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# AN APPLICATION OF THE ECF METHOD AND NUMERICAL INTEGRATION IN ESTIMATION OF THE STOCHASTIC VOLATILITY MODELS*

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**Abstract.** In this paper, the Empirical Characteristic Function (ECF) method is described in parameter estimations of the stochastic volatility (SV) models, as well as the original thresholds modification (and a generalization) of these models, named the Split-SV model. The estimation procedure is based on minimization of the objective function which represents the double integral with respect to some weight function $g : \mathbb{R}^2 \to \mathbb{R}$. Some typical, exponential classes of the weight functions $g(u_1, u_2)$ are considered, as well as the different types of cubature formulas. Estimation procedures are realized by the original authors' codes written in statistical programming language "R", and the performances of the ECF method are examined, by statistical and numerical aspects. The numerical simulation of the obtained estimates is also given. Finally, the standard SV model, and the Split-SV model as its alternative, are applied for fitting the empirical data: the daily returns of the exchange rates of GBP and USD per euro, and the efficiency of their fitting is compared.

**Keywords**: SV models, ECF estimation, numerical integration

## 1. Introduction and Definition of the Model

The stochastic volatility (SV) models represent a particularly important class of nonlinear stochastic models, commonly used in the stochastic modeling of financial sequences. Firstly introduced by Taylor [26], today there are many modifications and generalizations of these models (see, e.g., [3, 5, 20, 27]). In our interpretation, we used the so-called *Noise-Indicator model of Stochastic Volatility*, or the *Split-SV*

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*model*, defined in [17, 22] by the following relations:

$$\text{(1.1)} \qquad \begin{cases} X_t &= \sigma_t \varepsilon_t, \\ \sigma_t &= \sigma\, \mathrm{e}^{\frac{1}{2}\Delta_t}, \\ \Delta_t &= a\,\Delta_{t-1} + \xi_t\, q_{t-1}, \end{cases} \qquad t \in \mathbb{Z}.$$

Here, we denoted the following sequences:

- $(X_t)$ is an empirically (financial) sequence,
- $(\sigma_t)$ is the *volatility sequence*, well-known as a measure of uncertainty in the fluctuations of (financial) series $(X_t)$,
- $(\varepsilon_t)$ is a sequence of independent identically distributed (i.i.d.) random variables (RVs) with Gaussian $\mathcal{N}(0,1)$ distribution,
- $(\xi_t)$ is a i.i.d. sequence of RVs with Gaussian $\mathcal{N}(0,\delta^2)$ distribution, mutually independent of the $(\varepsilon_t)$,
- $(\Delta_t)$ is the autoregressive (AR) series with the coefficient $a \neq 0$, as well as with the "optional" noise $(\xi_t)$. Namely, we suppose that the sequence $(q_t)$, so-called *noise-indicator*, satisfies the equality

$$q_t(c) = I(\xi_t^2 \geq c) = \begin{cases} 1, & \xi_t^2 \geq c; \\ 0, & \xi_t^2 < c. \end{cases}$$

  In that way, the so-called *critical value of reaction* $c > 0$ indicates the realizations of the series $(\xi_t)$ which are sufficiently statistically significant so that their values are to be included in Eq. (1.1).

By its definition, the Split-SV model can explain the nonlinearity in the behavior of series $(X_t)$, i.e. its volatility series $(\sigma_t)$, similarly as it was done in the time series of autoregressive conditional heteroscedasticity (ARCH) type, described in [16, 21], as well as in the stochastic permanent breaking (STOPBREAK) processes, described in [23, 24]. Moreover, Split-SV model generalizes the standard Taylor's SV model, which is obtained from (1.1), when $c \to 0$. In the following, for the critical value $c$ we denote the mathematical expectation

$$m_c = E\Big[I(\xi_t^2 \geq c)\Big] = P\{\xi_t^2 \geq c\} = 1 - F_{\chi_1^2}\left(\frac{c}{\delta^2}\right),$$

where $F_{\chi_1^2}$ is the distribution function (DF) of the chi square random variable with one degree of freedom. In that way, $m_c$ is uniquely determined by the constant $c$, and vice versa. As could be seen bellow, it enables a "two-stage" estimation of the critical value $c$.

Let us remark that, unlike the Taylor's SV model, the RVs $\Delta_t$ have not "the usual" Gaussian distribution. Using the conditional probability, we can find the DF of the variable $\eta_t = \xi_t q_{t-1}$, which appears in the definition of $\Delta_t$, as

$$F_\eta(x) = m_c F_\xi(x) + (1 - m_c)F_0(x), \tag{1.2}$$

where $F_\xi(x)$ is the DF of $\xi_t : \mathcal{N}(0, \delta^2)$, and

$$F_0(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases} \tag{1.3}$$

is the DF of $Z_0 \overset{\text{as}}{=} 0$. So, the function $F_\eta(x)$ is continuous almost everywhere, with the discontinuity in point $x = 0$, where it has the jump of the order $1 - m_c$ (Fig. 1.1). Therefore, $\eta_t$ are mixtures of RVs of the Gaussian and discrete types, usually called the *Contaminated Gaussian Distributions* (*CGDs*). This is a very important difference between the Split-SV model and the Taylor's SV model, which prevents some of the standard procedures in the investigation of its properties. In the following, we give a short discussion on the properties of our model.

