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SOME COMMON FIXED POINT RESULTS FOR RATIONAL CONTRACTION TYPE VIA THE $\mathcal{C}$-CLASS FUNCTIONS ON METRIC SPACES

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Abstract. The purpose of this paper is to prove some common fixed point results for rational contraction type via the $\mathcal{C}$-class functions on metric spaces. As an application, we study the existence of solutions to the system of nonlinear integral equations.
Keywords. Common fixed point; rational contraction mappings; triangular α-orbital admissible mapping; contraction; integral equations.

1. Introduction

The Banach contraction principle [2] is a basic tool in studying the existence of solutions to many problems in mathematics and many different fields. In recent times, the contraction principle has been extended in many various directions. Geraghty's theorem [7] is one of the generalized result. In 2013, Cho *et al.* [5] introduced the notion of α-Geraghty contraction type maps and proved some fixed point theorems for such maps in complete metric spaces. In 2014, Popescu [14] extended the results in [5] by proving certain fixed point theorems for generalized α-Geraghty contraction type maps. Later, Karapina [12] introduced the notion of α-ψ-Geraghty contraction type maps and proved the existence and uniqueness of fixed points for such maps in metric spaces. In 2016, Chuadchawna *et al.* [6] improved and generalized the results in [12, 14] by proving some fixed point theorems for α-η-ψ-Greraghty contraction type maps in α-η complete metric spaces. Recently, Ansari and Kaewcharoen [1] extended the results in [12] and proved the fixed point theorems for α-η-ψ-φ-F contraction type maps in α-η complete metric spaces by using the $\mathcal{C}$-class function.

In 1977, Jaggi [11] also extended the Banach contraction principle by proving some fixed point theorems for a contractive condition of rational type in metric

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spaces. After that, some authors extended the main results in [11] by many different ways. Furthermore, certain fixed point results for rational contractions were established in metric spaces and generalized metric spaces (see, for example, [4, 8, 9, 13] and the references therein).

In this paper, we state some common fixed point theorems for rational contraction type via the $\mathcal{C}$-class functions on metric spaces. The obtained results are generalizations of the main results in [1, 12, 14]. In addition, we study the existence of solutions to the system of nonlinear integral equations.

2. Preliminaries

First, we recall some symbols that

1. $\mathcal{C}$ is the family of all functions $F : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ such that for all $s, t \in [0, \infty)$,
   (a) F is continuous.
   (b) $F(s, t) \leq s$.
   (c) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.
2. Ψ is the family of all functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that
   (a) ψ is nondecreasing and continuous.
   (b) $\psi(t) = 0$ if and only if $t = 0$.
3. Φ the family of all functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ such that
   (a) φ is continuous.
   (b) $\varphi(t) > 0$ for all $t > 0$.

In [1], the authors gave some functions which are elements in $\mathcal{C}$.

Example 2.1. ([1], Example 1.12) The following functions $F : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ are elements in $\mathcal{C}$.

1. $F(s, t) = s - t$ for all $s, t \in [0, \infty)$.
2. $F(s, t) = ms$ for all $s, t \in [0, \infty)$ where $0 < m < 1$.
3. $F(s, t) = \dfrac{s}{(1 + t)^r}$ for all $s, t \in [0, \infty)$ where $r \in (0, \infty)$.
4. $F(s, t) = s\beta(s)$ for all $s, t \in [0, \infty)$ where $\beta : [0, \infty) \longrightarrow [0, 1)$ is a continuous function.
5. $F(s, t) = s - \varphi(s)$ for all $s, t \in [0, \infty)$ where $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ iff $t = 0$.

In 2014, Popescu [14] introduced the notion of α-orbital admissible mappings and triangular α-orbital admissible mappings as follows.

Definition 2.1. ([14], Definition 5) Let X be a non-empty set and $f : X \longrightarrow X$, $\alpha : X \times X \longrightarrow [0, \infty)$ be two mappings. Then f is called an *α-orbital admissible mapping* if for all $x \in X$,

$$\alpha(x, fx) \geq 1 \text{ implies } \alpha(fx, f^2x) \geq 1.$$

Definition 2.2. ([14], Definition 6) Let X be a non-empty set and $f : X \longrightarrow X$, $\alpha : X \times X \longrightarrow [0, \infty)$ be two mappings. Then f is called a *triangular α-orbital admissible mapping* if

1. f is an α-orbital admissible.
2. For all $x, y \in X$, $\alpha(x, y) \geq 1, \alpha(y, fy) \geq 1$ imply $\alpha(x, fy) \geq 1$.

In 2016, Chuadchawna *et al* [6] introduced the notion of α-orbital admissible mappings respect to η and triangular α-orbital admissible mappings respect to η as follows.

Definition 2.3. ([6], Definition 2.1) Let X be a non-empty set and $f : X \longrightarrow X$, $\alpha : X \times X \longrightarrow [0, \infty)$ be two mappings. Then f is called an *α-orbital admissible mapping respect to η* if for all $x \in X$,

$$\alpha(x, fx) \geq \eta(x, fx) \text{ implies } \alpha(fx, f^2x) \geq \eta(fx, f^2x).$$

Definition 2.4. ([6], Definition 2.2) Let X be a non-empty set and $f : X \longrightarrow X$, $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings. Then f is called a *triangular α-orbital admissible mapping respect to η* if

1. f is an α-orbital admissible respect to η.
2. For all $x, y \in X$, $\alpha(x, y) \geq \eta(x, y), \alpha(y, fy) \geq \eta(y, fy)$ imply $\alpha(x, fy) \geq \eta(x, fy)$.

In 2014, Hussain *et al.* [10] introduced the notion of α-η-complete metric spaces and α-η-continuous functions.

Definition 2.5. ([10], Definition 4) Let (X, d) be a metric space, $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be mappings. Then

1. (X, d) is called *α-η-complete* if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ is a convergent sequence in (X, d).
2. (X, d) is called *α-complete* if X is α-η-complete with $\eta(x, y) = 1$ for all $x, y \in X$.

Remark 2.1. Every complete metric space is an α-η-complete metric space. However, [6, Example 1.12] proves that there exists an α-η-complete metric space which is not a complete metric space.

