

A SOLUTION ALGORITHM FOR p -MEDIAN LOCATION PROBLEM ON UNCERTAIN RANDOM NETWORKS

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Abstract. This paper investigates the classical p -median location problem in a network in which some of the vertex weights and the distances between vertices are uncertain and while others are random. For solving the p -median problem in an uncertain random network, an optimization model based on the chance theory is proposed first and then an algorithm is presented to find the p -median. Finally, a numerical example is given to illustrate the efficiency of the proposed method.

Keywords: Location problem; p -median; Chance theory; Uncertain random network.

1. Introduction

Location problems have received strong theoretical interest due to their relevance in practice. One of the well-known location problems which was considered in the literature is the p -median location problem which is stated as follows: Let $N = (V, E)$ be an undirected connected network with vertex set V , $|V| = n$, edge set E and let p be a constant with $p \leq n$. Every edge $e \in E$ has a positive length and each vertex $v_i \in V$ is associated with a nonnegative weight w_i that it is the demand of the client at this vertex. Let $d(x, y)$ denote the distance between $x, y \in N$ which is equal to the length of the shortest path connecting these two points. In the classical p -median problem, the aim is to locate p pairwise different facilities m_1, \dots, m_p on the network N (i.e., on vertices or edges) which minimize the sum of weighted distances from each vertex to its closest facility:

$$(\mathbf{P}_1) : \min_{X_p \subset N, |X_p|=p} \sum_{v_i \in V} w_i d(v_i, X_p)$$

where

$$d(v_i, X_p) = \min_{j=1,2,\dots,p} d(v_i, m_j), \quad X_p = \{m_1, \dots, m_p\}.$$

An optimal solution X_p^* is called a p -median.

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Hakimi [13] showed that there exists an optimal solution among the set of vertices. This property is called vertex optimality. Later, Kariv and Hakimi [16] proved that the classical p -median problem was NP-hard, even if N was a planar graph of maximum degree 3.

Now, let $d_{ij} = d(v_i, v_j)$ be the distance from demand vertex v_i to candidate facility at vertex v_j . Also, let $w = \{w_i \mid v_i \in V\}$ and $d = \{d_{ij} \mid v_i, v_j \in V\}$ denote the set of the vertex weights and the set of distances between vertices, respectively. Then in the network N , the optimal objective value of the p -median problem is a function of w and d , which is denoted as $f(w, d)$ in this paper.

We are going to present a 0-1 linear programming formulation of the classical discrete p -median problem. Let x_{ij} be the variable that is equal to 1 if the demands of the vertex v_i are served by a facility at the vertex v_j , and 0 otherwise. Also, let the variable x_j be equal to 1 if there is an open facility at the vertex v_j , and 0 otherwise. Then, the 0-1 linear programming formulation of the classical discrete p -median problem can be stated as follows:

$$\begin{aligned}
 (\mathbf{P}_2) : \quad & \min \quad \sum_{i=1}^n \sum_{j=1}^n w_i d_{ij} x_{ij} \\
 & \text{s.t.} \quad \sum_{j=1}^n x_{ij} = 1 \quad \forall i = 1, \dots, n, \\
 & \quad \quad \quad x_{ij} \leq x_j \quad \forall i, j = 1, \dots, n, \\
 & \quad \quad \quad \sum_{j=1}^n x_j = p, \\
 & \quad \quad \quad x_{ij}, x_j \in \{0, 1\} \quad \forall i, j = 1, \dots, n.
 \end{aligned}$$

This model minimizes the total weighted distance between each demand vertex and the nearest facility. The first set of constraints requires each demand vertex to be assigned to exactly one facility. The second set of constraints allows the demand of the vertex v_i to be assigned to a vertex v_j only if there is an open facility in this location. The third set of constraints states that exactly p facilities are to be located. Finally, the last constraints are the standard integrality conditions [8, 28].

The p -median location problems have been studied by many researchers. Kariv and Hakimi [16] presented an $O(p^2 n^2)$ time algorithm for the p -median location problem on tree networks. Gavish and Sridhar [10] proposed an algorithm for the 2-median problem on trees which is run in $O(n \log n)$ time. Tamir [36] improved the time complexity of the p -median problem on trees to $O(pn^2)$. Benkoczi and Bhattacharya [6] presented an algorithm with time complexity of $O(pn \log n)$ for solving the p -median problem on interval networks. Also, they designed an $O(n \log^{p+2} n)$ time algorithm to solve the p -median problem on trees [1]. For the 1-median problem on wheel networks, Hatzl [14] provided an algorithm with linear running time. In addition, he showed that the 2-median problem on the cactus networks can be calculated at $O(n^2)$ time. Chang et al. [7] proved that the connected p -median problem on block graphs was \mathcal{NP} -hard and for the case that the lengths of the edges were equal to one, he proposed an $O(n)$ time algorithm. In 2017, Nguyen et

al. [30] presented a simple algorithm with linear running time to find the 1-median location on a cactus network. In the real life, we are usually faced with various types of non-determinacy. For example, in location problems, we are usually not sure of the vertex weights and the distances between vertices of a network. For dealing with non-determinacy phenomena, probability theory was introduced by Kolmogorov [17] in 1933 for modeling frequencies, while in 2007, uncertainty theory was presented by Liu [18] for modeling belief degrees.