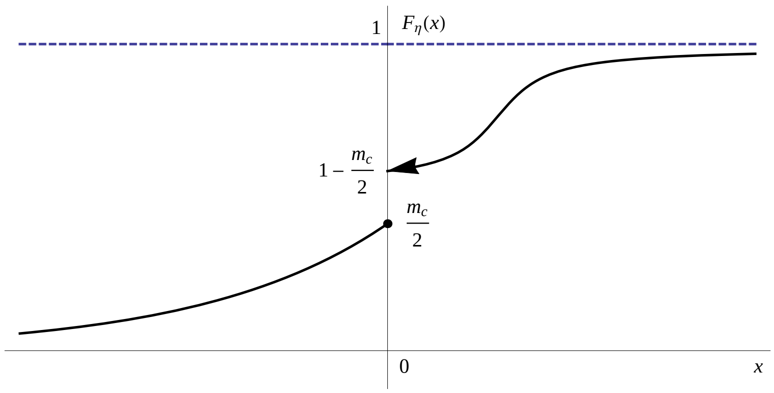


FIG. 1.1: The distribution function of CGD random variable.

## 2. Properties of the Model

In the sequel, we assume that the stationarity condition $|a| < 1$ is valid. Then, the RVs $\Delta_t$ have the mean $\mu := E(\Delta_t) = 0$, and the variance

$$\upsilon^2 := D(\Delta_t) = \frac{D(\xi_t q_{t-1})}{1 - a^2} = \frac{\delta^2 m_c}{1 - a^2}. \tag{2.1}$$

After some simple computations (see, for more details [22]), the autocovariance function of $\Delta_t$ can be obtain as

$$\text{Cov}(\Delta_t, \Delta_{t+k}) = \frac{\delta^2 m_c a^k}{1 - a^2},$$

as well as the autocorrelation function (ACF)

$$\mathrm{Corr}(\Delta_t, \Delta_{t+k}) = \frac{\mathrm{Cov}(\Delta_t, \Delta_{t+k})}{\sqrt{D(\Delta_t)D(\Delta_{t+k})}} = a^k,$$

which has the same form as in the case of standard linear autoregressive models.

Due to stationarity of the series $(\Delta_t)$, and applying the conditional probability, we can find the DF of this sequence

$$F_\Delta(x) = m_c(G \otimes F_\xi)(x) + (1 - m_c)G(x), \tag{2.2}$$

where $F_\xi$ denotes the DF of $\xi_t$,

$$G(x) := \begin{cases} 1 - F_\Delta(x/a), & a \in (-1, 0), \\ F_\Delta(x/a), & a \in (0, 1), \end{cases} \tag{2.3}$$

and "⊗" denotes the convolution of appropriate DFs, i.e.

$$(G \otimes F_\xi)(x) = \int_{-\infty}^{+\infty} G(x-u)F_\xi(du).$$

Finally, the characteristic function (CF) of $\Delta_t$ is

$$\varphi_\Delta(u) := \prod_{j=0}^{+\infty} \left[1 + m_c\left(\mathrm{e}^{-\frac{1}{2}a^{2j}u^2\delta^2} - 1\right)\right], \tag{2.4}$$

and the analytic expression of $\varphi_\Delta(u)$ in a closed form cannot be done. It can also be seen that the distribution of $\Delta_t$ is non-Gaussian by investigating its CF (see, for more details [17, 22]).

On the other hand, the sequence $(X_t)$ is the martingale difference and, therefore, the sequence of uncorrelated RVs. According to this, it holds

$$E(X_t) = E\Big[E\Big(X_t \,\Big|\, \mathcal{F}_{t-1}\Big)\Big] = 0,$$

but the determination of the variance of $X_t$ is somewhat complicated. Using the same procedure as in determining CF of $\Delta_t$, we obtain

$$D(X_t) = \sigma^2 \prod_{j=0}^{+\infty} \left[1 + m_c\left(\mathrm{e}^{\frac{1}{2}a^{2j}\delta^2} - 1\right)\right],$$

and the autocovariance function of $X_t^2$ is

$$\gamma(k) = R(k) - [D(X_t)]^2,$$

where

$$R(k) := E\left(X_t^2 X_{t-k}^2\right) = \sigma^4 E\left(\mathrm{e}^{\Delta_t + \Delta_{t-k}}\right).$$

Similarly, the ACF of this series is defined by

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \frac{R(k) - [D(X_t)]^2}{E(X_t^4) - [D(X_t)]^2} = \frac{S(k) - 1}{K_X - 1},$$

where $S(k) = R(k)\,[D(X_t)]^{-2}$ and $K_X$ is the kurtosis of $X_t$. When $k \to \infty$, it follows that

(2.5) $$\rho(k) \sim \frac{e^{\frac{m_c \delta^2 a^k}{1-a^2}} - 1}{K_X - 1} \longrightarrow 0, \qquad k \to \infty,$$

i.e., the ACF of the Split-SV model, when $|a| < 1$, has approximately the same decreasing rate as the standard SV models. The kurtosis satisfies the condition $K_X \geq 3$, and this points out the typical feature of fat tail distribution of the financial time series $(X_t)$ at mean $E(X_t) = 0$. Also, it is easy to show that the time series $(X_t)$ is ergodic in mean.

## 3. Estimation of Parameters by ECF Method

Parameters estimation procedures of SV models are much more complex than the other similar nonlinear stochastic models, because of their specific structure. The two most commonly used methods, introduced almost at the same time, are the simulated maximum likelihood method [2] and quasi-likelihood method [19], as well as their various modifications (see, e.g., [8, 9]). Unfortunately, in the case of our model, it can be proved that the likelihood function is unlimited at the point $x = 0$, and thus disenables the usage of all the well-known parameters estimation methods based on the maximum likelihood approach. For these reasons, we use the *method of Empirical Characteristic Function* (*ECF method*), which is conducive in the cases when the maximum likelihood approach encounters difficulties, as in the case of our model. The usage of the ECF method here was inspired by works [10, 28, 29] where, for the first time, an implementation of this method in econometrics analysis of financial series was described in details.

