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SOME NEW ESTIMTES OF APPROXIMATION OF FUNCTIONS BY FOURIER-JACOBI SUMS

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Abstract. In this paper, several direct and inverse theorems are proved concerning the approximation of one-variable functions from the space $\mathbb{L}_2^{(\alpha,\beta)}$ by partial sums of Fourier-Jacobi series.

Keywords: Partial sums of Fourier-Jacobi series; Generalized translation operator; Generalized modulus of continuity; Estimate of approximation.

1. Introduction and Preliminaries

It is well known that many problems for partial differential equations are reduced to a power series expansion of the desired solution in terms of special functions or orthogonal polynomials (such as Laguerre, Hermite, Jacobi, etc., polynomials). In particular, this is associated with the separation of variables as applied to problems in mathematical physics (see, e.g.,[2]-[3]). In [8], Abilov et al. proved several estimates for the Fourier-Bessel series in the space $\mathbb{L}_2([0,1], x^{2p+1})$, $p > -\frac{1}{2}$, on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.

In this paper, we also discuss this subject. More specifically, several direct and inverse theorems are proved concerning the approximation of functions form the space $\mathbb{L}_2^{(\alpha,\beta)}$ by partial sums of Fourier-Jacobi series, analogous of the statements proved in [8]. For this purpose, we use a generalized translation operator which was defined by Flensted-Jensen and Koornwinder (see [5]).

Throughout the paper, α and β are arbitrary real numbers with $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$. We put $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ and consider problems of the approximation of functions in the Hilbert spaces $L_2([-1,1], w(x)dx)$.
Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi orthogonal polynomials, $n \in \mathbb{N}_0 := \{0, 1, 2, ...\}$ (see [4] or [1]). The polynomials $P_n^{(\alpha,\beta)}(x)$, $n \in \mathbb{N}_0$, form a complete orthogonal system in the

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Hilbert space $L_2([-1,1], w(x)dx)$.
It is known (see [4], Ch. IV) that

$$\max_{-1\leq x\leq 1} |P_n^{(\alpha,\beta)}(x)| = P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{\alpha} = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}.$$

The polynomials

$$R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)},$$

are called normalized Jacobi polynomials.
In what follows it is convenient to change the variable by the formula $x = \cos t, t \in I := [0,\pi]$. We use the notation

$$\rho(t) = w(\cos t)\sin t = 2^{\alpha+\beta+1}\left(\sin\frac{t}{2}\right)^{2\alpha+1}\left(\cos\frac{t}{2}\right)^{2\beta+1},$$

$$\varphi_n(t) = \varphi_n^{(\alpha,\beta)}(t) := R_n^{(\alpha,\beta)}(\cos t), n \in \mathbb{N}_0.$$

Let $\mathbb{L}_2^{(\alpha,\beta)}$ denote the space of square integrable functions $f(t)$ on the closed interval I with the weight function $\rho(t)$ and the norm

$$\|f\| = \sqrt{\int_0^\pi |f(t)|^2\rho(t)dt}.$$

The Jacobi differential operator is defined as

$$\mathcal{B} := \frac{d^2}{dt^2} + \left(\left(\alpha+\frac{1}{2}\right)\cot\frac{t}{2} - \left(\beta+\frac{1}{2}\right)\tan\frac{t}{2}\right)\frac{d}{dt}.$$

The function $\varphi_n(t)$ satisfies the differential equation

$$\mathcal{B}\varphi_n = -\lambda_n\varphi_n, \qquad \lambda_n = n(n+\alpha+\beta+1), n \in \mathbb{N}_0,$$

with the initial conditions $\varphi_n(0) = 1$ and $\varphi_n'(0) = 0$.