The early works mainly focus on handling randomness, i.e., regarding distances between vertices and vertex weights as random variables. Berman and Krass [5] investigated the p -median problem with unreliable facilities and complete information on a line. They presented an approach for solving the problem that was based on representing the stochastic problem as a linear combination of deterministic median problems for which analytical results are available. Berman and Wang [3] studied the problem of locating p facilities to serve clients residing at the vertices of a network with discrete probabilistic demand weights. Tadei et al. [37] tried to find the location of p facilities when the cost for using a facility is a stochastic variable with unknown probability distribution. Berman and Drezner [4] investigated the p -median problem with stochastic objective value. Their aim was to find the location of p facilities such that the expected value of the objective function in the future is minimized. When the stochastic network is a tree, Mirchandani and Oudjit [27] used a selective enumeration approach for solving the 2-median problem. Various results on the p -median problem on stochastic networks are given in [2, 9, 23, 24, 25, 26].

Stochastic network optimization models work well when there are enough data to estimate the probability distributions of vertex weights and distances between the vertices. When we do not have enough samples to estimate the probability distributions of the vertex weights and the distances between vertices, we have to invite experts to give the belief degrees about the vertex weights and distances between vertices. Some researchers applied the uncertainty theory to deal with the location problems: for example the uncertain models for single facility location problems were investigated by Yuan Gao [12], the hierarchical facility location for the reverse logistics network design under uncertainty was studied by Wang and Yang [38], the capacitated facility location-allocation problem under uncertain environment was investigated by Wen et al. [39], the inverse 1-median problem on a tree under uncertain cost coefficients was solved by Nguyen and Chi [29] and the classical p -center location problem on a network with the uncertain vertex weights and the uncertain distances was studied by Soltanpour et al. [35].

In many cases, uncertainty and randomness simultaneously appear in a complex system. Specifically, for some non-deterministic phenomena, we have enough observational data to obtain their probability distribution functions, while for others, we can only estimate them by expert data. In order to describe this complex system, in 2013 the chance theory was developed by Liu [21] with the concepts of uncertain random variable, chance measure and chance distribution. Liu [21] also introduced the concepts of expected value and variance of uncertain random variables. For cal-

culating the variance of uncertain random variables, Guo and Wang [11] presented a formula based on uncertainty distribution. Sheng and Yao [34] verified a formula to calculate the variance using chance distribution and inverse chance distribution. As an important contribution to chance theory, Liu [22] presented an operational law of uncertain random variables. In addition, Hou [15] investigated the distance between uncertain random variables, and Yao and Gao [40] proved a law of large numbers for the uncertain random variables. In order to model the optimization problems with uncertainty and randomness, uncertain random programming was introduced by Liu [22] in 2013. For a survey on the uncertain random optimization problems, the reader is referred to [31, 32, 33].

In this paper, we will investigate the p -median location problem on a network with uncertain random vertex weights and distances. In the p -median problem too, there are some circumstances by which there are enough data for some of vertex weights and distances to estimate their probability distributions. On the other hand, there are no samples to estimate the probability distributions for some other vertex weights and distances so that we have to invite experts to give the belief degrees about them. Therefore, data are divided into categories; some of them have probability distributions and the others have uncertainty distributions.

The article is organized as follows: In the next section some basic concepts and properties of the uncertainty theory and chance theory will be introduced. In Section 3., we will introduce an uncertain random network and give an ideal chance distribution function of the p -median. Then, we propose an algorithm to calculate the ideal chance distribution function of the problem under investigation on uncertain random networks. Section 4. presents a model for finding the p -median and proposes an algorithm to seek the p -median of an uncertain random network. A numerical example is presented in Section 5. to illustrate the efficiency of our proposed method. Finally, Section 6. gives a brief summary to the paper.

2. Preliminary concepts and definitions

In this section, we will introduce some concepts and theorems of the uncertainty theory and chance theory.

2.1. Uncertainty theory

The uncertainty theory, introduced by Liu [18], provides a new approach to deal with non-determinacy factors. Nowadays, the uncertainty theory has become a branch of mathematics for modeling human uncertainty based on normality, duality, subadditivity, and product axioms.

