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NEW TYPE INEQUALITIES FOR $\mathbb{B}^{-1}$-CONVEX FUNCTIONS INVOLVING HADAMARD FRACTIONAL INTEGRAL

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Abstract. Abstract convexity is an important area of mathematics in recent years and it has very significant applications areas like inequality theory. The Hermite-Hadamard Inequality is one of these applications. In this article, we studied Hermite-Hadamard Inequalities for $\mathbb{B}^{-1}$-convex functions via Hadamard fractional integral.
Keywords: Hermite-Hadamard Inequality; Hadamard fractional integral; convexity.

1. Introduction

Abstract convexity has an important area in convexity theory and it becomes different convexity types. These abstract convexity types have significant applications variety fields like mathematical economy, operation research, inequality theory and optimization theory. Additionally, $\mathbb{B}^{-1}$-convexity is one of these abstract convexity types ([1, 4]). It has applications to mathematical economy and inequality theory ([4, 19]). There are many articles about $\mathbb{B}^{-1}$-convexity ([2, 3, 10, 16, 22]).

Hermite-Hadamard inequalities can be given as an application on inequality theory for abstract convexity types ([6, 7, 8, 9, 13, 14, 17, 20]). We proved this inequality for $\mathbb{B}^{-1}$-convex functions ([19]). Hermite-Hadamard inequality is an integral inequality and it has be given with classic integral operator up to now. But, recently, the studies are made with fractional integral operators that are more general([5, 11, 12, 15, 18, 21]). Therefore, we study Hermite-Hadamard inequalities involving Hadamard fractional integral operator for $\mathbb{B}^{-1}$-convexity.

The outline of paper as follows: Second section is given on two parts that are required definitions and theorems of $\mathbb{B}^{-1}$-convexity and Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex functions, respectively. In third section, we give Hermite-Hadamard type inequalities involving Hadamard fractional integral.

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2. Preliminaries

In this section, we give some required definition and theorems.

Lets recall the following definition of fractional integral type. Along the paper, let $f : [a, b] \to \mathbb{R}$ be a given function, where $0 \leq a < b < +\infty$ and $f \in L_1[a, b]$. Also, $\Gamma(\alpha)$ is the Gamma function.

Definition 2.1. [11] The left-sided Hadamard fractional integral $\mathbf{J}_{a+}^{\alpha}$ of order $\alpha > 0$ of f is defined by

$$\mathbf{J}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x > a \tag{2.1}$$

provided that the integral exists. The right-sided Hadamard fractional integral $\mathbf{J}_{b-}^{\alpha}$ of order $\alpha > 0$ of f is defined by

$$\mathbf{J}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x < b \tag{2.2}$$

provided that the integral exists.

2.1. $\mathbb{B}^{-1}$-convexity

For $r \in \mathbb{Z}^-$, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ to itself; $\boldsymbol{x} = (x_1, x_2, ..., x_n) \to \Phi_r(\boldsymbol{x}) = (\varphi_r(x_1), \varphi_r(x_2), ..., \varphi_r(x_n))$ is homeomorphism from $\mathbb{R}_*^n$ to itself.

For a finite nonempty set $A = \left\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)}\right\} \subset \mathbb{R}_*^n$ the Φ_r-convex hull (shortly r-convex hull) of A, which we denote $Co^r(A)$ is given by

$$Co^r(A) = \left\{\Phi_r^{-1}\left(\sum_{i=1}^{m} t_i \Phi_r(\boldsymbol{x}^{(i)})\right) : t_i \geq 0, \sum_{i=1}^{m} t_i = 1\right\} .$$

We denote by $\overset{m}{\underset{i=1}{\wedge}} \boldsymbol{x}^{(i)}$ the greatest lower bound with respect to the coordinate-wise order relation of $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)} \in \mathbb{R}^n$, that is:

$$\overset{m}{\underset{i=1}{\wedge}} \boldsymbol{x}^{(i)} = \left(\min\left\{x_1^{(1)}, x_1^{(2)}, ..., x_1^{(m)}\right\}, ..., \min\left\{x_n^{(1)}, x_n^{(2)}, ..., x_n^{(m)}\right\}\right)$$

where, $x_j^{(i)}$ denotes jth coordinate of the point $\boldsymbol{x}^{(i)}$.