Definition 2.6. ([10], Definition 7) Let (X, d) be a metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be mappings. Then f is called an *α-η-continuous mapping* on (X, d) if for all $x \in X$, $\lim\limits_{n\to\infty} x_n = x, \alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ imply $\lim\limits_{n\to\infty} fx_n = fx$.

Remark 2.2.

1. Every continuous mapping is an α-η-continuous mapping. However, there exists an α-η-continuous mapping is not a continuous mapping, (see [6, Example 1.14]).
2. T is called α-continuous if T is α-η-continuous with $\eta(x, y) = 1$ for all $x, y \in X$.

In 2016, Ansari and Kaewcharoen [1] introduced the notion of a generalized α-η-ψ-φ-F-contraction type and stated some fixed point results for such contraction type in metric spaces as follows.

Definition 2.7. ([1], Definition 2.1) Let (X, d) be a metric space, $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ and $f : X \longrightarrow X$ be mappings. Then f is called a *generalized α-η-ψ-φ-F-contraction type* if there exist $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that for all $x, y \in X$ with $\alpha(x, y) \geqslant \eta(x, y)$, we have

$$\psi\big(d(fx, fy)\big) \leqslant F\big(\psi(M(x, y)), \varphi(M(x, y))\big)$$

where

$$M(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy)\Big\}.$$

Theorem 2.1. ([1], Theorem 2.3, Theorem 2.4, Theorem 2.5) *Let (X, d) be a metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *(X, d) is an α-η-complete metric space.*
2. *f is triangular α-orbital admissible respect to η.*
3. *f is an α-η-ψ-φ-F-contraction type.*
4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.*
5. *(a) Either f is α-η-continuous or*
   (b) If $\{x_n\}$ is a sequence in X and $\lim\limits_{n\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant \eta(x_{n(k)}, x)$ for all $k \in \mathbb{N}$.

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$ there exists $z \in X$ such that $\alpha(z, fz) \geq \eta(z, fz), \alpha(x, z) \geq \eta(x, z)$ and $\alpha(y, z) \geq \eta(y, z)$, then f has a unique fixed point.

3. Main results

First, we generalize the notion of triangular α-orbital admissible mappings to a pair of mappings as follows.

Definition 3.1. Let X be a non-empty set, $f, g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings. Then the pair (f, g) is called *triangular α-orbital admissible* if for all $x, y, z \in X$,

1. (L1) $\alpha(x, fx) \geq 1$ implies $\alpha(fx, gfx) \geq 1$.
2. (L2) $\alpha(x, y) \geq 1$ and $\alpha(y, fy) \geq 1$ imply $\alpha(x, fy) \geq 1$.
3. (L3) $\alpha(x, gx) \geq 1$ implies $\alpha(gx, fgx) \geq 1$.
4. (L4) $\alpha(x, y) \geq 1$ and $\alpha(y, gy) \geq 1$ imply $\alpha(x, gy) \geq 1$.

Lemma 3.1. *Let X be a non-empty set, $f, g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *The pair (f, g) is triangular α-orbital admissible.*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*

Then the sequence $\{x_n\}$ defined by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ satisfies $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m \neq n$.

Proof. Since $\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1$ and the property (L1) of the pair (f, g), we obtain $\alpha(x_1, x_2) = \alpha(fx_0, gfx_0) \geq 1$. Since $\alpha(x_1, x_2) \geqslant 1$ and $x_2 = gx_1$, we get $\alpha(x_1, gx_1) \geq 1$. By using the property (L3) of the pair (f, g), we obtain $\alpha(gx_1, fgx_1) \geq 1$. This implies that $\alpha(x_2, x_3) \geqslant 1$. Since $x_3 = fx_2$, we obtain $\alpha(x_2, fx_2) \geq 1$. By using the property (L1) of the pair (f, g), we obtain $\alpha(fx_2, gfx_2) \geq 1$. This implies $\alpha(x_3, x_4) \geqslant 1$. By continuing the process as above, we obtain $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Now, suppose that $\alpha(x_n, x_m) \geq 1$ for $m > n$. We will prove that $\alpha(x_n, x_{m+1}) \geq 1$ for $m > n$. If m is odd, $\alpha(x_m, gx_m) = \alpha(x_m, x_{m+1}) \geq 1$. Note that $\alpha(x_n, x_m) \geq 1$. From the property (L4) of the pair (f, g), we have $\alpha(x_n, x_{m+1}) = \alpha(x_n, gx_m) \geq 1$. If m is even, $\alpha(x_m, fx_m) = \alpha(x_m, x_{m+1}) \geq 1$. Note that $\alpha(x_n, x_m) \geq 1$. From the property (L2) of the pair (f, g), we get $\alpha(x_n, x_{m+1}) = \alpha(x_n, fx_m) \geq 1$. Therefore, $\alpha(x_n, x_m) \geq 1$ for all $m > n$. □

Next, we introduce the notion of a pair of ψ-φ-F-rational contraction type mappings in metric space.

Definition 3.2. Let (X, d) be a metric space, $\alpha : X \times X \longrightarrow [0, \infty)$ and $f, g : X \longrightarrow X$ be mappings. Then the pair (f, g) is called a *ψ-φ-F-rational contraction type* if there exist $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that for all $x, y \in X, x \neq y$ with $\alpha(x, y) \geqslant 1$, we have

$$\psi\big(d(fx, gy)\big) \leqslant F\big(\psi(H(x, y)), \varphi(H(x, y))\big) \tag{3.1}$$

where

$$H(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, gy)}{d(x, y)}\Big\}.$$

Definition 3.3. Let (X, d) be a metric space, $\alpha : X \times X \longrightarrow [0, \infty)$ and $f, g : X \longrightarrow X$ be mappings. Then the pair (f, g) is called a *ψ-φ-F_k-rational contraction type* if there exist $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$, we have

$$\psi\big(d(fx, gy)\big) \leqslant F\big(\psi(H_k(x, y)), \varphi(H_k(x, y))\big) \tag{3.2}$$

where

$$H_k(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, gy)}{k + d(x, y)}\Big\}.$$

The first main result is a sufficient condition for the existence of a common fixed point of a pair of mappings satisfying ψ-φ-F-rational contraction type in metric spaces.