In the following part, we will introduce some foundational concepts and properties of the uncertainty theory, which will be used throughout this paper [18, 19, 20].

Definition 2.1. Let Γ be a nonempty set, \mathcal{L} a σ -algebra over Γ . A set function $\mathcal{M} : \mathcal{L} \rightarrow [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (*Normality Axiom*) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ .

Axiom 2: (*Duality Axiom*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in L$ (Λ^c is compliment of Λ).

Axiom 3: (*Subadditivity Axiom*) For every countable sequence of events $\Lambda_1, \Lambda_2, \dots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Moreover, in order to provide an operational law, Liu defined the product uncertain measure on the product σ -algebra L as follows.

Axiom 4: (*Product Axiom*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \dots$. Then the product uncertain measure \mathcal{M} is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}$$

where Λ_k are arbitrary chosen events from \mathcal{L}_k for $k = 1, 2, \dots$, respectively.

Definition 2.2. An uncertain variable is a measurable function ξ from an uncertainty space to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

In order to describe an uncertain variable, a concept of uncertainty distribution is defined as follows.

Definition 2.3. The uncertainty distribution function ϕ of an uncertain variable ξ is defined by

$$\phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x .

Definition 2.4. The uncertain variables $\xi_1, \xi_2, \dots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^n \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^n \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets B_1, B_2, \dots, B_n of real numbers.

Definition 2.5. Let ξ be an uncertain variable with regular uncertainty distribution function ϕ . Then the inverse function ϕ^{-1} is called the inverse uncertainty distribution function of ξ .

The distribution of a monotonous function of uncertain variables can be obtained by the following theorem.

Theorem 2.1. *Let $\xi_1, \xi_2, \dots, \xi_n$ be the independent uncertain variables with regular uncertainty distribution functions $\phi_1, \phi_2, \dots, \phi_n$, respectively. If the function $f(x_1, x_2, \dots, x_n)$ is strictly increasing with respect to x_1, x_2, \dots, x_n , then $\nu = f(\xi_1, \xi_2, \dots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution function*

$$\psi^{-1}(\alpha) = f(\phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \dots, \phi_n^{-1}(\alpha)).$$

2.2. Chance theory

In this subsection, we will introduce some foundational definitions and properties of the uncertain random variable, the chance measure, the chance distribution and the operational law, [21, 22].

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $(\Omega, \mathcal{A}, Pr)$ be a probability space. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ is called a chance space.

Definition 2.6. Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure of Θ is defined as

$$Ch\{\Theta\} = \int_0^1 Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\} dr.$$

Liu [21] proved that a chance measure satisfies normality, duality, and monotonicity properties, that is

- (1) $Ch\{\Gamma \times \Omega\} = 1$.
- (2) $Ch\{\Theta\} + Ch\{\Theta^c\} = 1$ for any event Θ (Θ^c is compliment of Θ).
- (3) $Ch\{\Theta_1\} \leq Ch\{\Theta_2\}$ for any real number set $\Theta_1 \subset \Theta_2$.

Moreover, Hou [15] proved the subadditivity of chance measure, that is,

$$Ch\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \leq \sum_{i=1}^{\infty} Ch\{\Theta_i\}$$

for a sequence of events $\Theta_1, \Theta_2, \dots$.

Theoretically, an uncertain random variable is a measurable function on the chance space. It is usually used to deal with measurable functions of uncertain variables and random variables.

Definition 2.7. An uncertain random variable is a measurable function θ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ to the set of real numbers, i.e., $\{\theta \in B\}$ is an event for any Borel set B .

Note that random variables and uncertain variables can be regarded as special cases of uncertain random variables.

Definition 2.8. The chance distribution function of an uncertain random variable θ is defined by

$$\Phi(x) = Ch\{\theta \leq x\}$$

for any real number x .

The chance distribution function of a random variable is just its probability distribution function, and the chance distribution function of an uncertain variable is just its uncertainty distribution function.

Theorem 2.2. Let $\eta_1, \eta_2, \dots, \eta_m$ be the independent random variables with probability distribution functions $\Psi_1, \Psi_2, \dots, \Psi_m$ respectively, and let $\tau_1, \tau_2, \dots, \tau_n$ be the uncertain variables. Then the uncertain random variable

$$\theta = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$$

has a chance distribution function

$$\Phi(x) = \int_{\mathcal{R}^m} F(x, y_1, y_2, \dots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \dots d\Psi_m(y_m)$$

where $F(x, y_1, y_2, \dots, y_m)$ is the uncertainty distribution function of uncertain variable $f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_n)$ for any real numbers y_1, y_2, \dots, y_m .