ECF method is based on the fact that CF of some RV has the same information as its DF and, consequently, the empirical characteristic function (ECF) preserves the whole information from the sample. In that way, the main goal of this method is the minimization of "the distance" between the CF and its appropriate ECF. What can be used, in general, for this purpose is the $p$-dimensional CF of the following logarithmic process

(3.1) $$Y_t := \log X_t^2 = \log \sigma^2 + \Delta_t + \nu_t,$$

where $t = 1, \dots, T$ and $\nu_t := \log \varepsilon_t^2$. If $\mathbf{u} = (u_1, \dots, u_p)' \in \mathbb{R}^p$ is some vector and $\mathbf{Y}_t^{(p)} := (Y_t, \dots, Y_{t+p-1})'$, $t = 1, \dots, T - p + 1$, so called, overlapping blocks, then the CF of the random vector $\mathbf{Y}_t^{(p)}$ is

(3.2) $$\varphi_Y^{(p)}(\mathbf{u}; \theta) := E\left[\exp\left(\mathrm{i}\mathbf{u}'\mathbf{Y}_t^{(p)}\right)\right].$$

The appropriate $p$-dimensional ECF of the random sample $Y_1, \ldots, Y_T$ is

$$\widetilde{\varphi}_T^{(p)}(\mathbf{u}) := \frac{1}{T-p+1} \sum_{t=1}^{T-p+1} \exp\left(\mathrm{i}\mathbf{u}'\mathbf{Y}_t^{(p)}\right),$$

and the objective function can be written as

$$S_T^{(p)}(\theta) := \int \cdots \int_{\mathbb{R}^p} g(\mathbf{u}) \left| \varphi_Y^{(p)}(\mathbf{u}; \theta) - \widetilde{\varphi}_T^{(p)}(\mathbf{u}) \right|^2 \mathbf{du}, \tag{3.3}$$

where $\mathbf{du} = du_1 \cdots du_p$ and $g : \mathbb{R}^p \to \mathbb{R}^+$ is a some weight function. Therefore, ECF estimates $\hat{\theta}_T^{(p)}$ are obtained by the minimization of the objective function $S_T^{(p)}(\theta)$ with respect to parameter $\theta := (a, \sigma, \delta, m_c) \in \mathbb{R}^4$, i.e.

$$\hat{\theta}_T^{(p)} = \arg\min_{\theta \in \Theta} S_T^{(p)}(\theta), \tag{3.4}$$

where $\Theta = (-1, 1) \times (0, +\infty) \times (0, +\infty) \times (0, 1)$ is the parameter space of the stationary Split-SV model. In [22] we investigated the asymptotic properties of the ECF estimates of the parameters of our model. Under some necessary regularity conditions, we proved a strong consistency and asymptotic normality (AN) of these estimates. We get the estimators of orders $p = 2$, i.e., we used the procedure based on two-dimensional vector $\mathbf{Y}_t^{(2)} = (Y_t, Y_{t+1})$. The analytic expression of its CF is

$$\begin{aligned} \varphi_Y^{(2)}(u_1, u_2; \theta) \quad = \quad & \frac{(2\sigma^2)^{\mathrm{i}(u_1+u_2)}}{\pi} \prod_{j=0}^{+\infty} \left[ 1 + m_c \left( \mathrm{e}^{-\frac{1}{2} a^{2j} (u_1 + a u_2)^2 \delta^2} - 1 \right) \right] \\ & \times \left[ 1 + m_c \left( \mathrm{e}^{-\frac{1}{2} u_2^2 \delta^2} - 1 \right) \right] \Gamma\left( \mathrm{i}u_1 + \frac{1}{2} \right) \Gamma\left( \mathrm{i}u_2 + \frac{1}{2} \right). \end{aligned}$$

The real and the imaginary part of this CF, when $a = 1/2$ and $\sigma = \delta = c = 1$, are shown in Fig. 3.1.

In this case, the objective function (3.3) represents a double integral with respect to the some weight function $g : \mathbb{R}^2 \to \mathbb{R}^+$. In our investigation we consider some typical, exponential weight functions $g(u_1, u_2)$, which put more weight around the origin. This is in accordance with the fact that characteristic functions (CFs) contain the most of information around this point. On the other hand, exponential weight functions have the numerical advantage, because the integral in (3.3) can be numerically approximated by using some $N$-point cubature formula

$$I(f; g) = \iint_{\mathbb{R}^2} g(u_1, u_2) f(u_1, u_2) \mathrm{d}u_1 \mathrm{d}u_2 \approx C_N(f) = \sum_{j=1}^{N} w_j f(u_{1j}, u_{2j}), \tag{3.5}$$

where the nodes $(u_{1j}, u_{2j})$ belong to $\mathbb{R}^2$ and $w_j$ are the corresponding weight coefficients.

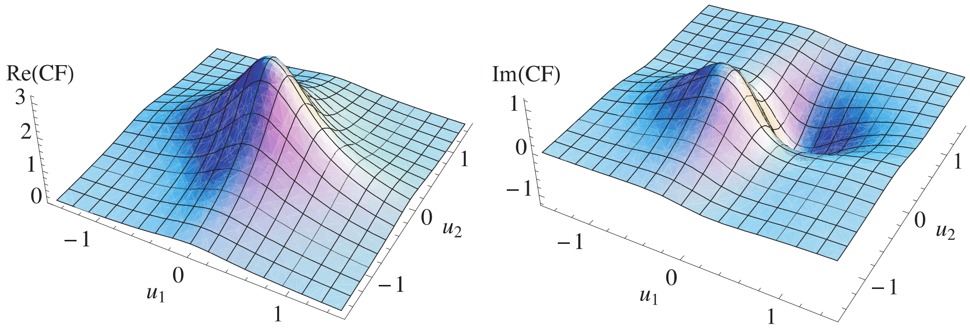


FIG. 3.1: Real and imaginary parts of two-dimensional CF of the sequence $\mathbf{Y}_t^{(2)} = (Y_t, Y_{t+1})$.