Thus, we can define $\mathbb{B}^{-1}$-polytopes as follows:

Definition 2.2. [1] The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r \in \mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$ where A is a finite subset of $\mathbb{R}_*^n$, is called $\mathbb{B}^{-1}$-polytope of A.

The definition of $\mathbb{B}^{-1}$-polytope can be expressed in the following form in $\mathbb{R}^n_{++}$.

Theorem 2.1. [1] For all nonempty finite subsets $A = \left\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)}\right\} \subset \mathbb{R}^n_{++}$ we have

$$Co^{-\infty}(A) = \lim_{r\to-\infty} Co^r(A) = \left\{ \mathop{\wedge}_{i=1}^{m} t_i \boldsymbol{x}^{(i)} : t_i \geq 1, \min_{1\leq i\leq m} t_i = 1 \right\} .$$

Next, we give the definition of $\mathbb{B}^{-1}$-convex sets.

Definition 2.3. [1] A subset U of $\mathbb{R}^n_*$ is called a $\mathbb{B}^{-1}$-convex if for all finite subsets $A \subset U$ the $\mathbb{B}^{-1}$-polytope $Co^{-\infty}(A)$ is contained in U.

By Theorem 2.1, we can reformulate the above definition for subsets of $\mathbb{R}^n_{++}$:

Theorem 2.2. [1] A subset U of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}$-convex if and only if for all $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in U$ and all $\lambda \in [1, \infty)$ one has $\lambda \boldsymbol{x}^{(1)} \wedge \boldsymbol{x}^{(2)} \in U$.

Remark 2.1. As a result of Theorem 2.2, we can say that $\mathbb{B}^{-1}$-convex sets in $\mathbb{R}_{++}$ are positive intervals.

Definition 2.4. [10] For $U \subset \mathbb{R}^n_*$, a function $f : U \to \mathbb{R}_*$ is called a $\mathbb{B}^{-1}$-convex function if $epi^*(f) = \{(\boldsymbol{x}, \mu) \,|\, \boldsymbol{x} \in U, \mu \in \mathbb{R}_*, \mu \geq f(\boldsymbol{x})\}$ is a $\mathbb{B}^{-1}$-convex set.

In $\mathbb{R}^n_{++}$, we can give the following fundamental theorem which provides a sufficient and necessary condition for $\mathbb{B}^{-1}$-convex functions [10].

Theorem 2.3. Let $U \subset \mathbb{R}^n_{++}$ and $f : U \to \mathbb{R}_{++}$. The function f is $\mathbb{B}^{-1}$-convex if and only if the set U is $\mathbb{B}^{-1}$-convex and one has the inequality

$$f(\lambda \boldsymbol{x} \wedge \boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) \wedge f(\boldsymbol{y}) \tag{2.3}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in U$ and all $\lambda \in [1, +\infty)$.

2.2. Hermite-Hadamard Inequality for $\mathbb{B}^{-1}$-convex Functions

We proved the following theorem that gives the Hermite-Hadamard inequality involving classic integral for $\mathbb{B}^{-1}$-convex functions in [19].

Theorem 2.4. Suppose $f : [a, b] \subset \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is a $\mathbb{B}^{-1}$-convex function. Then the following inequality holds

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \begin{cases} \frac{f(a)(a+b)}{2a}, & \frac{b}{a} \leq \frac{f(b)}{f(a)} \\ \frac{2bf(a)f(b)-a\left[(f(a))^2+(f(b))^2\right]}{2(b-a)f(a)}, & 1 \leq \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases} . \tag{2.4}$$

3. Hermite-Hadamard Type Inequalities Involving Hadamard Fractional Integral

Hadamard fractional integral is one of the important fractional integral types. So, we introduce Hermite-Hadamard type inequalities including Hadamard fractional integral in this section.