3. The ideal chance distribution function of the p -median

In this section, we will introduce the uncertain random network and the ideal chance distribution function of p -median. Then, we propose an algorithm for calculating the ideal chance distribution function. First, some assumptions are listed as follows.

- (1) The undirected uncertain random network is connected.
- (2) The weight of each vertex and the distances (shortest path length) between vertices are finite.
- (3) The weight of each vertex and the distances between vertices are positive uncertain variables or positive random variables.
- (4) All the uncertain variables and the random variables are independent.

Definition 3.1. The quartette $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ is said to be an uncertain random network if $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ is the vertex set and $\mathcal{U}, \mathcal{R}, \mathcal{W}$ are defined as follows.

$$\mathcal{U} = \{(v_i, v_j) \mid \text{the shortest distance between vertices } v_i \text{ and } v_j \text{ is uncertain}\}$$

$$\cup \{v_k \mid \text{the weight of the vertex } v_k \text{ is uncertain}\},$$

$$\mathcal{R} = \{(v_i, v_j) \mid \text{the shortest distance between vertices } v_i \text{ and } v_j \text{ is random}\} \\ \cup \{v_k \mid \text{the weight of the vertex } v_k \text{ is random}\},$$

and \mathcal{W} is the collection of uncertain and random vertex weights and uncertain and random vertex distances.

In this paper, all deterministic distances and weights are regarded as special uncertain distances and weights.

Let η_k and ξ_{ij} denote the weight of vertex $v_k, v_k \in \mathcal{U} \cup \mathcal{R}$, and the distance between two vertices v_i and $v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}$, respectively. Then η_k and ξ_{ij} are uncertain variables if $v_k, (v_i, v_j) \in \mathcal{U}$, and η_k and ξ_{ij} are random variables if $v_k, (v_i, v_j) \in \mathcal{R}$. Without loss of generality, we assume that the uncertain weight η_k and the distance ξ_{ij} for $v_k, (v_i, v_j) \in \mathcal{U}$ are defined on uncertainty spaces $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$, respectively, and also the random weight η_k and the distance ξ_{ij} for $v_k, (v_i, v_j) \in \mathcal{R}$ are defined on probability spaces $(\Omega_1, \mathcal{A}_1, Pr_1)$ and $(\Omega_2, \mathcal{A}_2, Pr_2)$, respectively. Since the weights and the distances are assumed to be finite, we have $a_k \leq \eta_k \leq b_k, a_{ij} \leq \xi_{ij} \leq b_{ij}$, where a_k, a_{ij} and b_k, b_{ij} are the lower bounds and the upper bounds, respectively.

Define $\eta = \{\eta_k \mid v_k \in \mathcal{U} \cup \mathcal{R}\}$ and $\xi = \{\xi_{ij} \mid (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$. We can denote the network with the uncertain random weights and distances as $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$. The optimal value of the p -median problem is a function of weights and distances which is denoted as f in this paper. Obviously, $f(\eta, \xi)$ is an uncertain random variable. For an uncertain random network, the optimal value of the p -median problem, $f(\eta, \xi)$, is an increasing function with respect to each component of η_k and ξ_{ij} . The chance distribution function of $f(\eta, \xi)$ is called an ideal chance distribution function associated with uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$. Note that the ideal chance distribution function is unique for a given uncertain random network. The following theorem explains how to calculate an ideal chance distribution function.

Theorem 3.1. *Let $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ be an uncertain random network. Assume that the uncertain weights η_k and the uncertain distances ξ_{ij} have regular uncertainty distribution functions Υ_k and Υ_{ij} for $v_k \in \mathcal{U}$ and $(v_i, v_j) \in \mathcal{U}$ and also the random weights η_k and the random distances ξ_{ij} have probability distribution functions Ψ_k and Ψ_{ij} for $v_k \in \mathcal{R}$ and $(v_i, v_j) \in \mathcal{R}$, respectively. Then the ideal chance distribution function associated with the uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ is*

$$(3.1) \quad \Phi(z) = \int_0^\infty \dots \int_0^\infty F(z; y_k, y_{ij} \mid v_k, (v_i, v_j) \in \mathcal{R}) \\ \prod_{v_k \in \mathcal{R}} d\Psi_k(y_k) \prod_{(v_i, v_j) \in \mathcal{R}} d\Psi_{ij}(y_{ij})$$

where $F(z; y_k, y_{ij} \mid v_k, (v_i, v_j) \in \mathcal{R})$ is the uncertainty distribution function of the uncertain variable $f(y_k, y_{ij} \mid v_k, (v_i, v_j) \in \mathcal{R}; \eta_k, \xi_{ij} \mid v_k, (v_i, v_j) \in \mathcal{U})$, and it is determined by its inverse uncertainty distribution function $F^{-1}(\alpha; y_k, y_{ij} \mid v_k, (v_i, v_j) \in \mathcal{R})$ which is equal to

$$f(y_k, y_{ij} \mid v_k, (v_i, v_j) \in \mathcal{R}; \Upsilon_k^{-1}(\alpha), \Upsilon_{ij}^{-1}(\alpha) \mid v_k, (v_i, v_j) \in \mathcal{U}),$$

and f can be calculated by using (\mathbf{P}_2) .