In our investigation the exponential weights $g(u_1, u_2)$ will be assumed to be symmetric in $u_1$ and $u_2$ and positive, i.e.,

$$g(-u_1, u_2) = g(u_1, -u_2) = g(u_1, u_2) > 0.$$

The construction of such cubature formulas can be done by constructive methods for Gaussian quadrature formulas and realized using the MATHEMATICA package `OrthogonalPolynomials` (see [1, 13] for more details). This package can be downloaded freely from Web Site `http://www.mi.sanu.ac.rs/~gvm/`.

## 4. Cubature Methods and Numerical Simulation

Using the above mentioned procedures, by different choices of weight functions, we examined the performance of the ECF method, by statistical and numerical aspects. We consider three weight functions $g(u_1, u_2)$:

1° $g_1(u_1, u_2) = \mathrm{e}^{-\gamma(u_1^2+u_2^2)}$,

2° $g_2(u_1, u_2) = \mathrm{e}^{-\gamma\sqrt{u_1^2+u_2^2}}$,

3° $g_3(u_1, u_2) = \mathrm{e}^{-\gamma(|u_1|+|u_2|)}$,

with a parameter $\gamma > 0$.

For the first weight $g_1(u_1, u_2)$, we use a product cubature formula based on the one-dimensional Gauss-Radau formula (cf. [11, pp. 329–330]) with respect to an exponential weight on $(0, +\infty)$, and trapezoidal quadrature in angular coordinate. Namely, introducing polar coordinates $u_1 = r\cos\theta$ and $u_2 = r\sin\theta$, the integral $I(f; g_1)$ in (3.5) reduces to

$$I(f; g_1) = \int_0^{+\infty} r\mathrm{e}^{-\gamma r^2} S(r)\,\mathrm{d}r, \tag{4.1}$$

where $S(r)$ is given by

$$S(r) = \int_{\pi}^{\pi} f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta. \tag{4.2}$$

The integral (4.2) can be approximated by the composite trapezoidal rule in $4m$ points $\theta_j = -\pi + j\pi/(2m)$, $j = 0, 1, \ldots, 4m$,

$$S(r) \approx S_m(r) = \frac{2\pi}{4m}\left\{\frac{1}{2}f(-r,0) + \sum_{j=1}^{4m-1} f(r\cos\theta_j, r\sin\theta_j) + \frac{1}{2}f(-r,0)\right\}.$$

Using the nodes

$$x_\nu = \cos\frac{\nu\pi}{2m}, \quad y_\nu = \sin\frac{\nu\pi}{2m}, \quad \nu = 1, \ldots, m,$$

after certain transformations, $S_m(r)$ can be represented in the form

$$S_m(r) = \frac{\pi}{2m}\sum_{\nu=1}^{m}\Big[f(rx_\nu, ry_\nu) + f(-rx_\nu, -ry_\nu) + f(ry_\nu, -rx_\nu) + f(-ry_\nu, rx_\nu)\Big].$$

Now, for calculating (4.1) we use the $(n+1)$-point Gauss-Radau formula with respect to the exponential weight $r \mapsto r\mathrm{e}^{-\gamma r^2}$ on $(0, +\infty)$,

$$\int_0^{+\infty} r\mathrm{e}^{-\gamma r^{\alpha}} S(r)\,\mathrm{d}r \approx Q_n(S) = A_0 S(0) + \sum_{k=1}^{n} A_k S(r_k), \tag{4.3}$$

where the nodes $r_k$ are zeros of the polynomial $\pi_n(r)$ orthogonalnal on $(0, +\infty)$ with respect to the weight function $r \mapsto r^2\mathrm{e}^{-\gamma r^2}$. The corresponding Christoffel numbers $w_k$ for this weight function give the weight coefficients in the formula (4.3), i.e.,

$$A_k = \frac{w_k}{r_k}, \quad k = 1, \ldots, n, \quad A_0 = \frac{1}{2\gamma} - \sum_{k=1}^{n} A_k.$$

For details see [11, pp. 329–330] (see also [12], [14]).

In this way, we obtain the cubature formula

$$I(f; g_1) \approx C_N(f) = Q_n(S_m) = A_0 S_m(0) + \sum_{k=1}^{n} A_k S_m(r_k) \quad (N = 4mn + 1),$$

i.e.,

$$\begin{aligned} C_N(f) &= 2\pi A_0 f(0,0) + \frac{\pi}{2m}\sum_{k=1}^{n} A_k \sum_{\nu=1}^{m}\Big[f(r_k x_\nu, r_k y_\nu) + f(-r_k x_\nu, -r_k y_\nu) \\ &\qquad + f(r_k y_\nu, -r_k x_\nu) + f(-r_k y_\nu, r_k x_\nu)\Big]. \end{aligned}$$

Numerical construction of the Gauss-Radau formula (4.3) for an arbitrary number of points can be done by the MATHEMATICA package ``OrthogonalPolynomials'' (see [1, 13]). In our calculations we use $C_N$ with $N = 81$ nodes ($n = 5$, $m = 4$), and with $\gamma \in \{1, 1/2, 3/2\}$.

Alternatively, for all weights 1°–3° we use, also, the so-called *perfectly symmetric two-dimensional integration formulas with minimal number of nodes* [18] (see also [25], [4], [15]). Such a cubature formula has the nodes of the form $(\pm u_{1j}, \pm u_{2j})$ and $(\pm u_{2j}, \pm u_{1j})$ with the same weight $w_j$. According to Rabinowitz and Richter [18] we call it a "good" formula if all of its weights are positive.

