FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 1 (2018), 125–140
https://doi.org/10.22190/FUMI1801125N

# ON COMMON FIXED POINTS THEOREMS FOR ORDERED $F$-CONTRACTIONS WITH APPLICATION

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**Abstract.** We study the conditions for existence of a unique common fixed point of two ordered $F$-contractions defined on an ordered partial metric space; in particular, we present a common fixed point result for a pair of ordered $F$-contractions satisfying a generalized rational type contractive condition and discuss its consequences. It is remarked that the notion of an $F$-contraction in partial metric spaces is more general than that in metric spaces. As application of our findings, we demonstrate the existence of common solution of the system of Volterra type integral equations.

**Keywords**: common fixed point; ordered F-contraction; partial metric space

## 1. Introduction

The Banach Contraction Principle plays a fundamental role in metric fixed point theory. A number of efforts have been made to enrich and generalize Banach Contraction Principle (see for example [1, 2, 3, 4, 6, 12, 16]). One of these generalizations is for $F$-contraction presented by Wardowski [18]: every $F$-contraction defined on a complete metric space has a unique fixed point. So the concept of an $F$-contraction proved to be a milestone in fixed point theory. Numerous research papers on $F$-contractions have been published (see for instant [5, 6, 11, 15, 17]). Recently, Minak *et al.*[13] presented a fixed point result for Círíc type generalized $F$-contraction.
In this paper, by introducing ordered $F$-contractions which satisfy a generalized rational type contractive condition defined on an ordered partial metric space, we investigate a unique common fixed point of ordered $F$-contractions. As an application, we show the existence of the common solution of operators satisfying an implicit integral equation.

The main result in this paper generalizes the results given by Durmaz [11] (Corollary 3.2) and Wardowski [18] (Corollary 3.4).

Received September 21, 2017; accepted March 31, 2018
2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H10, 54H25

## 2. Preliminaries

We denote the set of natural numbers, rational numbers, $(-\infty,+\infty)$, $(0,+\infty)$ and $[0,+\infty)$ by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$, respectively.

Matthews generalized the notion of metric, as follows:

**Definition 2.1.** [12] Let $M$ be a nonempty set. If the mapping $p: M\times M\to\mathbb{R}_0^+$ satisfies the following properties,

$(p_1)$ $r_1=r_2 \Leftrightarrow p(r_1,r_1)=p(r_1,r_2)=p(r_2,r_2)$,

$(p_2)$ $p(r_1,r_1)\le p(r_1,r_2)$,

$(p_3)$ $p(r_1,r_2)=p(r_2,r_1)$,

$(p_4)$ $p(r_1,r_3)\le p(r_1,r_2)+p(r_2,r_3)-p(r_2,r_2)$

for all $r_1,r_2,r_3\in M$, then $p$ is called a partial metric on $M$ and the pair $(M,p)$ is known as a partial metric space.

**Example 2.1.** Define the mapping $p:\mathbb{R}_0^+\times\mathbb{R}_0^+\to\mathbb{R}_0^+$ by $p(r_1,r_2)=\max\{r_1,r_2\}$. It is easy to check that $p$ satisfies $(p_1)-(p_4)$ and hence $p$ is a partial metric on $\mathbb{R}_0^+$. Note that $p$ does not define a metric on $\mathbb{R}_0^+$, since, $p(a,a)=a$ for all $a\in\mathbb{R}_0^+$.

Example 2.1 gives a classical example of a partial metric. We present a new non-trivial example of a partial metric as follows:

**Example 2.2.** We define $p:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ by

$$p(r,s)=\begin{cases} 1 & \text{if } r=s\in\mathbb{R}-\mathbb{Q};\\ \frac{3}{2} & \text{if } r\ne s\in\mathbb{R}-\mathbb{Q};\\ \frac{1}{3} & \text{if } r=s\in\mathbb{Q};\\ 1+\frac{1}{m}+\frac{1}{n} & \text{if } r=r_m, s=r_n \text{ and } m\ne n;\\ 1+\frac{1}{n} & \text{if } \{r,s\}\cap\mathbb{Q}=\{r_n\} \text{ and } \{r,s\}-\mathbb{Q}\ne\phi. \end{cases}$$

Clearly, $p$ satisfies $(p_1)-(p_3)$. To prove $p_4$, let $r,s,u\in\mathbb{R}-\mathbb{Q}$ and $m\ne n$. Then

$$\begin{aligned} p(r,s)+p(u,u) &\le p(r,u)+p(s,u);\\ p(r_n,s)+p(u,u) &= 2+\frac{1}{n}\le p(r_n,u)+p(s,u);\\ p(r_n,r_n)+p(u,u) &= \frac{4}{3}<p(r_n,u)+p(r_n,u);\\ p(r_m,r_n)+p(u,u) &= 2+\frac{1}{m}+\frac{1}{n}=p(r_m,u)+p(r_n,u); \end{aligned}$$

$$\begin{aligned}
p(r,s)+p(r_k,r_k) &< 2<p(r,r_k)+p(s,r_k);\\
p(r_n,s)+p(r_k,r_k) &= \frac{4}{3}+\frac{1}{n}<2+\frac{1}{n}+\frac{2}{k}=p(r_n,r_k)+p(s,r_k);\\
p(r_n,r_n)+p(r_k,r_k) &= \frac{2}{3}\leq p(r_n,r_k)+p(r_n,r_k);\\
p(r_m,r_n)+p(r_k,r_k) &= \frac{4}{3}+\frac{1}{m}+\frac{1}{n}\leq p(r_m,r_k)+p(r_n,r_k).
\end{aligned}$$

Note that $p$ is not a metric on $\mathbb{R}$.

Notice that a metric on a set $M$ is a partial metric $p$ such that $p(r,r)=0$ for all $r\in M$ and $p(r_1,r_2)=0$ implies $r_1=r_2$ ( using $(p_1)$ and $(p_2)$) but converse may not be true. The self distance $p(r,r)$ referred to as the size or weight of $r$, is a feature used to describe the amount of information contained in $r$.