Theorem 3.1. *Let (X, d) be a metric space, $f, g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *(X, d) is an α-complete metric space.*
2. *The pair (f, g) is triangular α-orbital admissible.*
3. *The pair (f, g) is a ψ-φ-F-rational contraction type.*
4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*
5. *f and g are α-continuous.*

Then f or g has a fixed point, or f and g have a common fixed point.

Proof. We define a sequence $\{x_n\}$ in X by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}$, where $\alpha(x_0, fx_0) \geq 1$. If there exists $n \in \mathbb{N}$ such that $x_{2n} = x_{2n+1}$, then $x_{2n} = fx_{2n}$, that is, x_{2n} is a fixed point of f. Similarly, if there exists $n \in \mathbb{N}$ such that $x_{2n+1} = x_{2n+2}$, then $x_{2n+1} = gx_{2n+1}$, that is, x_{2n+1} is a fixed point of g. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since the pair (f, g) is triangular α-orbital admissible, by using Lemma 3.1, we obtain the following for all $m,n\in\mathbb{N}, m>n$,

$$\alpha(x_n,x_m)\geq 1. \tag{3.3}$$

Since (f,g) is a ψ-φ-F-rational contraction type and using (3.3), we have

$$\begin{aligned}\psi\big(d(x_{2n+1},x_{2n+2})\big) &= \psi\big(d(fx_{2n},gx_{2n+1})\big)\\ &\leq F\big(\psi(H(x_{2n},x_{2n+1})),\varphi(H(x_{2n},x_{2n+1}))\big)\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}&H(x_{2n},x_{2n+1})\\ = &\max\Big\{d(x_{2n},x_{2n+1}),d(x_{2n},fx_{2n}),d(x_{2n+1},gx_{2n+1}),\\ &\frac{d(x_{2n},gx_{2n+1})+d(x_{2n+1},fx_{2n})}{2},\frac{d(x_{2n},fx_{2n})d(x_{2n+1},gx_{2n+1})}{d(x_{2n},x_{2n+1})}\Big\}\\ = &\max\Big\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\frac{d(x_{2n},x_{2n+2})}{2}\Big\}\\ \leq &\max\Big\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\frac{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}{2}\Big\}\\ = &\max\big\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\big\}.\end{aligned}$$

If there exists $n\in\mathbb{N}$ such that

$$\max\big\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\big\}=d(x_{2n+1},x_{2n+2})>\ 0,$$

then (3.4) becomes

$$\begin{aligned}\psi\big(d(x_{2n+1},x_{2n+2})\big) &\leq F\big(\psi(d(x_{2n+1},x_{2n+2})),\varphi(d(x_{2n+1},x_{2n+2}))\big)\\ &< \psi\big(d(x_{2n+1},x_{2n+2})\big).\end{aligned}$$

It is a contradiction. Therefore,

$$\max\big\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\big\}=d(x_{2n},x_{2n+1})>0$$

for all $n\in\mathbb{N}$. Then (3.4) becomes

$$\begin{aligned}\psi\big(d(x_{2n+1},x_{2n+2})\big) &\leq F\big(\psi(d(x_{2n},x_{2n+1})),\varphi(d(x_{2n},x_{2n+1}))\big)\\ &< \psi\big(d(x_{2n},x_{2n+1})\big)\end{aligned} \tag{3.5}$$

for all $n\in\mathbb{N}$. Moreover, since ψ is nondecreasing, we have

$$d(x_{2n+1},x_{2n+2})\leq d(x_{2n},x_{2n+1}) \tag{3.6}$$

for all $n\in\mathbb{N}$. Also, from (3.3) and(f,g) is a ψ-φ-F-rational contraction type, we have

$$\begin{aligned}\psi\big(d(x_{2n+1},x_{2n})\big) &= \psi\big(d(fx_{2n},gx_{2n-1})\big)\\ &\leq F\big(\psi(H(x_{2n},x_{2n-1})),\varphi(H(x_{2n},x_{2n-1}))\big).\end{aligned}$$

Similar to the above arguments, we also have

$$d(x_{2n+1}, x_{2n}) < d(x_{2n}, x_{2n-1}) \tag{3.7}$$

for all $n \in \mathbb{N}, n \geq 1$. Therefore, from (3.6) and (3.7), we obtain $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive numbers. Hence, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$. Then, taking the limit as $n \to \infty$ in (3.5), we obtain $\psi(r) \leq F\big(\psi(r), \varphi(r)\big)$. This implies that $F\big(\psi(r), \varphi(r)\big) = \psi(r)$. Then, $\psi(r) = 0$ or $\varphi(r) = 0$. So, we have $r = 0$. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. On the contrary, suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ where $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon. \tag{3.9}$$

It implies that

$$d(x_{2m(k)}, x_{2n(k)-2}) < \varepsilon. \tag{3.10}$$

Then, from (3.9) and (3.10), we have

$$\begin{aligned} \varepsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\ &< \varepsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}). \end{aligned} \tag{3.11}$$

Taking the limit as $k \to \infty$ in (3.11) and using (3.8), we get

$$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \varepsilon. \tag{3.12}$$

Moreover, we have

$$|d(x_{2m(k)}, x_{2n(k)-1}) - d(x_{2m(k)}, x_{2n(k)})| \leq d(x_{2n(k)-1}, x_{2n(k)}). \tag{3.13}$$

$$|d(x_{2m(k)}, x_{2n(k)-1}) - d(x_{2n(k)-1}, x_{2m(k)+1})| \leq d(x_{2m(k)}, x_{2m(k)+1}). \tag{3.14}$$

$$|d(x_{2m(k)+1}, x_{2n(k)}) - d(x_{2m(k)+1}, x_{2n(k)-1})| \leq d(x_{2n(k)}, x_{2n(k)-1}). \tag{3.15}$$

Taking the limit as $k \to \infty$ in (3.13), (3.14), (3.15) and using (3.8), (3.12) we obtain

$$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)-1}) = \varepsilon. \tag{3.16}$$

$$\lim_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1}) = \varepsilon. \tag{3.17}$$

$$\lim_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)}) = \varepsilon. \tag{3.18}$$