In order to obtain "good" cubature rules for the weights 1° and 2° of degree $d \leqslant 7$, Stroud and Secrest [25] used the nodes whose generators are of the form $(0, 0), (a, 0), (b, b)$. However, for rules of degree $d \geqslant 8$, it is necessary to include nodes whose generators are of the form $(c, d)$. As we can see $(c, d)$ generates eight nodes of the form: $(\pm c, \pm d), (\pm d, \pm c)$ with the same weight, while $(a, 0)$ and $(b, b)$ generate only four nodes: $(\pm a, 0)$, $(0, \pm a)$ and $(\pm b, \pm b)$, respectively. Of course, $(0, 0)$ gives only one node $(0, 0)$. We recall that a two-dimensional rule of degree $d$ integrates exactly all monomials $u_1^i u_2^j$ for which $i + j \leqslant d$. In our calculations, we use the corresponding 44-point cubature formulas of degree $d = 15$, with the following generators: $(a_j, 0)$, $j = 1, 2, 3, 4$; $(b_j, b_j)$, $j = 1, 2, 3$; $(c_j, d_j)$, $j = 1, 2$.

Using the previously mentioned MATHEMATICA package, these formulas can be obtained for all weights 1°–3°. Following [18], the method of construction needs the following "moments", i.e., integrals of the form

$$\mu_{jk}^{(\nu)} = \int_0^{+\infty}\int_0^{+\infty} g_\nu(u_1, u_2)(u_1^2 - u_2^2)^2(u_1^2u_2^2)^j(u_1^2 + u_2^2)^k \,\mathrm{d}u_1\mathrm{d}u_2 \quad (\nu = 1, 2, 3),$$

where $j \geqslant 1$, $k \geqslant 0$. They can be calculated in an analytic form for each of the weight functions $g_\nu$, $\nu = 1, 2, 3$. For example, for $\gamma = 1$,

$$\mu_{jk}^{(1)} = \frac{(2j + k + 2)!\pi}{2^{4j+3}(j + 1)}\binom{2j}{j} \qquad \text{and} \qquad \mu_{jk}^{(2)} = \frac{(4j + 2k + 5)!\pi}{2^{4j+2}(j + 1)}\binom{2j}{j}.$$

In the third case it can be expressed by the following integral

$$\mu_{jk}^{(3)} = \frac{(4j + 2k + 5)!}{2^{4j+k+1}} \int_0^1 \sqrt{z}\,(1 - z)^{2j}(1 + z)^k \mathrm{d}z$$

or in terms of hypergeometric functions as

$$\mu_{jk}^{(3)} = \frac{(4j + 2k + 5)!}{2^{k-2}(2j + 1)\binom{4(j + 1)}{2(j + 1)}}\Bigg\{ {}_2F_1\left(-\frac{1}{2}, -k; 2j + \frac{5}{2}; -1\right) + \frac{k - 2j - 1}{2(j + 1)}\, {}_2F_1\left(\frac{1}{2}, -k; 2j + \frac{5}{2}; -1\right)\Bigg\}.$$

For example, the obtained (generator) nodes in the 44-point cubature formula of degree $d = 15$ for the weight function $g_3$ ($\gamma = 1$) are:

```
{{16.75517334835192,0}, {9.520295794790188,0}, {4.451284933071043,0},
{1.326612922551803,0}, {10.40246868263913,10.40246868263913},
{6.307197292644404,6.307197292644404},{2.533316709591005,2.533316709591005},
{13.16709143114937,3.265192228507983},{6.770241049738993,2.369872911188105}},
```

and the corresponding weight coefficients are:

```
{8.186694686950403*(10ˆ-7), 0.0006529201474967032, 0.06663038092243385,
 0.8569723144924805, 6.812119062461652*(10ˆ-8), 0.00007773406088317548,
 0.07219519187714604, 2.913841882561950*(10ˆ-6), 0.001732372012567657}.
```

Distribution of nodes in this cubature formula is shown in Fig. 4.1.

**Remark 4.1.** Formulas for $\gamma \neq 1$ can be obtained by ones for $\gamma = 1$ by the simple changes of variables in (3.5), e.g., $u_1 := u_1/\sqrt{\gamma}$, $u_2 := u_2/\sqrt{\gamma}$ in the case of the first weight function $g_1$.

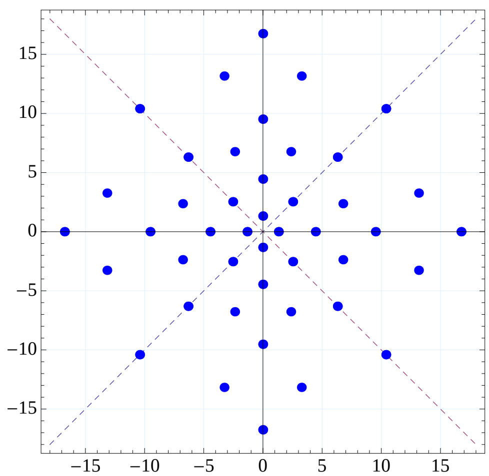


FIG. 4.1: Distribution of nodes in 44-point cubature formula of degree $d = 15$ for the weight function $g_3$ ($\gamma = 1$)

After numerical construction of cubature rules, the objective function is minimized by a Nelder-Mead method, and estimation procedures are realized by the original authors' codes written in statistical programming language "R". Using these procedures, by different choices of weights, we examine the performance of the ECF method in estimation of our model. For the true value of the parameter $\theta$ we choose the vector $\theta_0 = (a_0, \sigma_0, \delta_0, c_0) = (0.5, 1, 1, 1)$, and we calculate 120 independent Monte Carlo simulations of Split-SV model, i.e., the 120 independent realizations of the series $(X_t)$ and $(Y_t)$ of the length $T = 10\,000$. The starting points of the optimization procedures were the Bayesian estimates of parameters of SV models, obtained by the modification of Markov Chain Monte Carlo (MCMC) method, described in [7]. In the next step, using these estimates as the initial values, we obtained ECF estimates of parameters of the standard SV model (with $c = 0$). After that, we used the estimates of the SV model as the initial values, and we estimated parameters of the Split-SV model. As the initial value of the critical value $c$ we take $c = \delta$, the mean value of the random variable $\delta^{-1}\xi_1^2$, and the initial estimates of $\delta$ are obtained according to $\delta^2 = m_c^{-1}\delta_{SV}^2$, where $m_c = F_{\chi_1^2}^{-1}(1) \approx 0.3173$ and $\delta_{SV}^2$ is the variance of the appropriate Gaussian noise of the standard SV model.