Lemma 1.1. *The following inequalities are valid for Jacobi functions $\varphi_n(t)$*
1. *For $t \in (0,\pi/2]$ we have*

$$|\varphi_n(t)| < 1.$$

2. *For $t \in [0,\pi/2]$ we have*

$$1 - \varphi_n(t) \leq c_1\lambda_n t^2.$$

3. *For every γ there is a number $c_2 = c_2(\gamma,\alpha,\beta) > 0$ such that for all n and t with $\gamma \leq nt \leq \frac{\pi n}{2}$ we have*

$$|\varphi_n(t)| \leq c_2(nt)^{-\alpha-1/2}.$$

Proof. (See [7], Proposition 3.5.) □

Recall from [7], the Fourier-Jacobi series of a function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ is defined by

$$f(t) = \sum_{n=0}^{\infty} a_n(f)\tilde{\varphi}_n(t), \tag{1.1}$$

where

$$\tilde{\varphi}_n = \frac{\varphi_n}{\|\varphi_n\|} \quad , \quad a_n(f) = \langle f, \tilde{\varphi}_n \rangle = \int_0^{\pi} f(t)\tilde{\varphi}_n(t)\rho(t)dt.$$

Let

$$S_m f(t) = \sum_{n=0}^{m-1} a_n(f)\tilde{\varphi}_n(t),$$

be a partial sum of series (1.1), and let

$$E_m(f) = \inf_{P_m} \|f - P_m\|,$$

denote the best approximation of $f \in \mathbb{L}_2^{(\alpha,\beta)}$ by polynomials of the form

$$P_m(t) = \sum_{n=0}^{m-1} c_n\tilde{\varphi}_n(t), c_n \in \mathbb{R}.$$

It is well known that

$$\|f\| = \sqrt{\sum_{n=0}^{\infty} |a_n(f)|^2},$$

$$E_m(f) = \|f - S_m f\| = \sqrt{\sum_{n=m}^{\infty} |a_n(f)|^2}.$$

The Jacobi generalized translation is defined by the formula

$$T_h f(t) = \int_0^{\pi} f(\theta)K(t,h,\theta)\rho(\theta)d\theta, \quad 0 < t, h < \pi,$$

where $K(t,s,\theta)$ is a certain function (see [6]).
Below are some properties (see [7]):
(i) $T_h : \mathbb{L}_2^{(\alpha,\beta)} \to \mathbb{L}_2^{(\alpha,\beta)}$ is a continuous linear operator,
(ii) $\|T_h f\| \leq \|f\|$,
(iii) $T_h(\varphi_n(t)) = \varphi_n(h)\varphi_n(t)$,
(iv) $a_n(T_h f) = \varphi_n(h)a_n(f)$,
(v) $\|T_h f - f\| \to 0, \quad h \to 0$,
(vi) $\mathcal{B}(T_h f) = T_h(\mathcal{B}f)$.
For every function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ we define the differences $\Delta_h^k f$ of order, $k \in \mathbb{N} = \{1, 2, 3, ...\}$, with step h, $0 < h < \pi$, and the modulus of smoothness $\Omega_k(f, \delta)$ by the

formulae

$$\Delta_h^1 f(t) = \Delta_h f(t) = (T_h - I)f(t),$$

where I is the identity operator in $\mathbb{L}_2^{(\alpha,\beta)}$.

$$\Delta_h^k f(t) = \Delta_h(\Delta_h^{k-1} f(t)) = (T_h - I)^k f(t) = \sum_{i=0}^{k} (-1)^{k-1} \binom{k}{i} T_h^i f(t), \quad k > 1,$$

$$\Omega_k(f, \delta) = \sup_{0<h\leq\delta} \|\Delta_h^k f\|, \quad \delta > 0,$$

where

$$T_h^0 f(t) = f(t), \quad T_h^i f(t) = T_h(T_h^{i-1} f(t)), \quad i = 1, 2, ..., k.$$