Matthews [12] explored the following aspects of a partial metric $p$ on $M$:

(1) : The mapping $d: M\times M\to\mathbb{R}_0^+$ defined by

$$d(r_1,r_2)=2p(r_1,r_2)-p(r_1,r_1)-p(r_2,r_2) \tag{2.1}$$

for all $r_1,r_2\in M$ defines a metric on $M$ (called induced metric).

(2) : Open ball has the structure $B_p(r,\epsilon)=\{r_1\in M: p(r,r_1)<p(r,r)+\epsilon\}$ for all $r\in M$ and $\epsilon>0$.

(3) : Each partial metric $p$ on $M$ generates a $T_0$ topology $\mathcal{T}[p]$ on $M$. The base of topology $\mathcal{T}[p]$ consists of family of open balls

$$\{B_p(r,\epsilon)\ : r\in M,\ \epsilon>0\}.$$

(4) : A sequence $\{r_n\}_{n\in\mathbb{N}}$ in $(M,p)$ converges to a point $r\in M$ if and only if

$$p(r,r)=\lim_{n\to\infty}p(r,r_n).$$

(5) : A sequence $\{r_n\}_{n\in\mathbb{N}}$ in $(M,p)$ is called a Cauchy sequence if

$$\lim_{n,m\to\infty}p(r_n,r_m) \text{ exists and is finite.}$$

(6) : A partial metric space $(M,p)$ is said to be complete if every Cauchy sequence $\{r_n\}_{n\in\mathbb{N}}$ in $M$ converges, with respect to $\mathcal{T}[p]$, to a point $r\in M$ such that

$$p(r,r)=\lim_{n,m\to\infty}p(r_n,r_m).$$

Matthews [12] also evinced an analogue of Banach's fixed point theorem in partial metric spaces. This remarkable theorem led numerous authors to obtain various applicable fixed point results in partial metric spaces (see for example [2, 6, 7, 8, 14] and references therein).

The following lemma will be helpful in the sequel.

**Lemma 2.1.** [12]

(1) A sequence $\{r_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the partial metric space $(M,p)$ if and only if it is a Cauchy sequence in the metric space $(M,d)$.

(2) A partial metric space $(M,p)$ is complete if and only if the metric space $(M,d)$ is complete.

(3) A sequence $\{r_n\}_{n\in\mathbb{N}}$ in $M$ converges to a point $r\in M$, with respect to $\tau(d)$ if and only if $\lim_{n\to\infty} p(r,r_n)=p(r,r)=\lim_{n,m\to\infty} p(r_n,r_m)$.

(4) If $\lim_{n\to\infty} r_n=\upsilon$ such that $p(\upsilon,\upsilon)=0$,

$$\text{then } \lim_{n\to\infty} p(r_n,r)=p(\upsilon,r) \text{ for every } r\in M.$$

Wardowski [18] has introduced a useful class of mappings as follows:
Let $F:\mathbb{R}^+\to\mathbb{R}$ be a mapping satisfying the following conditions:

$(F_1)$ : F is strictly increasing;

$(F_2)$ : For each sequence $\{r_n\}$ of positive numbers, the following condition holds:

$$\lim_{n\to\infty} r_n=0 \text{ if and only if } \lim_{n\to\infty} F(r_n)=-\infty;$$

$(F_3)$ : There exists $\theta\in(0,1)$ such that $\lim_{\alpha\to 0^+}(\alpha)^\theta F(\alpha)=0$.

We denote the set of all mappings satisfying the conditions $(F_1)-(F_3)$ by $\Delta_F$.

**Example 2.3.** *Let $F:\mathbb{R}^+\to\mathbb{R}$ be defined by:*

*(a)* $F(r)=\ln(r)$.

*(b)* $F(r)=r+\ln(r)$.

*(c)* $F(r)=\ln(r^2+r)$.

*(d)* $F(r)=-\dfrac{1}{\sqrt{r}}$.

*It is easy to check that (a),(b),(c) and (d) are members of $\Delta_F$.*

**Definition 2.2.** [18] Let $(M,d)$ be a metric space. We say the mapping $T:M\to M$ is an $F_d$-contraction, if there exist $F\in\Delta_F$ and $\tau>0$ such that

$$d(T(r_1),T(r_2))>0 \Rightarrow \tau+F(d(T(r_1),T(r_2))\le F(d(r_1,r_2)), \text{ for all } r_1,r_2\in M.$$

**Definition 2.3.** Let $(M,p)$ be a partial metric space. We say the mapping $T:M\to M$ is an $F_p$-contraction, if there exist $F\in\Delta_F$ and $\tau>0$ such that

$$p(T(r_1),T(r_2))>0 \Rightarrow \tau+F(p(T(r_1),T(r_2))\le F(p(r_1,r_2)), \text{ for all } r_1,r_2\in M.$$

The following Example indicates that an $F_p$-contraction is more general than an $F_d$-contraction.