Since $\lim\limits_{k\to\infty} d(x_{2m(k)}, x_{2n(k)-1}) = \varepsilon > 0$, we have $d(x_{2m(k)}, x_{2n(k)-1}) > 0$ for all $k \geqslant k_0$ with some $k_0 \in \mathbb{N}$. For all $k \geqslant k_0$, since $2m(k) < 2n(k) - 1$ and using (3.3), we obtain $\alpha(x_{2m(k)}, x_{2n(k)-1}) \geq 1$. By using (3.1), we have

$$\begin{aligned} \psi\big(d(x_{2m(k)+1}, x_{2n(k)})\big) &= \psi\big(d(fx_{2m(k)}, gx_{2n(k)-1})\big) \\ &\leq F\big(\psi(H(x_{2m(k)}, x_{2n(k)-1})), \varphi(H(x_{2m(k)}, x_{2n(k)-1}))\big) \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} & H(x_{2m(k)}, x_{2n(k)-1}) \\ = \; & \max\Big\{ d(x_{2m(k)}, x_{2n(k)-1}), d(x_{2m(k)}, x_{2m(k)+1}), d(x_{2n(k)-1}, x_{2n(k)}), \\ & \qquad \frac{d(x_{2m(k)}, x_{2n(k)}) + d(x_{2n(k)-1}, x_{2m(k)+1})}{2}, \\ & \qquad \frac{d(x_{2m(k)}, x_{2m(k)+1}) d(x_{2n(k)-1}, x_{2n(k)})}{d(x_{2m(k)}, x_{2n(k)-1})} \Big\}. \end{aligned} \tag{3.20}$$

Taking the limit as $k \to \infty$ in (3.20) and using (3.8), (3.12), (3.16), (3.17), we obtain

$$\lim_{k\to\infty} H(x_{2m(k)}, x_{2n(k)-1}) = \max\Big\{\varepsilon, 0, 0, \frac{\varepsilon+\varepsilon}{2}, 0\Big\} = \varepsilon. \tag{3.21}$$

Taking the limit as $k \to \infty$ in (3.19), using the continuity of F, ψ, φ and (3.18), (3.21), we have

$$\psi(\varepsilon) \leq F\big(\psi(\varepsilon), \varphi(\varepsilon)\big).$$

It follows from the property of F that $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$. This implies that $\varepsilon = 0$ which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is an α-complete metric space and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$. Since f and g are α-continuous mappings, we have

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = f(\lim_{n\to\infty} x_{2n}) = fx$$

and

$$x = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} gx_{2n+1} = g(\lim_{n\to\infty} x_{2n+1}) = gx.$$

This implies that x is a common fixed point of f and g. $\square$

The second main result is a sufficient condition for the existence of a common fixed point of a pair of mappings satisfying ψ-φ-F_k-rational contraction type in metric spaces.

Theorem 3.2. *Let (X,d) be a metric space, $f,g : X \longrightarrow X$ and $\alpha : X\times X \longrightarrow [0,\infty)$ be mappings such that*

1. *(X,d) is an α-complete metric space.*
2. *The pair (f,g) is triangular α-orbital admissible.*
3. *The pair (f,g) is a ψ-φ-F_k-rational contraction type.*
4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*
5. *If $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_{2n}, x) \geq 1$ and $\alpha(x, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$.*

Then f or g has a fixed point, or f and g have a common fixed point.

Proof. As in the proof of Theorem 3.1, we conclude that either f or g has a fixed point or the sequence $\{x_n\}$ defined by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}$ satisfies

$$\alpha(x_n, x_m) \geqslant 1, \tag{3.22}$$

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{3.23}$$

for all $n,m \in \mathbb{N}$ with $n > m$ and there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x. \tag{3.24}$$

Then, from the assumption (5), we obtain $\alpha(x_{2n}, x) \geq 1$ and $\alpha(x, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$. Since $\alpha(x_{2n}, x) \geq 1$, (f,g) is triangular α-orbital admissible, we have

$$\psi\big(d(x_{2n+1}, gx)\big) = \psi\big(d(fx_{2n}, gx)\big) \leq F\big(\psi(H(x_{2n}, x), \varphi H(x_{2n}, x))\big) \tag{3.25}$$

where

$$\begin{aligned} H(x_{2n}, x) \quad = \quad & \max\Big\{d(x_{2n}, x), d(x_{2n}, x_{2n+1}), d(x, gx), \\ & \frac{d(x_{2n}, gx) + d(x, x_{2n+1})}{2}, \frac{d(x_{2n}, x_{2n+1}) d(x, gx)}{k + d(x, x_{2n})}\Big\}. \end{aligned} \tag{3.26}$$

Taking the limit as $n \to \infty$ in (3.26) and using (3.23), (3.24), we get

$$\lim_{n\to\infty} H(x_{2n}, x) = d(x, gx). \tag{3.27}$$

Taking the limit as $n \to \infty$ in (3.25), using the continuity of F, ψ, φ and (3.27), we obtain

$$\psi\big(d(x, gx)\big) \leq F\big(\psi(d(x, gx), \varphi(d(x, gx))\big).$$

By using the property of F, we have $\psi\big(d(x, gx)\big) = 0$ or $\varphi\big(d(x, gx)\big) = 0$. This implies that $d(x, gx) = 0$. Hence, $gx = x$. Similarly, we also have $fx = x$. Therefore, x is a common fixed point of f and g. $\square$

The following theorems are the sufficient conditions for the existence of a unique common fixed point of the pair of mappings satisfying ψ-φ-F-rational contraction type and ψ-φ-F_k-rational contraction type in metric spaces.

Theorem 3.3. *Suppose all assumptions of Theorem 3.1 hold. Assume that for all $x, y \in X, x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1$, $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Then f or g has a fixed point or f and g have a unique common fixed point.*

Proof. By Theorem 3.1, f or g has a fixed point or f and g have a common fixed point. Suppose that x, y are two common fixed point of f, g such that $x \neq y$. By the assumption, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x, z) \geq 1$. Since (f, g) is triangular α-orbital admissible, we have $\alpha(x, fz) \geq 1$. Since $\alpha(z, fz) \geq 1$ and using Theorem 3.1, we deduce that

$$\lim_{n\to\infty} z_n = z^* \tag{3.28}$$

where $z^* \in X$ and $\{z_n\}$ is defined by $z_0 = z, z_{2n+1} = fz_{2n}$ and $z_{2n+2} = gz_{2n+1}$ for all $n \in \mathbb{N}$.