Let W_2^r, $r \in \mathbb{N}_0$, denote the class of functions $f \in \mathbb{L}_2^{(\alpha,\beta)}$ that have generalized derivatives satisfying $\mathcal{B}^r f \in \mathbb{L}_2^{(\alpha,\beta)}$,
i.e.,

$$W_2^r := \{f \in \mathbb{L}_2^{(\alpha,\beta)} : \mathcal{B}^r f \in \mathbb{L}_2^{(\alpha,\beta)}\},$$

where $\mathcal{B}^0 f = f$, $\mathcal{B}^r f = \mathcal{B}(\mathcal{B}^{r-1} f)$, $r = 1, 2, ...$

Lemma 1.2. *If $f \in W_2^r$, then*

$$a_n(f) = (-1)^r \frac{1}{\lambda_n^r} a_n(\mathcal{B}^r f), \quad r \in \mathbb{N}_0.$$

Proof. Since $\mathcal{B}$ is self-adjoint (see [7]), we have

$$\begin{aligned} a_n(f) &= \langle f, \tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n} \langle f, \mathcal{B}\tilde{\varphi}_n \rangle \\ &= -\frac{1}{\lambda_n} \langle \mathcal{B}f, \tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n} a_n(\mathcal{B}f). \end{aligned}$$

This completes the proof. □

Lemma 1.3. *If*

$$f(t) = \sum_{n=0}^{\infty} a_n(f) \tilde{\varphi}_n(t),$$

then

$$T_h f(t) = \sum_{n=0}^{\infty} \varphi_n(h) a_n(f) \tilde{\varphi}_n(t).$$

Here, the convergence of the series on the right-hand side is understood in the sense of $\mathbb{L}_2^{(\alpha,\beta)}$.

Proof. By the definition of the operator T_h,

$$T_h(\tilde{\varphi}_n(t)) = \varphi_n(h)\tilde{\varphi}_n(t).$$

Therefore, for any polynomial

$$Q_N(t) = \sum_{n=0}^{N} a_n(f)\tilde{\varphi}_n(t),$$

since T_h is linear, we have

$$T_h Q_N(t) = \sum_{n=0}^{N} \varphi_n(h) a_n(f)\tilde{\varphi}_n(t). \tag{1.2}$$

Since T_h is a linear bounded operator in $\mathbb{L}_2^{(\alpha,\beta)}$ and the set of all polynomials $Q_N(t)$ is everywhere dense in $\mathbb{L}_2^{(\alpha,\beta)}$, passage to the limit in (1.2) gives the required equality. □

Remark. Since

$$T_h f(t) - f(t) = \sum_{n=0}^{\infty} (\varphi_n(h) - 1) a_n(f)\tilde{\varphi}_n(t),$$

the Parseval's identity gives

$$\|T_h f - f\|^2 = \sum_{n=0}^{\infty} (1 - \varphi_n(h))^2 |a_n(f)|^2.$$

If $f \in W_2^r$, from Lemma 1.2, we have

$$\|\Delta_h^k(\mathcal{B}^r f)\|^2 = \sum_{n=0}^{\infty} (1 - \varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2. \tag{1.3}$$

2. Estimate of Best Approximation

The goal of this work is to prove several estimates for $E_m(f)$ in certain classes of functions in $\mathbb{L}_2^{(\alpha,\beta)}$.

Theorem 2.1. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$. Then there is a constant $c > 0$ such that, for every $f \in W_2^r$,*

$$E_m(f) = O(\lambda_m^{-r} \Omega_k(\mathcal{B}^r f, cm^{-1})),$$

when $m \to +\infty$.