**Example 2.4.** Let $M = [0,1]$ and define partial metric by $p(r_1, r_2) = \max\{r_1, r_2\}$ for all $r_1, r_2 \in M$. The metric $d$ induced by partial metric $p$ is given by $d(r_1, r_2) = |r_1 - r_2|$ for all $r_1, r_2 \in M$. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ and $T$ by

$$T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0,1); \\ 0 & \text{if } r = 1 \end{cases}$$

Note that for all $r_1, r_2 \in M$ with $r_1 \leq r_2$ or $r_2 \leq r_1$, we have

$$\begin{aligned} \tau + F\left(p(T(r_1), T(r_2))\right) &\leq F\left(p(r_1, r_2)\right) \text{ implies} \\ \tau + F\left(\frac{r_1}{5}\right) &\leq F(r_1) \text{ or } \tau + F\left(\frac{r_2}{5}\right) \leq F(r_2). \end{aligned}$$

So $T$ is an $F_p$-contraction but $T$ is neither continuous nor an $F_d$-contraction. Indeed, for $r_1 = 1$ and $r_2 = \frac{5}{6}$, $d(T(r_1), T(r_2)) > 0$ and we have

$$\begin{aligned} \tau + F\left(d(T(r_1), T(r_2))\right) &\leq F\left(d(r_1, r_2)\right), \\ \tau + F\left(d(T(1), T(\frac{5}{6}))\right) &\leq F\left(d(1, \frac{5}{6})\right), \\ \tau + F\left(d(0, \frac{1}{6})\right) &\leq F\left(\frac{1}{6}\right), \\ \frac{1}{6} &< \frac{1}{6}, \end{aligned}$$

a contradiction for all possible values of $\tau$.

**Definition 2.4.** Let $(M \preceq)$ be a partially ordered set. We say the mappings $S, T\colon M \to M$ are weakly increasing, if $S(m) \preceq TS(m)$ and $T(m) \preceq ST(m)$ for all $m \in M$.

**Example 2.5.** Let $M = \mathbb{R}^+$ be endowed with usual order and usual topology. Let $S, T\colon M \to M$ be given by

$$S(m) = \begin{cases} m^{\frac{1}{2}} & \text{if } m \in [0,1] \\ m^2 & \text{if } m \in (1,\infty) \end{cases} \quad \text{and } T(m) = \begin{cases} m & \text{if } m \in [0,1] \\ 2m & \text{if } m \in (1,\infty) \end{cases}$$

Then, $S$ and$T$ are weakly increasing mappings.

## 3. Main Result

**Definition 3.1.** Let $(M, \preceq)$ be an ordered set and $p$ be a partial metric on $M$, then the triplet $(M, \preceq, p)$ is known as an ordered partial metric space. If $(M, p)$ is complete, then $(M, \preceq, p)$ is called an ordered complete partial metric space. Moreover, $(M, \preceq, p)$ is regular, if it satisfies the following condition:

$$\begin{cases} \text{If } \{r_n\} \subset M \text{ is a nondecreasing (nonincreasing) sequence with } r_n \to r, \\ \text{then } r_n \preceq r \ (r \preceq r_n) \text{ for all n} \end{cases}$$

**Definition 3.2.** Let $(M, \preceq, p)$ be an ordered partial metric space. Let

$$O = \{(\alpha, \beta) \in M \times M : \alpha \preceq \beta, p(S(\alpha), T(\beta)) > 0\}.$$

We say the weakly increasing mappings $S, T$ form a pair of generalized rational type ordered $F$-contractions, if there exist $F \in \Delta_F$ and $\tau > 0$ such that

$$\tau + F(p(S(\alpha), T(\beta))) \leq F(\mathcal{R}(\alpha, \beta)), \text{ for all } (\alpha, \beta) \in O \text{ and} \tag{3.1}$$

$$\mathcal{R}(\alpha, \beta) = \max \left\{ \begin{array}{l} p(\alpha, \beta), \dfrac{p(\alpha, S(\beta))p(\alpha, T(\beta))}{1 + p(\alpha, \beta)}, \\ \dfrac{p(\alpha, S(\alpha))p(\alpha, T(\beta))}{1 + p(S(\alpha), T(\beta))} \end{array} \right\}.$$

**Lemma 3.1.** *Let $(M, \preceq, p)$ be an ordered complete partial metric space and $S, T$ form a pair of generalized rational type ordered $F$-contractions. Then for all $i = 0, 1, 2, 3, ...$ $p(r_{2i}, r_{2i+1}) = 0$ implies $p(r_{2i+1}, r_{2i+2}) = 0$.*

*Proof.* Let $r_0 \in M$ be an initial point and take $r_1 = S(r_0)$ and $r_2 = T(r_1)$, then by induction we can construct an iterative sequence $r_n$ of points in $M$ such that, $r_{2i+1} = S(r_{2i})$ and $r_{2i+2} = T(r_{2i+1})$ where $i = 0, 1, 2, \ldots$. As there exists $r_0 \in M$ such that $r_0 \preceq S(r_0)$ and $S, T$ are weakly increasing self-mappings, we obtain

$$r_1 = S(r_0) \preceq TS(r_0) = T(r_1) = r_2 = T(r_1) \preceq ST(r_1) = S(r_2) = r_3.$$

Iteratively, we obtain

$$r_0 \preceq r_1 \preceq r_2 \preceq \cdots \preceq r_{n-1} \preceq r_n \preceq r_{n+1} \preceq \cdots.$$

We argue by contradiction that $p(r_{2i+1}, r_{2i+2}) > 0$. We note that

$$\begin{aligned} \mathcal{R}(r_{2i}, r_{2i+1}) &= \max \left\{ \begin{array}{l} p(r_{2i}, r_{2i+1}), \\ \dfrac{p(r_{2i}, S(r_{2i}))p(r_{2i+1}, T(r_{2i+1}))}{1 + p(r_{2i}, r_{2i+1})}, \\ \dfrac{p(r_{2i}, S(r_{2i}))p(r_{2i+1}, T(r_{2i+1}))}{1 + p(S(r_{2i}), T(r_{2i+1}))} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} p(r_{2i}, r_{2i+1}), \dfrac{p(r_{2i}, r_{2i+1})p(r_{2i+1}, r_{2i+2})}{1 + p(r_{2i}, r_{2i+1})}, \\ \dfrac{p(r_{2i}, r_{2i+1})p(r_{2i+1}, r_{2i+2})}{1 + p(r_{2i+1}, r_{2i+2})} \end{array} \right\} \\ &\leq \max \{0, p(r_{2i+1}, r_{2i+2})\} = p(r_{2i+1}, r_{2i+2}) \end{aligned}$$

Since $(r_{2i}, r_{2i+i}) \in O$, by inequality (3.1) we have

$$\begin{aligned} \tau + F(p(r_{2i+1}, r_{2i+2})) &= \tau + F(p(S(r_{2i}), T(r_{2i+1}))) \\ &\leq F(\mathcal{R}(r_{2i}, r_{2i+1})) \\ &\leq F(p(r_{2i+1}, r_{2i+2})) \end{aligned}$$

for all $i = 0, 1, 2, 3, ...$, which is a contradiction to $(F_1)$. Hence, $p(r_{2i+1}, r_{2i+2})) = 0$. □

The following theorem is our main result.