Moreover, $\alpha(x, z_1) = \alpha(x, fz) \geq 1$, and (f, g) is triangular α-orbital admissible, we have $\alpha(x, z_2) = \alpha(fx, gz_1) \geq 1$. This implies that $\alpha(x, z_3) = \alpha(gx, fz_2) \geq 1$. Continue this process, we have $\alpha(x, z_n) \geq 1$ for all $n \in \mathbb{N}$. We consider two following cases.

Case 1. If there exists $z_{n_0} \in X$ such that $z_{n_0} = x$, then

$$\lim_{n\to\infty} z_n = x. \tag{3.29}$$

By using (3.28) and (3.29), we obtain $x = z^*$.

Case 2. If $z_n \neq x$ for all $n \in \mathbb{N}$, then using (3.1), we obtain

$$\begin{aligned} \psi\big(d(x, z_{2n+2})\big) &= \psi\big(d(fx, gz_{2n+1})\big) \\ &\leq F\big(\psi(H(x, z_{2n+1})), \varphi(H(x, z_{2n+1}))\big) \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} H(x, z_{2n+1}) &= \max\Big\{d(x, z_{2n+1}), d(x, fx), d(z_{2n+1}, gz_{2n+1}), \\ &\quad \frac{d(x, gz_{2n+1}) + d(z_{2n+1}, fx)}{2}, \frac{d(x, fx)d(z_{2n+1}, gz_{2n+1})}{d(x, z_{2n+1})}\Big\} \\ &= \max\Big\{d(x, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(x, z_{2n+2}) + d(z_{2n+1}, x)}{2}\Big\}. \end{aligned} \tag{3.31}$$

Taking the limit as $n \to \infty$ in (3.31), we have

$$\lim_{n\to\infty} H(x, z_{n+1}) = d(x, z^*). \tag{3.32}$$

Taking the limit as $n \to \infty$ in (3.30) and (3.32), we have

$$\psi\big(d(x, z^*)\big) \leq F\big(\psi(d(x, z^*)), \varphi(d(x, z^*))\big). \tag{3.33}$$

By using the property of F, we have $\psi(d(x, z^*)) = 0$ or $\varphi(d(x, z^*)) = 0$. This implies that $d(x, z^*) = 0$. This means $x = z^*$.

From the above cases, we conclude that $x = z^*$. Similar, we also obtain $y = z^*$. Therefore, $x = y$ and hence the common fixed point of f and g is unique. □

Theorem 3.4. *Suppose all assumptions of Theorem 3.2 hold. Assume that for all $x, y \in X, x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1$, $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Then f or g has a fixed point or f and g have a unique common fixed point.*

Proof. The proof is similar to the proof of Theorem 3.3. □

By choosing $f = g$ in Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, we get the following results.

Corollary 3.1. *Let (X, d) be a metric space, $f : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *(X, d) is an α-complete metric space.*
2. *f is a triangular α-orbital admissible mapping.*
3. *For all $x, y \in X, x \neq y$ with $\alpha(x, y) \geq 1$, there exist $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that*

$$\alpha(x, y)\psi\big(d(fx, fy)\big) \leqslant F\big(\psi(H^f(x, y)), \varphi(H^f(x, y))\big)$$

where

$$H^f(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, fy)}{d(x, y)}\Big\}.$$

4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*
5. *f is α-continuous.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then f has a unique fixed point.

Corollary 3.2. *Let (X, d) be a metric space, $f : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *(X, d) is an α-complete metric space.*

2. *f is a triangular α-orbital admissible mapping.*

3. *For all $x, y \in X$ with $\alpha(x,y) \geq 1$, there exist $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that*
$$\alpha(x,y)\psi\big(d(fx,fy)\big) \leqslant F\big(\psi(H_k^f(x,y)), \varphi(H_k^f(x,y))\big)$$
where
$$H_k^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{k+d(x,y)}\Big\}.$$

4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*

5. *If $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$, then f has a unique fixed point.

By using the arguments as in the proof of [3, Theorem 2.2], from Corollary 3.1 and Corollary 3.2, we obtain the following results. These can be viewed as extending analogues of Theorem 2.1.

Corollary 3.3. *Let (X, d) be a metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *(X, d) is an α-η-complete metric space.*

2. *f is a triangular α-orbital admissible mapping respect to η.*

3. *For all $x, y \in X, x \neq y$ with $\alpha(x,y) \geq \eta(x,y)$, there exist $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that*
$$\psi\big(d(fx,fy)\big) \leqslant F\big(\psi(H^f(x,y)), \varphi(H^f(x,y))\big)$$
where
$$H^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{d(x,y)}\Big\}.$$

4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.*

5. *f is α-η-continuous.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq \eta(z, fz), \alpha(x,z) \geq \eta(x,z)$ and $\alpha(y,z) \geq \eta(y,z)$, then f has a unique fixed point.

Corollary 3.4. *Let (X,d) be a metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *(X,d) is an α-η-complete metric space.*
2. *f is a triangular α-orbital admissible mapping respect to η.*
3. *For all $x, y \in X$ with $\alpha(x,y) \geq \eta(x,y)$, there exist $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that*
$$\psi\big(d(fx,fy)\big) \leqslant F\big(\psi(H_k^f(x,y)), \varphi(H_k^f(x,y))\big)$$
where
$$H_k^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{k+d(x,y)}\Big\}.$$
4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.*
5. *If $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z,fz) \geq \eta(z,fz), \alpha(x,z) \geq \eta(x,z)$ and $\alpha(y,z) \geq \eta(y,z)$, then f has a unique fixed point.

In Corollary 3.1 and Corollary 3.2, by choosing $F(s,t) = s\beta(s)$ for all $s,t \in [0,\infty)$ where $\beta : [0,\infty) \longrightarrow [0,1)$ is a continuous function, we obtain the following corollaries. These results can be viewed as the extending analogues of [12, 14] with the condition " $\lim\limits_{n\to\infty} \beta(t_n) = 0$ implying that $\lim\limits_{n\to\infty} t_n = 1$" replaced by "$\beta$ is continuous".