Proof. Let $f \in W_2^r$. By the Hölder inequality, we have

$$\begin{aligned} E_m^2(f) - \sum_{n=m}^{\infty} \varphi_n(h)|a_n(f)|^2 &= \sum_{n=m}^{\infty}(1-\varphi_n(h))|a_n(f)|^2 \\ &= \sum_{n=m}^{\infty} |a_n(f)|^{2-\frac{1}{k}}(1-\varphi_n(h))|a_n(f)|^{\frac{1}{k}} \\ &\leq \left(\sum_{n=m}^{\infty}|a_n(f)|^2\right)^{\frac{2k-1}{2k}} \left(\sum_{n=m}^{\infty}|a_n(f)|^2(1-\varphi_n(h))^{2k}\right)^{\frac{1}{2k}} \\ &\leq \left(E_m^2(f)\right)^{\frac{2k-1}{2k}} \left(\lambda_m^{-2r}\sum_{n=m}^{\infty}\lambda_n^{2r}|a_n(f)|^2(1-\varphi_n(h))^{2k}\right)^{\frac{1}{2k}}. \end{aligned}$$

From (1.3), we have

$$\sum_{n=m}^{\infty}(1-\varphi_n(h))^{2k}\lambda_n^{2r}|a_n(f)|^2 \leq \|\Delta_h^k(\mathcal{B}^r f)\|^2.$$

Therefore

$$E_m^2(f) \leq \sum_{n=m}^{\infty}\varphi_n(h)|a_n(f)|^2 + \left(E_m^2(f)\right)^{\frac{2k-1}{2k}} \left(\lambda_m^{-2r}\|\Delta_h^k(\mathcal{B}^r f)\|^2\right)^{\frac{1}{2k}}. \tag{2.1}$$

From Lemma 1.1, we have

$$\sum_{n=m}^{\infty}\varphi_n(h)|a_n(f)|^2 \leq c_2(mh)^{-\alpha-1/2}E_m^2(f).$$

Choose a constant c_3 such that the number $c_4 = 1 - c_2 c_3^{-\alpha-1/2}$ is positive.
Setting $h = c_3/m$ in the inequality (2.1), we have

$$c_4 E_m^2(f) \leq (E_m(f))^{2-\frac{1}{k}} \left(\lambda_m^{-r}\|\Delta_{c_3 m^{-1}}^k(\mathcal{B}^r f)\|\right)^{\frac{1}{k}}.$$

By raising both sides to the power k and simplifying by $(E_m(f))^{2k-1}$ we finally obtain

$$c_4^k E_m(f) \leq \lambda_m^{-r}\Omega_k(\mathcal{B}^r f, c_3 m^{-1}),$$

for all $m > 0$. The theorem is proved with $c = c_3$. □

Theorem 2.2. *Let $f \in \mathbb{L}_2^{(\alpha,\beta)}$. Then, for each $k \in \mathbb{N}$,*

$$\Omega_k^2(f, m^{-1}) = O\left(m^{-4k}\sum_{n=1}^{m}\lambda_n^{2k-1}E_n^2(f)\right).$$

Proof. From (1.3), one can veify

$$\|\Delta_h^k f\|^2 = \sum_{n=0}^{\infty}(1-\varphi_n(h))^{2k}|a_n(f)|^2.$$

Let $m \in \mathbb{N}$ and $0 < h \le 1/m$. An application of Lemma 1.1 imples

$$\begin{aligned}
\sum_{n=0}^{\infty}(1-\varphi_n(h))^{2k}|a_n(f)|^2 &= \sum_{n=0}^{m-1}(1-\varphi_n(h))^{2k}|a_n(f)|^2 + \sum_{n=m}^{\infty}(1-\varphi_n(h))^{2k}|a_n(f)|^2 \\
&= O\left(h^{4k}\sum_{n=1}^{m-1}\lambda_n^{2k}|a_n(f)|^2 + \sum_{n=m}^{\infty}|a_n(f)|^2\right) \\
&= O(m^{-4k})\left(\sum_{n=1}^{m-1}\lambda_n^{2k}|a_n(f)|^2 + m^{4k}\sum_{n=m}^{\infty}|a_n(f)|^2\right) \\
&= O(m^{-4k})\left(\sum_{n=1}^{m-1}\lambda_n^{2k}\left(\sum_{l=n}^{\infty}|a_n(f)|^2 - \sum_{l=n+1}^{\infty}|a_n(f)|^2\right)\right) \\
&+ O(m^{-4k})m^{4k}\sum_{n=m}^{\infty}|a_n(f)|^2 \\
&= O(m^{-4k})\sum_{n=1}^{m}(\lambda_n^{2k}-\lambda_{n-1}^{2k})\sum_{l=n}^{\infty}|a_n(f)|^2.
\end{aligned}$$