**Theorem 3.1.** *Let $(M, \preceq, p)$ be an ordered complete partial metric space and $S, T : M \to M$ be generalized rational type ordered $F$-contractions. If there exists $r_0 \in M$ such that $r_0 \preceq S(r_0)$ and either*

*(a) one of $S, T$ is continuous or*

*(b) $(M, \preceq, p)$ is regular.*

*Then $S, T$ have a common fixed point.*

*Proof.* **(a)** We begin with the following observation:

$\mathcal{R}(r_{2i}, r_{2i+1}) = 0$ if and only if $r_{2i} = r_{2i+1}$ is a common fixed point of $S, T$.

Let $\mathcal{R}(r_{2i}, r_{2i+1}) > 0$ for all $i = 0, 1, 2, 3, ....$ Arguing as in Lemma 3.1, we have

$$r_0 \preceq r_1 \preceq r_2 \preceq \cdots \preceq r_{n-1} \preceq r_n \preceq r_{n+1} \preceq \cdots .$$

Now if $p(S(r_{2i}), T(r_{2i+1})) = 0$, then using Lemma 3.1, we can conclude that $r_{2i}$ is a common fixed point of $S, T$. Let $p(S(r_{2i}), T(r_{2i+1})) > 0$, so, $(r_{2i}, r_{2i+1}) \in O$. By contractive condition (3.1), we get

$$\begin{aligned} \tau + F\left(p(r_{2i+1}, r_{2i+2})\right) &= \tau + F\left(p(S(r_{2i}), T(r_{2i+1}))\right) \\ &\leq F\left(\mathcal{R}(r_{2i}, r_{2i+1})\right), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \mathcal{R}(r_{2i}, r_{2i+1}) &= \max\left\{ \begin{array}{l} p(r_{2i}, r_{2i+1}), \\ \dfrac{p(r_{2i}, S(r_{2i}))p(r_{2i+1}, T(r_{2i+1}))}{1 + p(r_{2i}, r_{2i+1})}, \\ \dfrac{p(r_{2i}, S(r_{2i}))p(r_{2i+1}, T(r_{2i+1}))}{1 + p(S(r_{2i}), T(r_{2i+1}))} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} p(r_{2i}, r_{2i+1}), \dfrac{p(r_{2i}, r_{2i+1})p(r_{2i+1}, r_{2i+2})}{1 + p(r_{2i}, r_{2i+1})}, \\ \dfrac{p(r_{2i}, r_{2i+1})p(r_{2i+1}, r_{2i+2})}{1 + p(r_{2i+1}, r_{2i+2})} \end{array} \right\} \\ &\leq \max\left\{p(r_{2i}, r_{2i+1}), p(r_{2i+1}, r_{2i+2})\right\} \end{aligned}$$

If $\mathcal{R}(r_{2i}, r_{2i+1}) \leq p(r_{2i+1}, r_{2i+2})$, then by (3.2), we get a contradiction to $F_1$). Thus, for $\mathcal{R}(r_{2i}, r_{2i+1}) \leq p(r_{2i}, r_{2i+1})$, we have

$$F\left(p(r_{2i+1}, r_{2i+2})\right) \leq F\left(p(r_{2i}, r_{2i+1})\right) - \tau, \tag{3.3}$$

for all $i = 0, 1, 2, 3, ....$ As $(r_{2i+1} \preceq r_{2i+2}$ and $p(S(r_{2i+2}), T(r_{2i+1})) > 0$ because otherwise by Lemma 3.1, $r_{2i+1}$ is a common fixed point of $S$ and $T$. Thus,

$(r_{2i+1}, r_{2i+2}) \in O$ and

$$\begin{aligned}
\mathcal{R}(r_{2i+2}, r_{2i+1}) &= \max\left\{\begin{array}{l} p(r_{2i+2}, r_{2i+1}), \\ \dfrac{p(r_{2i+2}, S(r_{2i+2}))p(r_{2i+1}, T(r_{2i+1}))}{1+p(r_{2i+2}, r_{2i+1})}, \\ \dfrac{p(r_{2i+2}, S(r_{2i+2}))p(r_{2i+1}, T(r_{2i+1}))}{1+p(S(r_{2i+2}), T(r_{2i+1}))} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} p(r_{2i+2}, r_{2i+1}), \\ \dfrac{p(r_{2i+2}, r_{2i+3})p(r_{2i+1}, r_{2i+2})}{1+p(r_{2i+2}, r_{2i+1})}, \\ \dfrac{p(r_{2i+2}, r_{2i+3})p(r_{2i+1}, r_{2i+2})}{1+p(r_{2i+3}, r_{2i+2})} \end{array}\right\} \\
&\leq \max\left\{p(r_{2i+2}, r_{2i+1}), p(r_{2i+2}, r_{2i+3})\right\}
\end{aligned}$$

Again the case $\mathcal{R}(r_{2i+2}, r_{2i+1}) \leq p(r_{2i+2}, r_{2i+3})$ is not possible. So, for the other case, contractive condition (3.1) implies

$$F\left(p(r_{2i+2}, r_{2i+3})\right) < F\left(p(r_{2i+1}, r_{2i+2})\right) - \tau, \tag{3.4}$$

for all $i = 0, 1, 2, 3, ....$ By (3.3) and (3.4), we have

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_{n-1}, r_n)\right) - \tau, \tag{3.5}$$

for all $n \in \mathbb{N}$. Repeating these steps, we get

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_0, r_1)\right) - n\tau. \tag{3.6}$$