Proof. Let $f(\eta_k, \xi_{ij} | v_k, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R})$ be the sum of the weighted distance of the p -median. By Definitions 2.6 and 2.8 and also Theorem 2.2, we can obtain the ideal chance distribution function as follows.

$$\begin{aligned}
\Phi(z) &= Ch\{f(\eta_k, \xi_{ij} | v_k, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}) \leq z\} \\
&= \int_0^1 Pr\{\omega_1 \in \Omega_1, \omega_2 \in \Omega_2 | \mathcal{M}\{f(\eta_k(\omega_1), \xi_{ij}(\omega_2) | v_k, (v_i, v_j) \in \mathcal{R}, \\
&\quad \eta_k, \xi_{ij} | v_k, (v_i, v_j) \in \mathcal{U}) \leq z\} \geq r\} dr \\
&= \int_0^\infty \dots \int_0^\infty \mathcal{M}\{f(\eta_k(\omega_1), \xi_{ij}(\omega_2) | v_k, (v_i, v_j) \in \mathcal{R}, \eta_k, \xi_{ij} | v_k, (v_i, v_j) \in \mathcal{U}) \leq z\} \\
&\quad \prod_{v_k \in \mathcal{R}} d\Psi_k(y_k) \prod_{(v_i, v_j) \in \mathcal{R}} d\Psi_{ij}(y_{ij}) \\
&= \int_0^\infty \dots \int_0^\infty F(z; y_k, y_{ij} | v_k, (v_i, v_j) \in \mathcal{R}) \prod_{v_k \in \mathcal{R}} d\Psi_k(y_k) \prod_{(v_i, v_j) \in \mathcal{R}} d\Psi_{ij}(y_{ij})
\end{aligned}$$

where $F(z; y_k, y_{ij} | v_k, (v_i, v_j) \in \mathcal{R})$ is the uncertainty distribution function of uncertain variable $f(y_k, y_{ij} | v_k, (v_i, v_j) \in \mathcal{R}; \eta_k, \xi_{ij} | v_k, (v_i, v_j) \in \mathcal{U})$ for any real numbers $y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R}$, and it is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R})$. By Theorem 2.1, for given $\alpha \in (0, 1)$, we have

$$\begin{aligned}
&F^{-1}(\alpha; y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R}) \\
&= f(y_k, y_{ij} | v_k, (v_i, v_j) \in \mathcal{R}; \Upsilon_k^{-1}(\alpha), \Upsilon_{ij}^{-1}(\alpha) | v_k, (v_i, v_j) \in \mathcal{U}),
\end{aligned}$$

which is just the optimal value of the p -median problem of a determinacy network and f is a strictly increasing function with respect to η_k and ξ_{ij} , where η_k and ξ_{ij} denote the weight of the vertex v_i and the shortest distance between two vertices v_i and v_j , respectively. We can calculate f by using (\mathbf{P}_2) . Thus the theorem is proved. \square

Note that, it is difficult to calculate the ideal chance distribution function by using formula (3.1). Hence, in order to calculate the ideal chance distribution function of the p -median in an uncertain random network, we propose the following algorithm:

Algorithm 1

1. For any y_k and y_{ij} , $v_k, (v_i, v_j) \in \mathcal{R}$, give partitions \prod_k and \prod_{ij} of intervals $[a_k, b_k]$ and $[a_{ij}, b_{ij}]$ with step $\Delta = 0.01$. Let random variables η_k and ξ_{ij} take values in $\{y_k | y_k = a_k + 0.01 * i \text{ for } i = 1, 2, \dots, (b_k - a_k) * 100\}$ and $\{y_{ij} | y_{ij} = a_{ij} + 0.01 * i \text{ for } i = 1, 2, \dots, (b_{ij} - a_{ij}) * 100\}$, respectively.
2. Calculate $F^{-1}(\alpha; y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R})$ by using (\mathbf{P}_2) for given y_k and y_{ij} and each $\alpha \in \{0.01, 0.02, \dots, 0.99\}$.
3. Obtain the uncertainty distribution function of $F(z; y_k, y_{ij} | v_k, (v_i, v_j) \in \mathcal{R})$, from its discrete form via linear interpolation.
4. Input $F(z; y_k, y_{ij} | v_k, (v_i, v_j) \in \mathcal{R})$ into formula (3.1) to calculate the chance distribution function $\Phi(z)$ for each z .