By the equality $\lambda_n^{2k} - \lambda_{n-1}^{2k} = O(\lambda_n^{2k-1})$, we obtain

$$\|\Delta_h^k f\|^2 = O\left(m^{-4k}\sum_{n=1}^{m}\lambda_n^{2k-1}E_n^2(f)\right),$$

which implies

$$\Omega_k^2(f, m^{-1}) = O\left(m^{-4k}\sum_{n=1}^{m}\lambda_n^{2k-1}E_n^2(f)\right).$$

This completes the proof of the theorem. □

Theorem 2.3. *Let $f \in \mathbb{L}_2^{(\alpha,\beta)}$. If the series*

$$\sum_{n=1}^{\infty} n^{2r-1}E_n(f), \quad r \in \mathbb{N},$$

converges, then $f \in W_2^r$ and

$$\Omega_k(\mathcal{B}^r f, m^{-1}) = O\left(\left(m^{-4k}\sum_{n=1}^{m}\lambda_n^{2r+2k-1}E_n^2(f)\right)^{\frac{1}{2}} + \sum_{n=m}^{\infty} n^{2r-1}E_n(f)\right),$$

where $r, k \in \mathbb{N}$.

Proof. Let $f \in \mathbb{L}_2^{(\alpha,\beta)}$. By the equality $\lambda_n^{2r} - \lambda_{n-1}^{2r} = O(\lambda_n^{2r-1})$, we obtain

$$\begin{aligned}
\|\mathcal{B}^r f\| &= \left(\sum_{n=1}^{\infty} \lambda_n^{2r} |a_n(f)|^2\right)^{1/2} \\
&= \left(\sum_{n=1}^{\infty} \lambda_n^{2r} \left(\sum_{i=n}^{\infty} |a_n(f)|^2 - \sum_{i=n+1}^{\infty} |a_n(f)|^2\right)\right)^{1/2} \\
&= \left(\sum_{n=1}^{\infty} (\lambda_n^{2r} - \lambda_{n-1}^{2r}) \sum_{i=n}^{\infty} |a_n(f)|^2\right)^{1/2} = O\left(\sum_{n=1}^{\infty} \lambda_n^{2r-1} E_n^2(f)\right)^{1/2} \\
&= O\left(\sum_{n=1}^{\infty} \lambda_n^{r-1/2} E_n(f)\right) = O\left(\sum_{n=1}^{\infty} n^{2r-1} E_n(f)\right).
\end{aligned}$$

Since the series

$$\sum_{n=1}^{\infty} n^{2r-1} E_n(f), \quad r \in \mathbb{N},$$

converges, we see that $f \in W_2^r$.
Let $j, m \in \mathbb{N}$ be such that

$$\frac{1}{2^{j+1}} < h \leq \frac{1}{2^j}, \quad m = [h^{-1}].$$

From (1.3), we have

$$\|\Delta_h^k(\mathcal{B}^r f)\|^2 = \sum_{n=0}^{m} (1-\varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2 + \sum_{n=m+1}^{\infty} (1-\varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2 = I_1 + I_2.$$