By (3.6), we obtain $\lim_{n\to\infty} F\left(p(r_n, r_{n+1})\right) = -\infty$ and $(F_2)$, implies

$$\lim_{n\to\infty} p(r_n, r_{n+1}) = 0. \tag{3.7}$$

By the property $(F_3)$, there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} \left((p(r_n, r_{n+1}))^{\kappa} F\left(p(r_n, r_{n+1})\right)\right) = 0. \tag{3.8}$$

Multiplying (3.6) by $(p(r_n, r_{n+1}))^{\kappa}$, we obtain

$$(p(r_n, r_{n+1}))^{\kappa}\left(F\left(p(r_n, r_{n+1})\right) - F\left(p(r_0, x_1)\right)\right) \leq -\left(p(r_n, r_{n+1})\right)^{\kappa} n\tau \leq 0. \tag{3.9}$$

Using (3.7), (3.8) and letting $n \to \infty$ in (3.9), we have

$$\lim_{n\to\infty} \left(n\left(p(r_n, r_{n+1})\right)^{\kappa}\right) = 0. \tag{3.10}$$

There exists $n_1 \in \mathbb{N}$, such that $n\left(p(r_n, r_{n+1})\right)^{\kappa} \leq 1$ for all $n \geq n_1$ or,

$$p(r_n, r_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{3.11}$$

By (3.11), for $m > n \geq n_1$, we have

$$\begin{aligned} p(r_n, r_m) &\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + ... + p(r_{m-1}, r_m) \\ &- \sum_{j=n+1}^{m-1} p(r_j, r_j) \\ &\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + ... + p(r_{m-1}, r_m) \\ &= \sum_{i=n}^{m-1} p(r_i, r_{i+1}) \leq \sum_{i=n}^{\infty} p(r_i, r_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty} p(r_n, r_m) = 0$. Hence $\{r_n\}$ is a Cauchy sequence in $(M, p)$. By Lemma 2.1(1), $\{r_n\}$ is a Cauchy sequence in $(M, d)$. Since $(M, d)$ is a complete metric space, there exists $\upsilon \in M$ such that $\lim_{n\to\infty} d(r_n, \upsilon) = 0$. By Lemma 2.1(3)

$$\lim_{n\to\infty} p(\upsilon, r_n) = p(\upsilon, \upsilon) = \lim_{n,m\to\infty} p(r_n, r_m). \tag{3.12}$$

Since $\lim_{n,m\to\infty} p(r_n, r_m) = 0$, by (3.12) we have

$$p(\upsilon, \upsilon) = 0 = \lim_{n\to\infty} p(\upsilon, r_n). \tag{3.13}$$

By equation (3.13) it follows that $r_{2n+1} \to \upsilon$ and $r_{2n+2} \to \upsilon$ as $n \to \infty$ with respect to $\tau(p)$. Suppose that $T$ is continuous then

$$\upsilon = \lim_{n\to\infty} r_n = \lim_{n\to\infty} r_{2n+1} = \lim_{n\to\infty} r_{2n+2} = \lim_{n\to\infty} T(r_{2n+1}) = T(\lim_{n\to\infty} r_{2n+1}) = T(\upsilon).$$

Now we show that $\upsilon = S(\upsilon)$. Suppose on contrary that $p(\upsilon, S(\upsilon)) > 0$. Since $\upsilon \preceq \upsilon$, by contractive condition (3.1), we obtain

$$\begin{aligned} \tau + F(p(\upsilon, S(\upsilon))) &= \tau + F(p(S(\upsilon), T(\upsilon))) \\ &\leq F(\mathcal{R}(\upsilon, \upsilon)) \\ F(p(\upsilon, S(\upsilon))) &< F(p(\upsilon, S(\upsilon))), \end{aligned}$$

a contradiction to $(F_1)$. Thus, $p(\upsilon, S(\upsilon)) = 0$. The axioms $(p_1)$ and $(p_2)$ implies $\upsilon = S(\upsilon)$. Consequently, $S(\upsilon) = T(\upsilon) = \upsilon$ that is $(S, T)$ have a common fixed point $\upsilon$.

**(b)** In the other case, if $M$ is regular then we have $r_n \preceq \upsilon$ for all $n \in \mathbb{N}$. To show that $\upsilon$ is a common fixed point of $S, T$, we split the proof into two cases.

**Case. 1**

If $r_n = \upsilon$ for some $n$, then there exists $i_0 \in \mathbb{N}$ such that $r_{2i_0} = \upsilon$ and $S(\upsilon) = S(r_{2i_0}) = r_{2i_0+1} \preceq \upsilon$ also $\upsilon = r_{2i_0} \preceq r_{2i_0+1} = S(\upsilon)$. Thus, $\upsilon = S(\upsilon)$ and by (3.1) we have $\upsilon = T(\upsilon)$. This completes the proof.