4. The p -median model in uncertain random networks

In this section, we will consider the p -median problem in an uncertain random network and present an algorithm for finding the p -median.

Given a connected and undirected uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$. Let $\mathcal{V}_p \subseteq \mathcal{V}$, $|\mathcal{V}_p| = p$, and let the variable x_j be equal to 1 if $v_j \in \mathcal{V}_p$, and 0 otherwise. Also let x_{ij} be the variable that is equal to 1 if the nearest vertex to the vertex v_i in \mathcal{V}_p is the vertex v_j , and 0 otherwise. A p -facility location is represented by $\{x_j, x_{ij} \mid v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$, where

$$\begin{aligned} \sum_{(v_i, v_j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} &= 1 \quad \forall v_i \in \mathcal{U} \cup \mathcal{R}, \\ x_{ij} &\leq x_j \quad \forall v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}, \\ \sum_{v_j \in \mathcal{U} \cup \mathcal{R}} x_j &= p, \\ x_{ij}, x_j &\in \{0, 1\} \quad \forall v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}. \end{aligned}$$

Therefore the sum of weighted distances of a p -facility location $\{x_j, x_{ij} \mid v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$ is

$$\sum_{v_i \in \mathcal{U} \cup \mathcal{R}} \sum_{(v_i, v_j) \in \mathcal{U} \cup \mathcal{R}} \eta_i \xi_{ij} x_{ij}$$

which is obviously an uncertain random variable. Its chance distribution function is denoted by $\Psi(z)$, i.e.,

$$\Psi(z) = Ch \left\{ \sum_{v_i \in \mathcal{U} \cup \mathcal{R}} \sum_{(v_i, v_j) \in \mathcal{U} \cup \mathcal{R}} \eta_i \xi_{ij} x_{ij} \leq z \right\}.$$

Based on Theorem 2.2, we suggest the following algorithm to calculate the chance distribution function which corresponds to the p -facility location

$$\{x_j, x_{ij} \mid v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}.$$

Algorithm 2

1. Step 1 as described in Algorithm 1.
2. For given y_k and y_{ij} and each $\alpha \in \{0.01, 0.02, \dots, 0.99\}$, calculate $F^{-1}(\alpha; y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R})$, which is the sum of the weighted distances of the corresponding p -facility location.
3. Use discrete form of linear interpolation to obtain the uncertainty distribution function of $F(z; y_k, y_{ij} \mid v_k, (v_i, v_j) \in \mathcal{R})$.

4. Input $F(z; y_k, y_{ij} | v_k, (v_i, v_j) \in \mathcal{R})$ into formula

$$\int_0^\infty \cdots \int_0^\infty F(z; y_k, y_{ij} | v_k, (v_i, v_j) \in \mathcal{R}) \prod_{v_k \in \mathcal{R}} d\Psi_k(y_k) \prod_{(v_i, v_j) \in \mathcal{R}} d\Psi_{ij}(y_{ij})$$

and calculate the chance distribution function $\Psi(z)$ for each z .

Theorem 4.1. *Let $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ be an uncertain random network and let $\Phi(z)$ be an ideal chance distribution function associated with it. Assume that $\{x_j, x_{ij} | v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$ is a given p -facility location. Then we have*

$$\Psi(z) \leq \Phi(z).$$

Proof. Obviously, by Theorem 3.1, the ideal chance distribution function of the p -median $\Phi(z)$ is the smallest sum distribution. So we can obtain the chance distribution function of any p -facility location

$$\Psi(z) \leq \Phi(z).$$

The theorem is proved. \square

In order to find the p -median location of an uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$, we give the following definition.

Definition 4.1. Let $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ be an uncertain random network and let $\Phi(z)$ be the ideal chance distribution function of the p -median. Assume that $\Psi(z)$ is the chance distribution function of a p -facility location $\{x_j, x_{ij} | v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$. If $\Psi(z)$ is the closest the $\Phi(z)$, i.e.,

$$\int_0^\infty \{\Phi(z) - \Psi(z)\} dz$$

is minimum, then the p -facility location $\{x_j, x_{ij} | v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$ is called the p -median in an uncertain random network.

