$, we obtain:

$$S(\widetilde{J}V, W) = \widetilde{S_{M_2}}(\widetilde{J}V, W) - [f\Delta(f) + (m-1)|grad(f)|^2] g_2(J_2 V, W) =$$

$$= \widetilde{S_{M_2}}(V, \widetilde{J}W) - [f\Delta(f) + (m-1)|grad(f)|^2] g_2(V, J_2 W) = S(V, \widetilde{J}W).$$

□

Example 2.1. Consider $M := \{(u, \alpha_1, \alpha_2, ..., \alpha_n), u > 0, \alpha_i \in [0, \frac{\pi}{2}], i \in \{1, ..., n\}\}$ and let $f : M \to \mathbb{R}^{2n}$ be the immersion given by:

$$f(u, \alpha_1, ..., \alpha_n) := (u \cos \alpha_1, u \sin \alpha_1, ..., u \cos \alpha_n, u \sin \alpha_n). \tag{2.17}$$

We can find a local orthonormal frame of the submanifold M in $\mathbb{R}^{2n}$, spanned by the vectors:

$$Z_0 = \sum_{i=1}^{n} \left(\cos \alpha_i \frac{\partial}{\partial x_i} + \sin \alpha_i \frac{\partial}{\partial y_i} \right), \quad Z_i = -u \sin \alpha_i \frac{\partial}{\partial x_i} + u \cos \alpha_i \frac{\partial}{\partial y_i}, \tag{2.18}$$

for any $i \in \{1, ..., n\}$.

We remark that $\|Z_0\|^2 = n$, $\|Z_i\|^2 = u^2$, $Z_0 \perp Z_i$, for any $i \in \{1, ..., n\}$ and $Z_i \perp Z_j$, for $i, j \in \{1, ..., n\}$ with $i \neq j$.

In the next considerations, we shall denote by:

$$(X^1, Y^1, ..., X^k, Y^k, X^{k+1}, Y^{k+1}, ..., X^n, Y^n) =: (X^i, Y^i, X^j, Y^j),$$

for any $k \in \{2, ..., n-1\}$, $i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$.

Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the $(1,1)$-tensor field defined by:

$$J(X^i, Y^i, X^j, Y^j) := (\sigma X^i, \sigma Y^i, \overline{\sigma} X^j, \overline{\sigma} Y^j), \tag{2.19}$$

for any $k \in \{2, ..., n-1\}$, $i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$, where $\sigma := \sigma_{p,q}$ is the metallic number and $\overline{\sigma} = 1 - \sigma$. It is easy to verify that J is a metallic structure on $\mathbb{R}^{2n}$ (i.e. $J^2 = pJ + qI$).

Moreover, the metric $\overline{g}$, given by the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n}$, is J-compatible and $(\mathbb{R}^{2n}, \overline{g}, J)$ is a metallic Riemannian manifold.

From (2.18) we get:

$$JZ_0 = \sigma \sum_{i=1}^{k} \left(\cos \alpha_i \frac{\partial}{\partial x_i} + \sin \alpha_i \frac{\partial}{\partial y_i} \right) + \overline{\sigma} \sum_{j=k+1}^{n} \left(\cos \alpha_j \frac{\partial}{\partial x_j} + \sin \alpha_j \frac{\partial}{\partial y_j} \right)$$

and, for any $k \in \{2, ..., n-1\}, i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$ we get:

$$JZ_i = \sigma Z_i, \quad JZ_j = \overline{\sigma} Z_j.$$

We can verify that JZ_0 is orthogonal to $span\{Z_1, ..., Z_n\}$ and

$$\cos(\widehat{JZ_0, Z_0}) = \frac{k\sigma + (n-k)\bar{\sigma}}{\sqrt{n(k\sigma^2 + (n-k)\bar{\sigma}^2)}}. \tag{2.20}$$

Consider the manifolds M_1 and M_2 with $TM_1 = span\{Z_0\}$ and $TM_2 = span\{Z_1, ..., Z_n\}$. Then $M := M_1 \times_u M_2$ with the Riemannian metric tensor $g = ndu^2 + u^2 \sum_{i=1}^{n} d\alpha_i^2$ is a warped product (semi-slant) submanifold of the metallic Riemannian manifold $(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle, J)$.

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Adara M. Blaga
West University of Timişoara
Timişoara, 300223, Romania
adarablaga@yahoo.com

Cristina E. Hreţcanu
Stefan cel Mare University of Suceava
Suceava, 720229, Romania
criselenab@yahoo.com