Theorem 3.1. *Let $\alpha > 0$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then*

$$\boldsymbol{J}^{\alpha}_{a+}f\left(b\right) \leq \begin{cases} \frac{f(a)}{\Gamma(\alpha)}\int_1^{\frac{b}{a}}\left(\ln\frac{b}{\lambda a}\right)^{\alpha-1}d\lambda, & \frac{b}{a}\leq\frac{f(b)}{f(a)} \\ \frac{f(a)}{\Gamma(\alpha)}\int_1^{\frac{f(b)}{f(a)}}\left(\ln\frac{b}{\lambda a}\right)^{\alpha-1}d\lambda+\frac{f(b)\left(\ln\frac{bf(a)}{af(b)}\right)^{\alpha}}{\Gamma(\alpha+1)}, & 1\leq\frac{f(b)}{f(a)}<\frac{b}{a}\end{cases}. \tag{3.1}$$

Proof. The inequality (2.3) is satisfy for f, because of its $\mathbb{B}^{-1}$-convexity of f. Lets' multiply both sides of this inequality by $\frac{(\ln(\min\{\lambda a,b\}))'}{[\ln b-\ln\lambda a]^{1-\alpha}}$ then integrate with respect to λ over $[1,+\infty)$. For the left sided of inequality, we have that

$$\begin{aligned}&\int_1^{+\infty}\frac{\left(\ln\left(\min\left\{\lambda a,b\right\}\right)\right)'}{\left[\ln b-\ln\lambda a\right]^{1-\alpha}}f\left(\min\left\{\lambda a,b\right\}\right)d\lambda\\
&=\int_1^{\frac{b}{a}}\frac{\left(\ln\left(\min\left\{\lambda a,b\right\}\right)\right)'}{\left[\ln b-\ln\lambda a\right]^{1-\alpha}}f\left(\min\left\{\lambda a,b\right\}\right)d\lambda+\int_{\frac{b}{a}}^{+\infty}\frac{\left(\ln\left(\min\left\{\lambda a,b\right\}\right)\right)'}{\left[\ln b-\ln\lambda a\right]^{1-\alpha}}f\left(\min\left\{\lambda a,b\right\}\right)d\lambda\\
&=\int_1^{\frac{b}{a}}\frac{1}{\lambda\left(\ln\frac{b}{\lambda a}\right)^{1-\alpha}}f\left(\lambda a\right)d\lambda\\
&=\int_a^b\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{f\left(t\right)}{t}dt=\Gamma\left(\alpha\right)\mathbf{J}^{\alpha}_{a+}f\left(b\right).\end{aligned}$$

For the right sided of the inequality, we have to examine two cases of $\frac{f(b)}{f(a)}$. One of these cases is $\frac{b}{a}\leq\frac{f(b)}{f(a)}$ and in this situation the equation is

$$\begin{aligned}&\int_1^{+\infty}\frac{\left(\ln\left(\min\left\{\lambda a,b\right\}\right)\right)'}{\left[\ln b-\ln\lambda a\right]^{1-\alpha}}\min\left\{\lambda f\left(a\right),f\left(b\right)\right\}d\lambda\\
&=\int_1^{\frac{b}{a}}\frac{\left(\ln\left(\min\left\{\lambda a,b\right\}\right)\right)'}{\left[\ln b-\ln\lambda a\right]^{1-\alpha}}\min\left\{\lambda f\left(a\right),f\left(b\right)\right\}d\lambda+\\
&+\int_{\frac{b}{a}}^{+\infty}\frac{\left(\ln\left(\min\left\{\lambda a,b\right\}\right)\right)'}{\left[\ln b-\ln\lambda a\right]^{1-\alpha}}\min\left\{\lambda f\left(a\right),f\left(b\right)\right\}d\lambda\\
&=f\left(a\right)\int_1^{\frac{b}{a}}\left(\ln\frac{b}{\lambda a}\right)^{\alpha-1}d\lambda.\end{aligned}$$

Thus, with these calculations the first part of requested inequality is

$$\mathbf{J}_{a+}^{\alpha} f(b) \leq \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} \left(\ln \frac{b}{\lambda a} \right)^{\alpha-1} d\lambda \,. \tag{3.2}$$

The other case is $1 \leq \frac{f(b)}{f(a)} < \frac{b}{a}$, thence we obtain the followings:

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} \min\{\lambda f(a), f(b)\} \, d\lambda \\
&= \int_1^{\frac{f(b)}{f(a)}} \frac{1}{\lambda \left(\ln \frac{b}{\lambda a}\right)^{1-\alpha}} \lambda f(a) \, d\lambda + \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{1}{\lambda \left(\ln \frac{b}{\lambda a}\right)^{1-\alpha}} \lambda f(a) \, d\lambda \\
&= f(a) \int_1^{\frac{f(b)}{f(a)}} \left(\ln \frac{b}{\lambda a} \right)^{\alpha-1} d\lambda + \frac{f(b) \left(\ln \frac{b f(a)}{a f(b)} \right)^{\alpha}}{\alpha}.
\end{aligned}$$

Hence, we attain that

$$\mathbf{J}_{a+}^{\alpha} f(b) \leq \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} \left(\ln \frac{b}{\lambda a} \right)^{\alpha-1} d\lambda + \frac{f(b) \left(\ln \frac{b f(a)}{a f(b)} \right)^{\alpha}}{\Gamma(\alpha+1)} \,. \tag{3.3}$$

Finally, the inequality (3.1) can be get from (3.2) and (3.3). $\square$

Theorem 3.2. *Let $\alpha > 0$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then*

$$\boldsymbol{J}_{b-}^{\alpha} f(a) \leq \begin{cases} \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} (\ln \lambda)^{\alpha-1} d\lambda, & \frac{b}{a} \leq \frac{f(b)}{f(a)} \\ \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} (\ln \lambda)^{\alpha-1} d\lambda + \frac{f(b)}{\Gamma(\alpha+1)} \left[\left(\ln \frac{b}{a}\right)^{\alpha} - \left(\ln \frac{f(b)}{f(a)} \right)^{\alpha} \right], & 1 \leq \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases}. \tag{3.4}$$

Proof. Let the function f be $\mathbb{B}^{-1}$-convex. Thus, the inequality (2.3) holds. If we multiply by $\frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}}$ and integrate with respect to λ over $[1, +\infty)$ to this inequality, then we have

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} f(\min\{\lambda a, b\}) \, d\lambda \\
&= \int_1^{\frac{b}{a}} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} f(\min\{\lambda a, b\}) \, d\lambda + \\
&+ \int_{\frac{b}{a}}^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} f(\min\{\lambda a, b\}) \, d\lambda \\
&= \int_a^b \left(\ln \frac{t}{a} \right)^{\alpha-1} \frac{f(t)}{t} dt = \Gamma(\alpha) \, \mathbf{J}_{b-}^{\alpha} f(a) \,.
\end{aligned}$$

There are two following situations that have to been examined for the right sided of inequality. The first is $\frac{b}{a} \leq \frac{f(b)}{f(a)}$. Hence,

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda \\
&= \int_1^{\frac{b}{a}} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda + \\
&+ \int_{\frac{b}{a}}^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda \\
&= f(a) \int_1^{\frac{b}{a}} (\ln \lambda)^{\alpha-1}\, d\lambda.
\end{aligned}$$

Therefore, the inequality is obtained

$$\mathbf{J}_{b^-}^{\alpha} f(a) \leq \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} (\ln \lambda)^{\alpha-1}\, d\lambda\ . \tag{3.5}$$

The second case is $1 \leq \frac{f(b)}{f(a)} < \frac{b}{a}$. At this stage, we get that

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda \\
&= \int_1^{\frac{f(b)}{f(a)}} \frac{1}{(\ln \lambda)^{1-\alpha}} f(a)\, d\lambda + \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{1}{\lambda (\ln \lambda)^{1-\alpha}} f(b)\, d\lambda \\
&= f(a) \int_1^{\frac{f(b)}{f(a)}} (\ln \lambda)^{\alpha-1}\, d\lambda + \frac{f(b)}{\alpha} \left[\left(\ln \frac{b}{a} \right)^{\alpha} - \left(\ln \frac{f(b)}{f(a)} \right)^{\alpha} \right].
\end{aligned}$$

So, the inequality is in that form:

$$\mathbf{J}_{b^-}^{\alpha} f(a) \leq \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} (\ln \lambda)^{\alpha-1}\, d\lambda + \frac{f(b)}{\Gamma(\alpha+1)} \left[\left(\ln \frac{b}{a} \right)^{\alpha} - \left(\ln \frac{f(b)}{f(a)} \right)^{\alpha} \right]\ . \tag{3.6}$$

Consequently, we have proven the inequality (3.4) by using the inequalities (3.5) and (3.6). □

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