Corollary 3.5. *Let (X,d) be a complete metric space, $f : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *f is a triangular α-orbital admissible mapping.*
2. *For all $x, y \in X, x \neq y$ with $\alpha(x,y) \geq 1$, there exist $\psi \in \Psi, \varphi \in \Phi$ and $\beta : [0,\infty) \longrightarrow [0,1)$ is a continuous function such that*
$$\alpha(x,y)\psi\big(d(fx,fy)\big) \leqslant \psi\big(H^f(x,y)\big).\beta\big(\psi(H^f(x,y)\big)$$
where
$$H^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{d(x,y)}\Big\}.$$
3. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*

4. f is continuous.

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then f has a unique fixed point.

Corollary 3.6. *Let (X, d) be a complete metric space, $f : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *f is a triangular α-orbital admissible mapping.*
2. *For all $x, y \in X$ with $\alpha(x, y) \geq 1$, there exist $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ and $\beta : [0, \infty) \longrightarrow [0, 1)$ is a continuous function such that*
$$\alpha(x, y)\psi\big(d(fx, fy)\big) \leqslant \psi\big(H_k^f(x, y)\big).\beta\big(\psi(H_k^f(x, y)\big)$$
where
$$H_k^f(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, fy)}{k + d(x, y)}\Big\}.$$
3. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*
4. *If $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then f has a unique fixed point.

In Corollary 3.3 and Corollary 3.4, by choosing $F(s, t) = s\beta(s)$ for all $s, t \in [0, \infty)$ where $\beta : [0, \infty) \longrightarrow [0, 1)$ is a continuous function, we obtain the following corollaries. These results can be viewed as extending analogues of [6, Theorem 2.7, Theorem 2.8, Theorem 2.9] with the condition "$\lim\limits_{n\to\infty} \beta(t_n) = 0$ implies $\lim\limits_{n\to\infty} t_n = 1$" replaced by "$\beta$ is continuous" .

Corollary 3.7. *Let (X, d) be a complete metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *f is a triangular α-orbital admissible mapping respect to η.*
2. *For all $x, y \in X, x \neq y$ with $\alpha(x, y) \geq \eta(x, y)$, there exist $\psi \in \Psi, \varphi \in \Phi$ and $\beta : [0, \infty) \longrightarrow [0, 1)$ is a continuous function such that*
$$\psi\big(d(fx, fy)\big) \leqslant \psi\big(H^f(x, y)\big).\beta\big(\psi(H^f(x, y)\big)$$
where
$$H^f(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, fy)}{d(x, y)}\Big\}.$$

3. *There exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.
4. f *is continuous.*

Then f *has a fixed point. Moreover, if for all* $x, y \in X$, $x \neq y$, *there exists* $z \in X$ *such that* $\alpha(z, fz) \geq \eta(z, fz), \alpha(x, z) \geq \eta(x, z)$ *and* $\alpha(y, z) \geq \eta(y, z)$ *then* f *has a unique fixed point.*

Corollary 3.8. *Let* (X, d) *be a complete metric space,* $f : X \longrightarrow X$ *and* $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ *be mappings such that*

1. f *is a triangular* α*-orbital admissible mapping respect to* η.
2. *For all* $x, y \in X$ *with* $\alpha(x, y) \geq \eta(x, y)$, *there exist* $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ *and* $\beta : [0, \infty) \longrightarrow [0, 1)$ *is a continuous function such that*
$$\psi\big(d(fx, fy)\big) \leqslant \psi\big(H_k^f(x, y)\big).\beta\big(\psi(H_k^f(x, y))\big)$$
where
$$H_k^f(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, fy)}{k + d(x, y)}\Big\}.$$
3. *There exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.
4. *If* $\{x_n\}$ *is a sequence in* X *such that* $\lim\limits_{n\to\infty} x_n = x$ *and* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$, *then* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N}$.

Then f *has a fixed point. Moreover, if for all* $x, y \in X$, $x \neq y$, *there exists* $z \in X$ *such that* $\alpha(z, fz) \geq \eta(z, fz), \alpha(x, z) \geq \eta(x, z)$ *and* $\alpha(y, z) \geq \eta(y, z)$, *then* f *has a unique fixed point.*

The following example shows that there exist $f, F, \alpha, \eta, \psi, \varphi$ such that Corollary 3.3 can be applied.

Example 3.1. Let $X = \{1, 2, 3, 4, 5\}$ and metric d on X as follows.
$$d(x, y) = \begin{cases} \frac{1}{4} & \text{if } (x, y) \in \{(2, 4); (3, 4); (3, 5); (4, 2); (4, 3); (4, 5); (5; 3); (5, 4)\} \\ 0 & \text{if } x = y \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Define $f : X \longrightarrow X, \alpha, \eta : X \times X \longrightarrow [0, \infty)$, $F : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ and $\varphi, \psi : [0, \infty) \longrightarrow [0, \infty)$ by
$$f1 = f4 = 1; f2 = 3; f3 = f5 = 2,$$
$$\alpha(x, y) = \begin{cases} \frac{1}{2} & \text{if } (x, y) \in \{(1, 1); (3, 5); (4, 1); (4, 2); (4, 3); (4, 5); (5, 3)\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta(x,y) = \frac{1}{2} \text{ for all } x, y \in X,$$
$$F(s,t) = s - t \text{ for all } s, t \in [0,\infty),$$
$$\psi(t) = t, \varphi(t) = \frac{t^2}{4} \text{ for all } t \in [0,\infty).$$

Then Corollary 3.3 can be applied to $f, F, \alpha, \eta, \psi, \varphi$.

Proof. For all $x, y \in X, x \neq y$ with $\alpha(x,y) \geqslant \eta(x,y)$, we obtain

$$(x,y) \in \{(3,5);(4,1);(4,2);(4,3);(4,5);(5,3)\}.$$

We consider the following cases.