**Case. 2**

If $r_n \neq \upsilon$ for all $n$, suppose that $p(\upsilon, S(\upsilon)) > 0$. Since $\lim_{n\to\infty} r_{2i} = \upsilon$, there exists $\eta \in \mathbb{N}$ such that

$$p(r_{2i+1}, S(\upsilon)) > 0 \text{ and } p(r_{2i}, \upsilon) < \frac{p(\upsilon, S(\upsilon))}{2} \text{ for all } i \geq \eta.$$

$$\begin{aligned} \mathcal{R}(r_{2i}, \upsilon) &= \max \left\{ \begin{array}{l} p(r_{2i}, \upsilon), \\ \dfrac{p(r_{2i}, S(r_{2i}))p(\upsilon, T(\upsilon))}{1 + p(r_{2i}, \upsilon)}, \\ \dfrac{p(r_{2i}, S(r_{2i}))p(\upsilon, T(\upsilon))}{1 + p(S(r_{2i}), T(\upsilon))} \end{array} \right\}, \\ \mathcal{R}(r_{2i}, \upsilon) &\leq \frac{p(\upsilon, S(\upsilon))}{2} \text{ for all } i \geq \eta \end{aligned}$$

As $(r_{2i}, \upsilon) \in O$. By contractive condition (3.1), we have

$$\begin{aligned} \tau + F(p(r_{2i+1}, S(\upsilon))) &= \tau + F(p(S(r_{2i}), T(\upsilon))) \\ &\leq F(\mathcal{R}(r_{2i}, \upsilon)), \\ F(p(\upsilon, S(\upsilon))) &< F(\frac{p(\upsilon, S(\upsilon))}{2}) \text{ as } i \to \infty, \end{aligned}$$

a contradiction to $(F_1)$. Therefore, $p(\upsilon, S(\upsilon)) = 0$. The axioms $(p_1)$ and $(p_2)$ implies $\upsilon = S(\upsilon)$ and hence by (3.1), we have $\upsilon = T(\upsilon)$. Thus, the mappings $S$ and $T$ have a common fixed point $\upsilon$. □

We denote set of common fixed points of $S, T$ by $C_{fp}$.

**Theorem 3.2.** *In addition to the assumptions in Theorem 3.1, if $C_{fp}$ is a chain, then it is singleton set (common fixed point is unique).*

*Proof.* As $\upsilon \in C_{fp}$ and if $\omega$ is another common fixed point of $S, T$, then $\omega \preceq \upsilon$, also $p(S(\upsilon), T(\omega)) > 0$ (other wise $\upsilon = \omega$) so, $(\upsilon, \omega) \in O$ and the contractive condition (3.1) implies

$$\begin{aligned} \tau + F(p(\upsilon, \omega)) &= \tau + F(p(S(\upsilon), T(\omega))) \\ &\leq F(\mathcal{R}(\upsilon, \omega)) = F(p(\upsilon, \omega)) \end{aligned}$$

which is a contradiction to $(F_1)$. Hence, $\upsilon = \omega$ and $\upsilon$ is a unique common fixed point of mappings $(S, T)$. □

**Theorem 3.3.** *In addition to the assumptions in Theorem 3.1, if there exists $z$ in $M$ such that every element in the orbit $O_T(z) = \{z, T(z), T^2(z), \ldots\}$ is comparable to the element(s) in $C_{fp}$. Then common fixed point is unique.*

*Proof.* As $\upsilon \in C_{fp}$ and if $\omega$ is another common fixed point of $S, T$, then there exists an element $z \in M$ such that every element in $O_T(z) = \{z, T(z), T^2(z), \ldots\}$

is comparable to $\upsilon$ and $\omega$. Thus, $(T^{n-1}(z), S^{n-1}(\upsilon))$ and $(T^{n-1}(z), S^{n-1}(\omega))$ are elements of $O$ for each $n \geq 1$. By (3.1), we have

$$\begin{aligned} \tau + F(p(\upsilon, T^n(z))) &= \tau + F(p(S^n(\upsilon), T^n(z)) \\ &\leq F(\mathcal{R}\left(S^{n-1}(\upsilon), T^{n-1}(z)\right)) \end{aligned} \tag{3.14}$$

As

$$\begin{aligned} \mathcal{R}\left(S^{n-1}(\upsilon), T^{n-1}(z)\right) &= \max \left\{ \begin{array}{l} p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right), \\ \dfrac{p\left(S^{n-1}(\upsilon), S^n(\upsilon)\right) p\left(T^{n-1}(z), T^n(z)\right)}{1 + p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right)}, \\ \dfrac{p\left(S^{n-1}(\upsilon), S^n(\upsilon)\right) p\left(T^{n-1}(z), T^n(z)\right)}{1 + p\left(S^n(\upsilon), T^n(z)\right)} \end{array} \right\} \\ &= p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right) = p\left(\upsilon, T^{n-1}(z)\right) \end{aligned}$$

thus, the inequality (3.14) implies $\{p(\upsilon, T^n(z))\}$ is a non-negative decreasing sequence which in turn converges to 0. Similarly, we can show that $\{p(\omega, T^n(z))\}$ is a non-negative decreasing sequence converges to 0. Consequently, $\upsilon = \omega$. □

Let

$$\xi = \{(\alpha, \beta) \in M \times M : \alpha \preceq \beta, p(T(\alpha), T(\beta)) > 0\}.$$

**Definition 3.3.** Let $(M, \preceq, p)$ be an ordered partial metric space. We say the nondecreasing mapping $T : M \to M$ is a generalized rational type ordered $F$-contraction, if there exist $F \in \Delta_F$ and $\tau > 0$ such that

$$\tau + F(p(T(\alpha), T(\beta))) \leq F\left(\mathcal{R}_1(\alpha, \beta)\right), \tag{3.15}$$

for all $(\alpha, \beta) \in \xi$ and

$$\mathcal{R}_1(\alpha, \beta) = \max \left\{ \begin{array}{l} p(\alpha, \beta), \dfrac{p(\alpha, T(\alpha))p(\beta, T(\beta))}{1 + p(\alpha, \beta)}, \\ \dfrac{p(\alpha, T(\alpha))p(\beta, T(\beta))}{1 + p(T(\alpha), T(\beta))} \end{array} \right\}.$$

**Corollary 3.1.** *Let $(M, \preceq, p)$ be an ordered complete partial metric space and $T : M \to M$ be a generalized rational type ordered $F$-contraction. If there exists $r_0 \in M$ such that $r_0 \preceq T(r_0)$ and either*