The results of these estimation procedures are shown in Fig. 4.2. The estimates with respect to the weight $g_1(u_1, u_2)$ have a slightly smaller values of estimated errors, especially for the parameters $\sigma$ and $\delta$. The corresponding results, with respect to the weights $g_2(u_1, u_2)$ and $g_3(u_1, u_2)$, in our simulations study, have a similar properties, according to the corresponding estimated errors.

In the following, we investigate the asymptotic properties (strong consistency and asymptotic normality) of the ECF estimates of the parameters of our model. They are, as we have already pointed out, formally proved in [22]. In the light of calculated simulations, the consistency is a particularly confirmed for all parameters. Especially, it is most pronounced in the case of estimated values of parameter $\sigma$. According to these, it can be examined and some conclusions can be made about the asymptotic normality of the obtained estimates. It can be easily seen that the asymptotic normality is confirmed in the cases of the parameters $a$ and $\sigma$, but it varies for the rest of the parameters. This is understandable because of the specific threshold structure of our model and because these estimates are obtained in a two-stage procedure, based on the previously determined estimates of parameter $m_c \in (0, 1)$. Let us point out that this fact is also verified in [22], where the Anderson-Darling's and Cramer-von Mises's tests of normality were used.

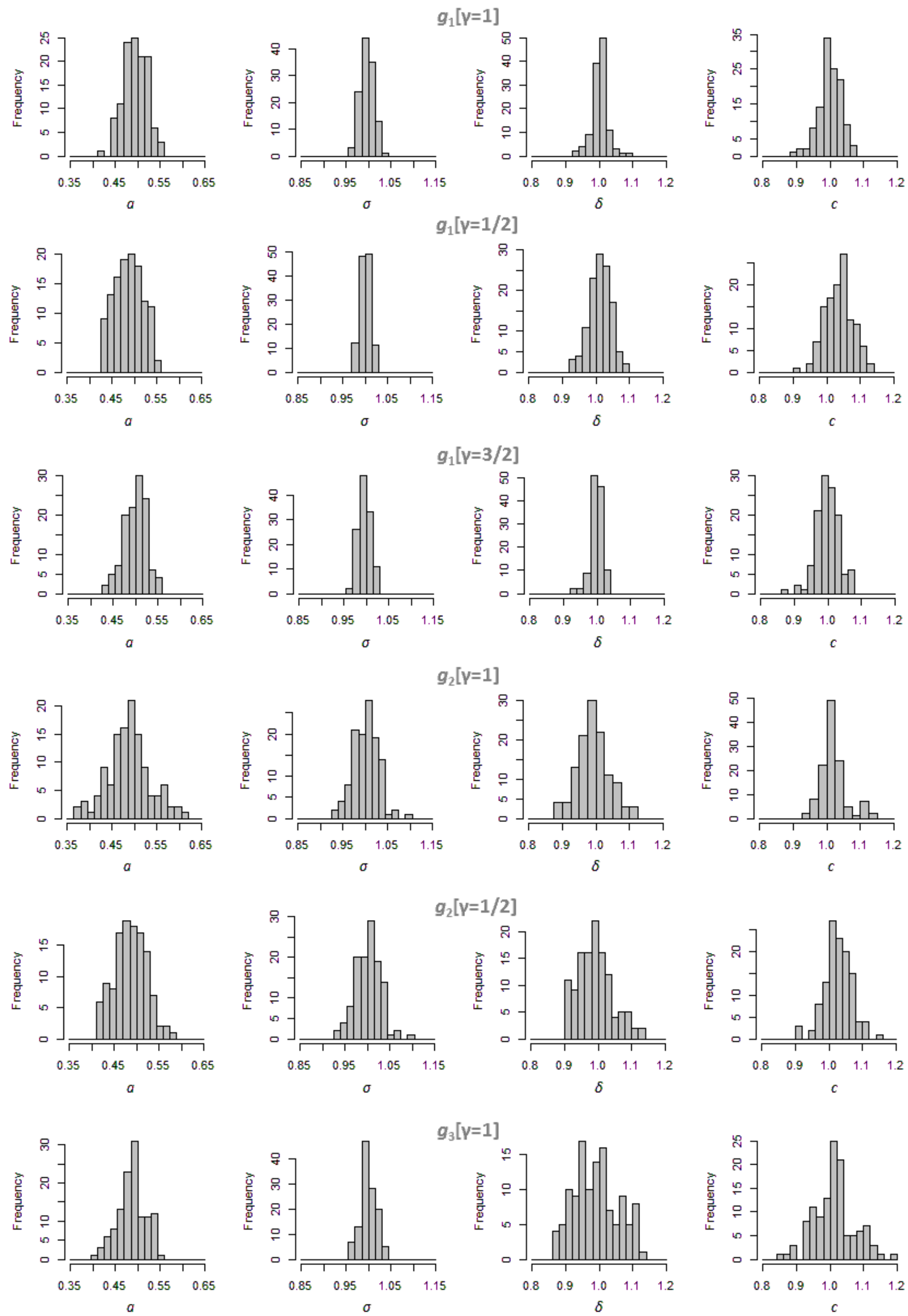


FIG. 4.2: Empirical distributions of estimated parameters, based on 120 MC simulations of Split-SV model

## 5. Application of the ECF Method

In this, the final part of our paper, the standard SV model and the Split-SV model are applied for fitting the empirical data: the daily returns of the exchange rates of British pound (GBP) and U.S. dollar (USD) against the euro, in the period 2003 to 2013 (Fig. 5.1).