Case 1. $(x,y) \in \{(3,5);(4,1);(5,3)\}$. Then $\psi\big(d(f(x), f(y))\big) = 0$ and

$$\begin{aligned} H^f(3,5) &= \max\Big\{d(3,5), d(3,f3), d(5,f5), \frac{d(3,f5)+d(5,f3)}{2}, \frac{d(3,f3)d(5,f5)}{d(3,5)}\Big\} \\ &= \max\Big\{\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\Big\} \\ &= 1, \end{aligned}$$

$$\begin{aligned} H^f(4,1) &= \max\Big\{d(4,1), d(4,f4), d(1,f1), \frac{d(4,f1)+d(1,f4)}{2}, \frac{d(4,f4)d(1,f1)}{d(4,1)}\Big\} \\ &= \max\Big\{\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{4}, 0\Big\} \\ &= \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} H^f(5,3) &= \max\Big\{d(5,3), d(5,f5), d(3,f3), \frac{d(5,f3)+d(3,f5)}{2}, \frac{d(5,f5)d(3,f3)}{d(5,3)}\Big\} \\ &= \max\Big\{\frac{1}{4}, \frac{1}{2}, , \frac{1}{2}, , \frac{1}{2}, 1\Big\} \\ &= 1. \end{aligned}$$

Therefore, $F\big(\psi(H^f(x,y)), \varphi(H^f(x,y))\big) = H^f(x,y) - \frac{(H^f(x,y))^2}{4} > 0 = \psi\big(d(f(x), f(y))\big)$.

Case 2. $(x,y) \in \{(4,2);(4,3);(4,5)\}$. Then

$$\psi\big(d(f4, f2)\big) = \psi\big(d(1,3)\big) = \frac{1}{2},$$

$$\psi\big(d(f4, f3)\big) = \psi\big(d(1,2)\big) = \frac{1}{2},$$

$$\psi\big(d(f4,f5)\big)=\psi\big(d(1,2)\big)=\frac{1}{2}.$$

$$\begin{aligned}H^f(4,2) &= \max\Big\{d(4,2),d(4,f4),d(2,f2),\frac{d(4,f2)+d(2,f4)}{2},\frac{d(4,f4)d(2,f2)}{d(4,2)}\Big\}\\ &= \max\Big\{\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{8},1\Big\}\\ &= 1.\end{aligned}$$

$$\begin{aligned}H^f(4,3) &= \max\Big\{d(4,3),d(4,f4),d(3,f3),\frac{d(4,f3)+d(3,f4)}{2},\frac{d(4,f4)d(3,f3)}{d(4,3)}\Big\}\\ &= \max\Big\{\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{8},1\Big\}\\ &= 1.\end{aligned}$$

$$\begin{aligned}H^f(4,5) &= \max\Big\{d(4,5),d(4,f4),d(5,f5),\frac{d(4,f5)+d(5,f4)}{2},\frac{d(4,f4)d(5,f5)}{d(4,5)}\Big\}\\ &= \max\Big\{\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{8},1\Big\}\\ &= 1.\end{aligned}$$

Therefore

$$F\big(\psi(H^f(x,y)),\varphi(H^f(x,y))\big)=H^f(x,y)-\frac{(H^f(x,y))^2}{4}=\frac{3}{4}>\frac{1}{2}=\psi\big(d(f(x),f(y))\big).$$

Hence, the inequality (3.1) is satisfied for all $x,y\in X, x\neq y$ with $\alpha(x,y)\geqslant\eta(x,y)$.

Next, we claim that f is a triangular α-orbital admissible respect to η. Indeed, since $\alpha(x,fx)\geqslant\eta(x,fx)$, we have $x=1$ or $x=4$. Then

$$\alpha(f1,f^21)=\alpha(1,1)\geqslant\eta(1,1)=\eta(f1,f^21),\alpha(f4,f^24)=\alpha(1,1)\geqslant\eta(1,1)=\eta(f4,f^24).$$

Hence, f is an α-orbital admissible respect to η. Since $\alpha(x,y)\geqslant\eta(x,y)$, $\alpha(y,fy)\geqslant\eta(y,fy)$ implies $(x,y)=(1,1)$ or $(x,y)=(4,1)$. Then,

$$\alpha(4,f1)=\alpha(4,1)\geqslant\eta(4,1)=\eta(4,f1),\alpha(1,f4)=\alpha(1,1)\geqslant\eta(1,1)=\eta(1,f1).$$

Hence, f is a triangular α-orbital admissible respect to η. Furthermore, all assumptions in Corollary 3.3 are satisfied. Then Corollary 3.3 can be applied to $f,F,\alpha,\eta,\psi,\varphi$ given. $\square$

Finally, we apply Theorem 3.2 to study the existence of solutions to the system of nonlinear integral equations.

Theorem 3.5. *Let $C[a,b]$ be a set of all continuous functions on $[a,b]$ and d be a metric defined by*

$$d(u,v) = \sup_{t\in[a,b]} |u(t) - v(t)|$$

for all $u, v \in C[a,b]$. Consider the system of nonlinear integral equations

$$\left\{ \begin{array}{l} u(t) = \varphi(t) + \displaystyle\int_a^b K_1(t,s,u(s))ds \\ u(t) = \varphi(t) + \displaystyle\int_a^b K_2(t,s,u(s))ds \end{array} \right. \tag{3.34}$$

where $t \in [a,b], \varphi : [a,b] \longrightarrow \mathbb{R}, K_1, K_2 : [a,b] \times [a,b] \times [a,b] \longrightarrow \mathbb{R}$. Suppose that the following statements hold.

1. *$K_1(t,s,u(s))$ and $K_2(t,s,u(s))$ are integrable with respect to s on $[a,b]$.*

2. *$fu, gu \in C[a,b]$ for all $u \in C[a,b]$, where*

$$fu(t) = \varphi(t) + \int_a^b K_1(t,s,u(s))ds,$$

$$gu(t) = \varphi(t) + \int_a^b K_2(t,s,u(s))ds$$

   for all $t \in [a,b]$.