*(a) $T$ is continuous or*

*(b) $(M, \preceq, p)$ is regular.*

*Then $T$ has a fixed point.*

*Proof.* Set $S = T$ in Theorem 3.1. □

**Corollary 3.2.** *[11] Let $(M, \preceq, d)$ be an ordered complete metric space and $T : M \to M$ be an ordered $F$-contraction. Let $T$ is nondecreasing mapping and there exists $r_0 \in M$ such that $r_0 \preceq T(r_0)$. If $T$ is continuous or $M$ is regular, then $T$ has a fixed point.*

*Proof.* Set $\mathcal{R}_1(\alpha, \beta) = p(\alpha, \beta)$, $p(\alpha, \alpha) = 0$ for all $\alpha, \beta \in M$ in Corollary 3.1. □

**Remark 3.1.** The Theorem 3.1 is valid if $\mathcal{R}(\alpha, \beta)$ is given any value from following.

$$\begin{aligned}
\mathcal{R}(\alpha,\beta) &= \frac{p(\alpha,S\alpha)\cdot p(\beta,T\beta)}{1+p(\alpha,\beta)};\\
\mathcal{R}(\alpha,\beta) &= p(\alpha,S\alpha);\\
\mathcal{R}(\alpha,\beta) &= p(\beta,T\beta);\\
\mathcal{R}(\alpha,\beta) &= \max\left\{p(\alpha,\beta),\frac{p(\alpha,S\alpha)\cdot p(\beta,T\beta)}{1+p(\alpha,\beta)}\right\};\\
\mathcal{R}(\alpha,\beta) &= \max\left\{p(\alpha,\beta),p(\alpha,S\alpha)\right\};\\
\mathcal{R}(\alpha,\beta) &= \max\left\{p(\alpha,\beta),p(\beta,T\beta)\right\};\\
\mathcal{R}(\alpha,\beta) &= \max\left\{\frac{p(\alpha,S\alpha)\cdot p(\beta,T\beta)}{1+p(\alpha,\beta)},p(\alpha,S\alpha)\right\};\\
\mathcal{R}(\alpha,\beta) &= \max\left\{\frac{p(\alpha,S\alpha)\cdot p(\beta,T\beta)}{1+p(\alpha,\beta)},p(\beta,T\beta)\right\};\\
\mathcal{R}(\alpha,\beta) &= \max\left\{p(\alpha,S\alpha),p(\beta,T\beta)\right\};\\
\mathcal{R}(\alpha,\beta) &= \max\left\{p(\alpha,\beta),\frac{p(\alpha,S\alpha)\cdot p(\beta,T\beta)}{1+p(\alpha,\beta)},p(\alpha,S\alpha)\right\};\\
\mathcal{R}(\alpha,\beta) &= \max\left\{p(\alpha,\beta),\frac{p(\alpha,S\alpha)\cdot p(\beta,T\beta)}{1+p(\alpha,\beta)},p(\beta,T\beta)\right\};\\
\mathcal{R}(\alpha,\beta) &= \max\left\{p(\alpha,\beta),p(\alpha,S\alpha),p(\beta,T\beta)\right\}.
\end{aligned}$$

We extend the Definition 3.2 for all $\alpha, \beta \in M$ as follows:

**Definition 3.4.** Let $(M, p)$ be partial metric space. We say the mappings $S, T$ form a pair of generalized rational type $F$-contractions. If there exist $F \in \Delta_F$ and $\tau > 0$ such that

$$\tau + F(p(S(\alpha), T(\beta))) \leq F\left(\mathcal{R}(\alpha, \beta)\right), \text{ for all } \alpha, \beta \in M \text{ and}$$

$$\mathcal{R}(\alpha,\beta) = \max\left\{\begin{array}{l} p(\alpha,\beta), \dfrac{p(\alpha,S(\beta))p(\alpha,T(\beta))}{1+p(\alpha,\beta)}, \\ \dfrac{p(\alpha,S(\alpha))p(\alpha,T(\beta))}{1+p(S(\alpha),T(\beta))} \end{array}\right\}.$$

**Theorem 3.4.** *Let $(M, p)$ be complete partial metric space and $S, T : M \to M$ be generalized rational type $F$-contractions. If one of $S$ and $T$ is continuous, then $S$ and $T$ have a common fixed point.*

*Proof.* The arguments follow the same lines as in proof of Theorem 3.1. □

**Corollary 3.3.** *Let $(M, p)$ be complete partial metric space and $T : M \to M$ be generalized rational type $F$-contraction. Then $T$ has a unique fixed point.*

*Proof.* Set $S = T$ in Theorem 3.4. □

**Corollary 3.4.** [18] Let $(M, d)$ be a complete metric space and $T : M \to M$ be an $F$-contraction. Then $T$ has a unique fixed point $\upsilon \in M$ and for every $r_0 \in M$ the sequence $\{T^n(r_0)\}$ for all $n \in \mathbb{N}$ is convergent to $\upsilon$.

*Proof.* Set $\mathcal{R}_1(\alpha, \beta) = p(\alpha, \beta)$, $p(\alpha, \alpha) = 0$ for all $\alpha, \beta \in M$ in Corollary 3.3 □

The following example illustrates Theorem 3.1 and shows that condition (3.1) is more general than contractivity condition given by Durmaz *et al.* ([11]).