Estimate the summands I_1 and I_2.
By Lemma 1.1, it is easy to see that

$$\begin{aligned}
I_1 &= \sum_{n=0}^{m} (1-\varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2 \\
&= O(h^{4k}) \sum_{n=1}^{m} \lambda_n^{2k} \lambda_n^{2r} |a_n(f)|^2 = O(h^{4k}) \sum_{n=1}^{m} \lambda_n^{2r+2k} |a_n(f)|^2 \\
&= O(h^{4k}) \sum_{n=1}^{m} \lambda_n^{2r+2k} \left(\sum_{i=n}^{\infty} |a_n(f)|^2 - \sum_{i=n+1}^{\infty} |a_n(f)|^2\right) \\
&= O(m^{-4k}) \sum_{n=1}^{m} (\lambda_n^{2k+2r} - \lambda_{n-1}^{2k+2r}) \sum_{i=n}^{\infty} |a_n(f)|^2 \\
&= O\left(m^{-4k} \sum_{n=1}^{m} \lambda_n^{2r+2k-1} E_n^2(f)\right).
\end{aligned}$$

By (1.3), we have

$$\begin{aligned} I_2 &= \sum_{n=m+1}^{\infty} (1-\varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2 \\ &= O\left(\sum_{n=j+1}^{\infty} \sum_{l=2^{n-1}}^{2^n-1} (1-\varphi_l(h))^{2k} \lambda_l^{2r} |a_n(f)|^2 \right) \\ &= O\left(\sum_{n=j+1}^{\infty} \|\Delta_h^k \mathcal{B}^r (S_{2^n}(f) - S_{2^{n-1}}(f))\|^2 \right), \end{aligned}$$

i.e.,

$$I_2^{1/2} = O\left(\sum_{n=j+1}^{\infty} \|\Delta_h^k \mathcal{B}^r (S_{2^n}(f) - S_{2^{n-1}}(f))\| \right).$$

In view of

$$\|\Delta_h^k f\| \leq 2^k \|f\|, \quad \|\mathcal{B}^r (S_{2^n} f - S_{2^{n-1}} f)\| \leq \lambda_{2^n}^r \|S_{2^n} f - S_{2^{n-1}} f\|.$$

we obtain

$$\begin{aligned} \|\Delta_h^k \mathcal{B}^r (S_{2^n}(f) - S_{2^{n-1}}(f))\| &\leq 2^k \lambda_{2^n}^r \|S_{2^n}(f) - S_{2^{n-1}}(f)\| \\ &\leq 2^k \lambda_{2^{n+1}}^r (\|f - S_{2^n}(f)\| + \|f - S_{2^{n-1}}(f)\|) \\ &\leq 2^k \lambda_{2^{n+1}}^r (E_{2^n}(f) + E_{2^{n-1}}(f)) \\ &\leq 2^k 2 \lambda_{2^{n+1}}^r E_{2^{n-1}}(f). \end{aligned}$$

Therefore

$$I_2^{1/2} = O\left(\sum_{n=j+1}^{\infty} 2^{2r(n+1)+1} E_{2^{n-1}}(f) \right) = O\left(2^{2r+1} \sum_{n=j}^{\infty} 2^{2r(n+1)} E_{2^n}(f) \right).$$

Note that for $n \in \mathbb{N}$ we derive

$$2^{4r} \sum_{l=2^{n-1}}^{2^n} l^{2r-1} \geq 2^{4r} (2^{n-1})^{2r-1} 2^{n-1} = 2^{2r(n+1)}. \tag{2.2}$$

Using (2.2) and the fact that $E_l(f)$ is monotone decreasing with respect to l, we get the following inequality for $n \in \mathbb{N}$:

$$2^{2r(n+1)} E_{2^n}(f) \leq 2^{4r} \sum_{l=2^{n-1}}^{2^n} l^{2r-1} E_l(f).$$

Consequently, we find that

$$I_2^{1/2} = O\left(\sum_{n=m}^{\infty} n^{2r-1} E_n(f) \right).$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^k(\mathcal{B}^r f)\| = O\left(\left(m^{-4k}\sum_{n=1}^{m}\lambda_n^{2r+2k-1}E_n^2(f)\right)^{\frac{1}{2}} + \sum_{n=m}^{\infty} n^{2r-1}E_n(f)\right).$$

Theorem 2.3 is proved. □

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