3. *For all $u \in C[a,b]$ such that $u(t) \geq 0$ for all $t \in [a,b]$, we have $fu(t) \geq 0$ and $gu(t) \geq 0$ for all $t \in [a,b]$.*

4. *For all $s,t \in [a,b]$ and $u, v \in C[a,b]$ such that $u(t) \neq v(t)$ and $u(t), v(t) \in [0,\infty)$, we have*

$$\begin{aligned} & |K_1(t,s,u(s)) - K_2(t,s,v(s))| \\ \leq \ & \phi(t,s) \max\Big\{|u(s)-v(s)|, |u(s)-fu(s)|, |v(s)-gv(s)|, \\ & \frac{|u(s)-gv(s)| + |v(s)-fu(s)|}{2}, \frac{|u(s)-fu(s)||v(s)-gv(s)|}{1+|u(s)-v(s)|}\Big\} \end{aligned}$$

   where $\phi : [a,b] \times [a,b] \longrightarrow [0,\infty)$ is a continuous function satisfying

$$0 < \sup_{t\in[a,b]} \Big(\int_a^b \phi(t,s)ds \Big) < 1.$$

5. *There exists $u_0 \in C[a,b]$ such that $u_0(t) \geq 0$ for all $t \in [a,b]$.*

Then the system of nonlinear integral equations (3.34) has a solution $u \in C[a,b]$.

Proof. Consider $f, g : C[a,b] \longrightarrow C[a,b]$ defined by

$$fu(t) = \varphi(t) + \int_a^b K_1(t,s,u(s))ds \quad \text{and} \quad gu(t) = \varphi(t) + \int_a^b K_2(t,s,u(s))ds$$

for all $u \in C[a,b]$ and $t \in [a,b]$. It follows from assumptions (1) and (2) that f and g are well-defined. Notice that the existence of a solution to (3.34) is equivalent to the existence of the common fixed point of f and g. Now, we shall prove that all assumptions of Theorem 3.2 are satisfied.

Define a mapping $\alpha : C[a,b] \times C[a,b] \longrightarrow \mathbb{R}$ by

$$\alpha(u,v) = \begin{cases} 1 & \text{if } u(t), v(t) \in [0,\infty) \text{ for all } t \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

(1) Since $(C[a,b], d)$ is a complete metric space, $(C[a,b], d)$ is a α-complete metric space.

(2) We claim that the pair (f,g) is triangular α-orbital admissible. Indeed,

(L1) For all $u \in C[a,b]$ such that $\alpha(u, fu) \geq 1$, we have $u(t), fu(t) \in [0,\infty)$ for all $t \in [a,b]$. It follows from assumption (3), we conclude that $gfu(t) \geq 0$ for all $t \in [a,b]$. Therefore, $\alpha(fu, gfu) \geq 1$.

(L2) For all $u, v \in C[a,b]$ such that $\alpha(u,v) \geq 1$ and $\alpha(v, fv) \geq 1$, we obtain $u(t), fv(t) \in [0,\infty)$. Thus, $\alpha(u, fv) \geq 1$.

(L3) For all $u \in C[a,b]$ such that $\alpha(u, gu) \geq 1$, we have $u(t), gu(t) \in [0,\infty)$ for all $t \in [a,b]$. It follows from assumption (3), we conclude that $fgu(t) \geq 0$ for all $t \in [a,b]$. Therefore, $\alpha(gu, fgu) \geq 1$.

(L4) For all $u, v \in C[a,b]$ such that $\alpha(u,v) \geq 1$ and $\alpha(v, gv) \geq 1$, we obtain $u(t), gv(t) \in [0,\infty)$. Thus, $\alpha(u, gv) \geq 1$.

From the above, we conclude that the pair (f,g) is triangular α-orbital admissible.

(3) We claim that the pair (f,g) is a ψ-φ-F-rational contraction mapping with $F(s,t) = \lambda s$ for all $s, t \in [a,b]$ and $0 < \lambda < 1$. Indeed, let $u, v \in C[a,b]$ with $u \neq v$ and $\alpha(u,v) \geq 1$. Then $u(t), v(t) \in [0,\infty)$ for all $t \in [a,b]$. Therefore, from assumption (4), we have

$$\begin{aligned} |fu(t) - gv(t)| &\leq \int_a^b |K_1(t,s,u(s)) - K_2(t,s,v(s))|ds \\ &\leq \int_a^b \Big(\phi(t,s) \max\Big\{|u(s)-v(s)|, |u(s)-fu(s)|, |v(s)-gv(s)|, \\ &\quad \frac{|u(s)-gv(s)| + |v(s)-fu(s)|}{2}, \frac{|u(s)-fu(s)||v(s)-gv(s)|}{1+|u(s)-v(s)|}\Big\}\Big)ds \\ &\leq H(u,v) \int_a^b \phi(t,s)ds \\ &\leq \lambda H(u,v). \end{aligned}$$

where $\lambda = \sup_{t\in[a,b]} \Big(\int_a^b \alpha(t,s)ds \Big)$ and $H(u,v)$ defined by (3.1). It implies that

$$d(fu, fv) \le \lambda H(u,v).$$

Therefore, the pair (f,g) is a ψ-φ-F-rational contraction mapping with $\psi(t) = t$, $F(s,t) = \lambda s$ for all $s,t \in [0,\infty)$, $0 < \lambda < 1$.

(4) We claim that there exists $u_0 \in C[a,b]$ such that $\alpha(u_0, fu_0) \ge 1$. Indeed, from assumption (5), there exists $u_0 \in C[a,b]$ such that $u_0(t) \ge 0$ for all $t \in [a,b]$. By using assumption (3), we see that $fu_0(t) \ge 0$ for all $t \in [a,b]$. Therefore, $\alpha(u_0, fu_0) \ge 1$.

(5) We claim that assumption (5) in Theorem 3.2 holds. Indeed, let $\{u_n\}$ be a sequence in $C[a,b]$ such that $\lim_{n\to\infty} u_n = u$ and $\alpha(u_n, u_{n+1}) \ge 1$. Then $u(t) \ge 0$ and $u_n(t) \ge 0$ for all $t \in [a,b]$ and $n \in \mathbb{N}$. Therefore, $\alpha(u_{2n}, u) \ge 1$ and $\alpha(u, u_{2n+1}) \ge 1$.

By the above, all assumptions of Theorem 3.2 are satisfied. Then, f and g have a common fixed point $u \in C[a,b]$ and the system of integral equations (3.34) has a solution $u \in C[a,b]$. $\square$

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