**Example 3.1.** Let $M = [0, 1]$ and define $p(r_1, r_2) = \max\{r_1, r_2\}$. Let $\prec_1$ be defined by $r_1 \prec_1 r_2$ if and only if $r_2 \leq r_1$ for all $r_1, r_2 \in M$, then $r_1 \prec_1 r_2$ is a partial order on $M$ and $(M, \prec_1, p)$ is a complete ordered partial metric space. Induced metric is given by $d(r_1, r_2) = |r_1 - r_2|$, so, $(M, \prec_1, d)$ is a complete ordered metric space. Define mappings $S, T : M \to M$ by

$$T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0, 1); \\ 0 & \text{if } r = 1 \end{cases} \quad \text{and } S(r) = \frac{3r}{14} \text{ for all } r \in M$$

Clearly $S, T$ are weakly increasing self mappings with respect to $\prec_1$. Define the mapping $F : R^+ \to R$ by $F(r) = \ln(r)$, for all $r \in R^+ > 0$. Let $r_1, r_2 \in M$ such that $p(S(r_1), T(r_2)) > 0$ and suppose that $r_2 \prec_1 r_1$. Then

$$\mathcal{R}(r_1, r_2) = \max\left\{ r_2, \frac{r_1 r_2}{1 + r_1}, \frac{r_1 r_2}{1 + \max\left\{\frac{3r_1}{14}, \frac{r_2}{5}\right\}} \right\}.$$

Since $\frac{r_1}{1+r_1} < 1$ and $\frac{r_1}{1+\max\left\{\frac{3r_1}{14}, \frac{r_2}{5}\right\}} < 1$, we have that $\mathcal{R}(r_1, r_2) = r_2$. In a similar way, if $r_1 \prec_1 r_2$, we obtain that $\mathcal{R}(r_1, r_2) = r_1$, i.e., $\mathcal{R}(r_1, r_2) = p(r_1, r_2)$. Let $\tau \leq \ln(\frac{14}{3})$. Then, since $(r_1, r_2) \in O$

$$\begin{aligned} \tau + \ln(p(S(r_1), T(r_2))) &= \tau + \ln\left(\max\left\{\frac{3r_1}{14}, \frac{r_2}{5}\right\}\right) \\ &\leq \ln(\frac{14}{3}) + \ln\left(\max\left\{\frac{3p(r_1, r_2)}{14}, \frac{p(r_1, r_2)}{5}\right\}\right) \\ &= \ln(\frac{14}{3}) + \ln\left(\frac{3p(r_1, r_2)}{14}\right) = \ln(p(r_1, r_2)) \\ &= \ln(\mathcal{R}(r_1, r_2)). \end{aligned}$$

Thus, for $F(r) = \ln(r)$, the contractive condition (3.1) is satisfied for all $r_1, r_2 \in M$. Hence, all the hypotheses of the Theorem 3.1 are satisfied, note that the mappings $S$ and $T$ have a unique common fixed point $r = 0$. As we have seen in Example 2.4, $T$ is not an $F_d$-contraction in $(M, \prec_1, d)$ and $p$ is not a metric on $M$. Thus, we can not apply Corollary 3.2 and hence Corollary 3.4.

## 4. Application of Theorem 3.4

Let M=$C([a,b],\mathbb{R})$ be the space of all continuous real valued mappings defined on $[a,b]$, $a>0$. Define $p: M\times M\to\mathbb{R}_0^+$ by

$$p(u,v)=\sup_{t\in[a,b]}|u(t)-v(t)|+\eta,\ \text{for all } u,v\in C([a,b],\mathbb{R}) \text{ and } \eta\geq 0. \tag{4.1}$$

Obviously, $(M,p)$ is a complete partial metric space. We shall apply Theorem 3.1 to show the existence of a common solution of the system of Volterra type integral equations given by

$$u(t) = q(t)+\int_a^t K(t,r,u(t))dr, \tag{4.2}$$

$$v(t) = q(t)+\int_a^t J(t,r,v(t))dr, \tag{4.3}$$

for all $t\in[a,b]$, where $q: M\to\mathbb{R}$ is a continuous mapping and $K,J:[a,b]\times[a,b]\times M\to\mathbb{R}$ are lower semi continuous operators. We prove the following theorem to ensure the existence of a solution of the system of integral equations given by (4.2) and (4.3).

**Theorem 4.1.** Let M=$C([a,b],\mathbb{R})$. Define the mappings $f,g: M\to M$ by

$$f(u(t)) = q(t)+\int_a^t K(t,r,u(t))dr;$$

$$g(v(t)) = q(t)+\int_a^t J(t,r,v(t))dr$$

where $q: M\to\mathbb{R}$ is a continuous mapping and $K,J:[a,b]\times[a,b]\times M\to\mathbb{R}$ are lower semi continuous operators. Assume that the following conditions hold:

(i) there exists a number $\tau>0$ such that

$$|K(t,r,u(t))-J(t,r,v(t))|\leq\frac{1}{b}\ln(p(u(t),v(t))e^{-\tau})$$

for each $t,r\in[a,b]$ and $u(t),v(t)\in M$ such that

$$p(u(t),v(t))>e^{\tau};$$

(ii) $\ln(p(f(u(t)),g(v(t))))\leq|f(u(t))-g(v(t))|$, for all $t\in[a,b]$.

Then the system of integral equations given by (4.2) and (4.3) has a solution.

*Proof.* By assumption (i), we have

$$\begin{aligned} |fu(t) - gv(t))| &= \int_a^t |K(t,r,u(t)) - J(t,r,v(t))|dr \\ &\leq \int_a^t \frac{1}{b}\ln(p(u(t),v(t))e^{-\tau})dr \\ &= \frac{1}{b}\ln(p(u(t),v(t))e^{-\tau})(t-a) \\ &\leq \frac{1}{b}\ln(p(u(t),v(t))e^{-\tau})b \\ &= \ln(p(u(t),v(t))e^{-\tau}). \end{aligned}$$

By condition (ii), we have

$$\ln(p(fu(t),gv(t))) \leq \ln(p(u(t),v(t))e^{-\tau}) = \ln(p(u(t),v(t))) - \tau,$$

which implies

$$\tau + \ln(p(fu(t),gv(t))) \leq \ln(p(u(t),v(t))) \leq \ln(\mathcal{R}(u(t),v(t))).$$

Now $F(r) = \ln(r)$ satisfies all the hypotheses of Theorem 3.4 and so the system of integral equations given by (4.2) and (4.3) has a unique common solution. □

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