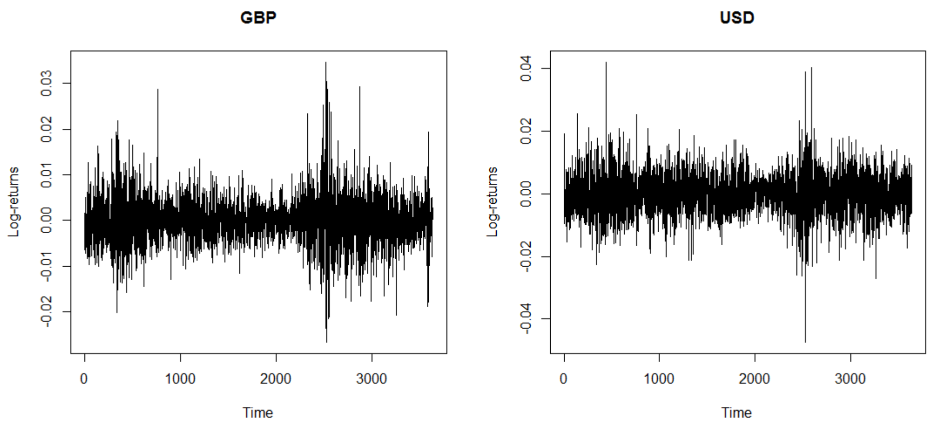


FIG. 5.1: The daily returns of the exchange rates of GBP and USD per one euro.

In the first step, similarly as in the previous simulation study, for both series were estimated values of parameters, assuming that their dynamics is subjected to the standard SV model. For this purpose, we first estimated parameters using the Bayesian estimates of SV models. The estimation algorithm, implemented in R by G. Kastner [6], realizes MCMC-simulates from the posterior distribution of the SV-parameters, along with the volatility estimates, via R-procedure ”svsample()” in R-package ”stochvol” (Figures 5.2 and 5.3).

After that, we applied the ECF method based on “usual” Gauss-Hermitian cubature, e.g., the weight function $g_1(u_1; u_2)$ with $\gamma = 1$, and we obtained estimates of SV model, also. They are used as the initial values for estimating parameters of Split-SV model, where we applied (again) the ECF method with the weights $g_i(u_1, u_2)$, $i = 1, 2, 3$. In that way, for both of the data series, we were able to compare the obtained results, i.e. the efficiency of their fittings, using the standard SV model, and the Split-SV model as its alternative.

Some conclusions were made by using several well known statistical procedures. First of all, we generated 1 500 simulations of the standard SV model, as well as of the Split-SV model, for all estimated values of their parameters. After that, the “distances” of the PDF’s of the simulated and empirical data were calculated using the corresponding mean-squared errors (MSE), the Kullback-Leibler

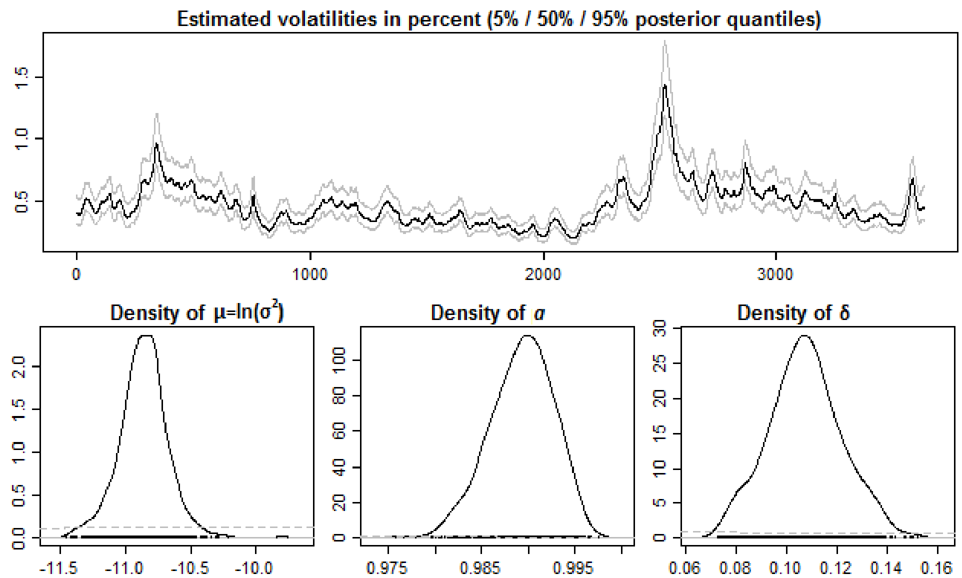


FIG. 5.2: Estimations of SV-parameters obtained by MCMC method (dataset is time series of daily returns of the exchange rates of GBP per one euro).