Based on Definition 4.1 and the above statements, we formulate the following optimization model to determine a p -median for an uncertain random network.

$$\begin{aligned} (\mathbf{P}_3) : \quad & \min \int_0^\infty \{\Phi(z) - \Psi(z)\} dz \\ & \text{s.t.} \quad \sum_{(v_i, v_j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} = 1 \quad \forall v_i \in \mathcal{U} \cup \mathcal{R}, \\ & \quad \quad x_{ij} \leq x_j \quad \quad \quad \forall v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}, \\ & \quad \quad \sum_{v_j \in \mathcal{U} \cup \mathcal{R}} x_j = p, \\ & \quad \quad x_{ij}, x_j \in \{0, 1\} \quad \quad \forall v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}, \end{aligned}$$

where $\Phi(z)$ is the ideal chance distribution function and

$$\Psi(z) = Ch \left\{ \sum_{v_i \in \mathcal{U} \cup \mathcal{R}} \sum_{(v_i, v_j) \in \mathcal{U} \cup \mathcal{R}} \eta_i \xi_{ij} x_{ij} \leq z \right\}$$

is the chance distribution function of any p -facility location.

In order to find the p -median in an uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$, we propose the following algorithm.

Algorithm 3

1. Calculate the ideal chance distribution function for the uncertain random network by using Algorithm 1.
2. Consider all the p -facility locations in the uncertain random network.
3. Calculate the chance distribution function of the sum of weighted distances for each p -facility location by using Algorithm 2.
4. Calculate the objective function of Model (\mathbf{P}_3) for given p -facility location, and choose the minimum value of objective function, which corresponds to the desired p -median.

5. An illustrative example

In this section, we give an example for the 2-median location problem in an uncertain random network to illustrate the proposed algorithms.

Consider goods distribution system. Assume that the system is given as the network N in Figure 5.1 where the vertices denote urban areas. In this system, warehouses of the distribution company and supermarkets are facilities and clients, respectively. There is a supermarket at each area. Suppose that the weight η_i of the vertex v_i is equal to the average monthly purchase of residents of this area from the supermarket located at vertex v_i . Some of the vertex weights and the distances between vertices in the network are uncertain, while others are random (see Table 5.1). Our aim is to find two vertices on the network N to locate warehouses of the distribution company which will minimize the sum of weighted distances from each supermarket to its closest warehouse.

$$\begin{aligned} \mathcal{V} &= \{v_1, v_2, v_3, v_4, v_5, v_6\}, \\ \mathcal{U} &= \{v_1, v_2, v_4, v_5, v_6\} \cup \{(v_1, v_2), (v_1, v_4), (v_2, v_5), (v_3, v_4)(v_4, v_6)\}, \\ \mathcal{R} &= \{v_3\} \cup \{(v_2, v_3)\}, \\ \mathcal{W} &= \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\} \cup \{\xi_{12}, \xi_{14}, \xi_{23}, \xi_{25}, \xi_{34}, \xi_{46}\}. \end{aligned}$$

Note that if the distance between some vertices or weight of some vertices are considered as constant values, then we regard them as special uncertain variables.

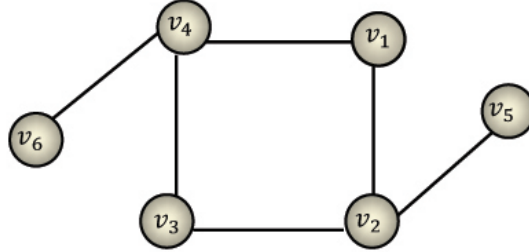


FIG. 5.1: An uncertain random network

Table 5.1: Uncertainty and probability distributions

vertex	η_i	arc	ξ_{ij}
v_1	2	(v_1, v_2)	$\mathcal{L}(2, 3)$
v_2	$\mathcal{L}(2, 4)$	(v_1, v_4)	$\mathcal{L}(3, 4)$
v_3	$U(2, 3)$	(v_2, v_3)	$U(4, 6)$
v_4	3	(v_2, v_5)	2
v_5	$\mathcal{L}(1, 2)$	(v_3, v_4)	$\mathcal{L}(2, 3)$
v_6	$\mathcal{L}(2, 3)$	(v_4, v_6)	5

From Theorem 3.1, we can obtain the ideal chance distribution function associated with the network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ as follows:

$$\Phi(z) = \int_0^\infty \int_0^\infty F(z; y_3, y_{23}) d\Psi_3(y_3) d\Psi_{23}(y_{23})$$

where $F(z; y_3, y_{23})$ is determined by its inverse uncertainty distribution function

$$F^{-1}(\alpha; y_3, y_{23}) = f(\Upsilon_1^{-1}(\alpha), \Upsilon_2^{-1}(\alpha), y_3, \Upsilon_4^{-1}(\alpha), \Upsilon_5^{-1}(\alpha), \Upsilon_6^{-1}(\alpha); \Upsilon_{12}^{-1}(\alpha), \Upsilon_{14}^{-1}(\alpha), y_{23}, \Upsilon_{25}^{-1}(\alpha), \Upsilon_{34}^{-1}(\alpha), \Upsilon_{46}^{-1}(\alpha)).$$

Give a partition Π_3 on interval $[2, 3]$ with step 0.01, and let random variable η_3 take values in

$$\{y_3 \mid y_3 = 2 + 0.01i \text{ for } i = 1, \dots, 100\}.$$

Also give a partition Π_{23} on interval $[4, 6]$ with step 0.01, and let random variable η_{23} take values in

$$\{y_{23} \mid y_{23} = 4 + 0.01i \text{ for } i = 1, \dots, 200\}.$$

For any $y_3 \in \prod_3$, $y_{23} \in \prod_{23}$, given $\alpha \in \{0.01, 0.02, \dots, 0.99\}$, we calculate $F^{-1}(\alpha; y_3, y_{23})$ by using the model (\mathbf{P}_2) .

We may first calculate the ideal chance distribution function $\Phi(z)$ associated with the network by Algorithm 1.

Then we will calculate the chance distribution function of the total weighted distance of each 2-facility location(i.e., $\Psi(z)$) by Algorithm 2, which is shown in Figure 5.2.

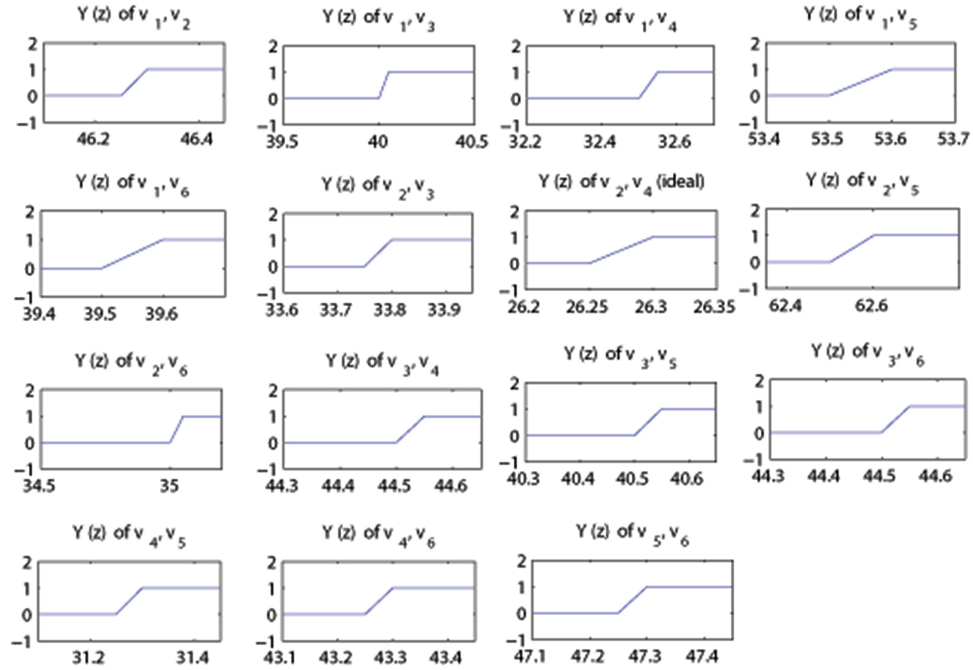


FIG. 5.2: The shapes of chance distribution functions of all 2-facilities for the example

Using Algorithm 3, we will calculate the difference between the chance distribution function of total weighted distance of each 2-facility location and the ideal one, which is given as Table 5.2.

Model (\mathbf{P}_3) implies that the vertices v_2 and v_4 are the desired warehouses locations since the difference between the chance distribution function and the ideal one is minimum.

Table 5.2: Difference between the chance distribution function and the ideal chance distribution function

2-facility location	$\{v_1, v_2\}$	$\{v_1, v_3\}$	$\{v_1, v_4\}$	$\{v_1, v_5\}$	$\{v_1, v_6\}$
Difference	20.015	13.759	6.259	27.234	13.234
2-facility location	$\{v_2, v_3\}$	$\{v_2, v_4\}$	$\{v_2, v_5\}$	$\{v_2, v_6\}$	$\{v_3, v_4\}$
Difference	7.511	0	36.234	8.759	18.261
2-facility location	$\{v_3, v_5\}$	$\{v_3, v_6\}$	$\{v_4, v_5\}$	$\{v_4, v_6\}$	$\{v_5, v_6\}$
Difference	14.261	18.261	5.009	17.009	21.006

6. Conclusions

In this paper, we have investigated the p -median location problem in an uncertain random network, i.e., a network in which the weights of vertices and the distances between vertices are uncertain random variables. We first introduced the concept of the ideal chance distribution function and then presented an algorithm to calculate the ideal chance distribution function of the p -median associated with the uncertain random network. We have formulated the discrete p -median location problem in an uncertain random network and presented an algorithm to find the p -median. Finally, to illustrate the efficiency of the proposed method, we gave a numerical example.

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