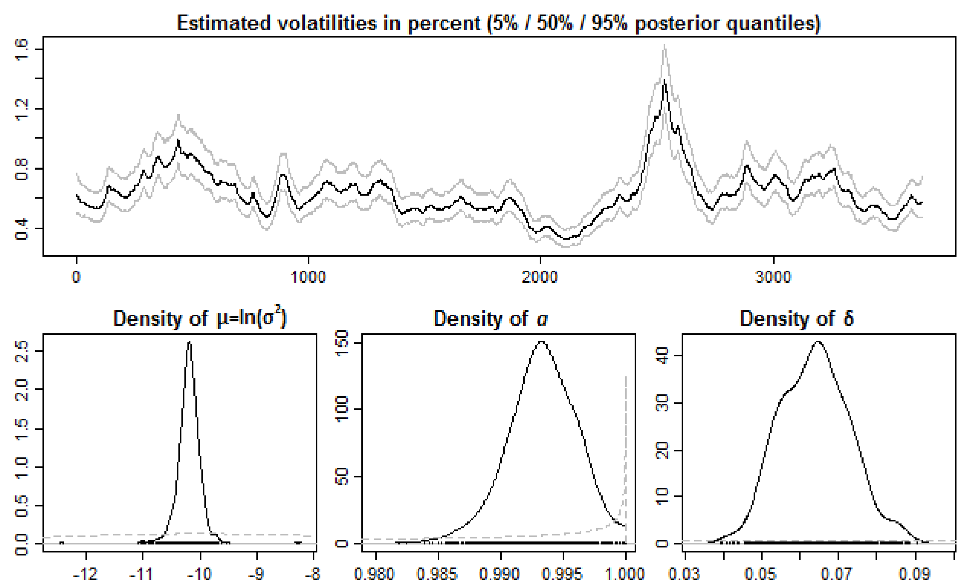


FIG. 5.3: Estimations of SV-parameters obtained by MCMC method (dataset is time series of daily returns of the exchange rates of USD per one euro).

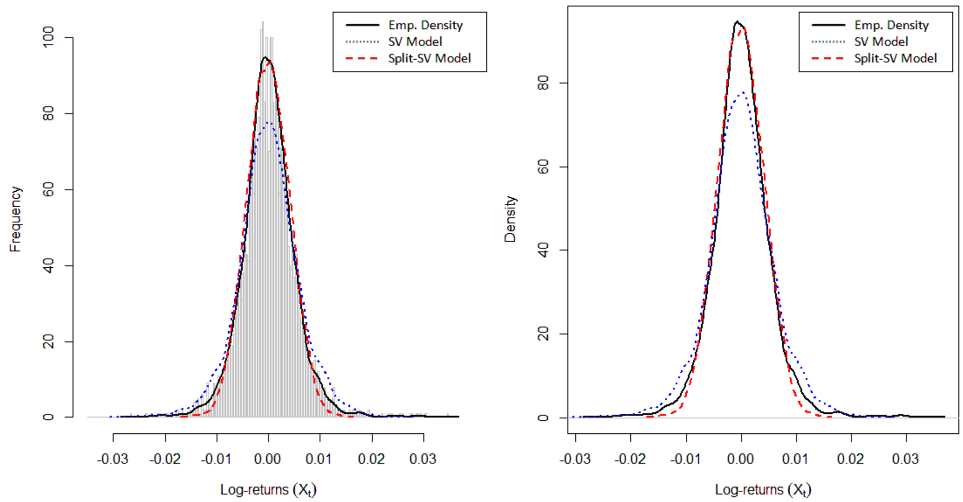


FIG. 5.4: Empirical and fitted log-returns densities of the exchange rates of GBP per one euro

divergence (KLD), and the two sample Kolmogorov-Smirnov (KS) test of the distributional equivalency. The detailed results of these procedures are given in [22], also.

The values of all of these statistics suggest on the relatively high degree of PDF's agreement of the empirical and simulated data. Therefore, the quality of the theoretical models for both the empirical data series, generated for all obtained estimates, is shown.

As an illustration, Figure 5.4 shows the empirical PDF of the original, GBP series, along with the PDFs of the fitting data, generated using both theoretical models. The estimated values of parameters in the case of daily log-returns of the GBP series are generally better when the Split-SV model was applied than with the standard SV model. Therefore, it seems that the Split-SV model is more adequate model than standard SV model in the case of this series.

On the other hand, in the case of daily log-returns of USD series, the estimates of the standard SV model and the Split-SV model have similar efficiency. However, there is (noticeably) better fitting of the appropriate PDFs in the most cases of the obtained estimates of the Split-SV model, which can be seen in Figure 5.5, also. It represents, as in the previous the GBP-series, the fitting using the both theoretical models, that provide the best match to empirical PDF.

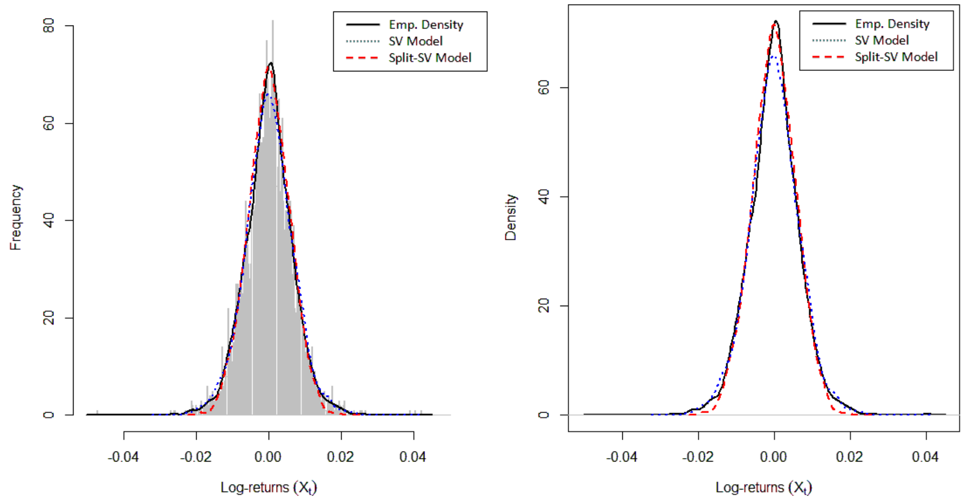


FIG. 5.5: Empirical and fitted log-returns densities of the exchange rates of USD per one euro

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