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BOUNDEDNESS FOR MULTILINEAR COMMUTATOR ASSOCIATED WITH LITTLEWOOD-PALEY OPERATOR ON TWO SPACES *

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Abstract. In this paper, the $(H^p_{\vec{b}}, L^p)$ and $(H\dot{K}^{\alpha,p}_{q,\vec{b}}, \dot{K}^{\alpha,p}_q)$ type boundedness for the multilinear commutator associated with the Littlewood-Paley operator on Hardy and Herz-Hardy spaces are obtained, using some techniques for classical inequalities.

Keywords: Littlewood-Paley operator; Multilinear commutator; BMO; Hardy space; Herz-Hardy space.

1. Introduction and definition

Let T be the Calderón-Zygmund operator and $b \in BMO(R^n)$. Then we can define the commutator $[b, T]$ generated by b and T as follows,

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In [2], Coifman, Rochberg and Weiss prove the boundedness of the commutator $[b, T]$ on $L^p(R^n)$ $(1 < p < \infty)$. However, it is well known that the $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$. But if a suitable atomic space $H^p_{\vec{b}}(R^n)$ or $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ substituted for $H^p(R^n)$, then $[b, T]$ maps continuously $H^p_{\vec{b}}(R^n)$ into $L^p(R^n)$ and $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ into $\dot{K}^{\alpha,p}_q$. Moreover, it was observed that $H^p_{\vec{b}}(R^n) \subset H^p(R^n)$, $\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n) \subset H\dot{K}^{\alpha,p}_q(R^n)$. In this paper, we will establish the continuity of the multilinear commutators related to the Littlewood-Paley operators and $BMO(R^n)$ functions on certain Hardy and Herz-Hardy spaces.

At first, let us introduce some definitions (see [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14], [17],). Suppose a positive integer m and $1 \leqslant j \leq m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements.

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For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

Definition 1. Let b_i $(i = 1, \cdots, m)$ be a locally integrable function and $0 < p \le 1$. A bounded measurable function a on R^n is said to be a $(p, \vec{b})$ atom if it satisfies the following condition,

(1) $suppa \subset B = B(x_0, r)$,

(2) $\|a\|_{L^\infty} \le |B|^{-1/p}$,

(3) $\int_B a(y)dy = \int_B a(y) \prod_{l\in\sigma} b_l(y)dy = 0$ for any $\sigma \in C_j^m$, $1 \le j \le m$.

We say that a temperate distribution f belongs to $H^p_{\vec{b}}(R^n)$ if in the Schwartz distribution sense it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x).$$

where $a_j's$ are $(p, \vec{b})$ atoms, $\lambda \in C$ and $\sum_{j=1}^\infty |\lambda|^p < \infty$. Moreover, $\|f\|_{H^p_{\vec{b}}(R^n)} \approx (\sum_{j=1}^\infty |\lambda_j|^p)^{1/p}$.

Definition 2. Let $0 < p, q < \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leqslant 2^k\}$ and $C_k = B_k \backslash B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \left\{ f \in L^q_{loc}(R^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_{B_k}\|^p_{L^q} \right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \left\{ f \in L^q_{loc}(R^n) : \|f\|_{K_q^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q} + \|f\chi_{B_0}\|^p_{L^q} \right]^{1/p}.$$

Definition 3. Let

$$\alpha \in R^n,\ 1 < q < \infty,\ \alpha \ge n(1 - \frac{1}{q}),\ b_i \in BMO(R^n),\ 1 \le i \le m.$$

A function $a(x)$ is called a central $(\alpha, q, \vec{b})$-atom (or a central $(\alpha, q, \vec{b})$-atom of restrict type) if

(1) $suppa \in B = B(x_0, r)$(or for some $r \geqslant 1$),

(2) $||a||_{L^q} \leq |B|^{-\alpha/n}$,

(3) $\int_B a(x)x^\beta dx = \int_B a(x)x^\beta \prod_{i\in\sigma} b_i(x)dx = 0$ for any $\sigma \in C_j^m$, $1 \leq j \leq m$.

We say that a temperate distribution f belongs to $H\dot{K}^{\alpha,}_{q,\vec{b}}(R^n)$ (or $HK^{\alpha,p}_{q,\vec{b}}(R^n)$) if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$), in the $S'(R^n)$ sense, where a_j is a central $(\alpha, q, \vec{b})$-atom(or a central $(\alpha, q, \vec{b})$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{-\infty}^{\infty} |\lambda_j|^p < \infty$(or $\sum_{j=0}^{\infty} |\lambda_j| < \infty$). Moreover,

$$||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}(\text{ or } ||f||_{HK^{\alpha,p}_{q,\vec{b}}}) = \inf(\sum_j |\lambda_j|^p)^{1/p},$$

where the infimum is taken over all the decompositions of f as above.

Definition 4. Let $\varepsilon > 0$, $\mu > 1$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leqslant C(1 + |x|)^{-(n+1)}$,

(3) $|\psi(x + y) - \psi(x)| \leqslant C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

The Littlewood-Paley multilinear commutator is defined by

$$g_\mu^{\vec{b}}(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|}\right)^{n\mu} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x, y) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z))\right] \psi_t(y - z)f(z)dz.$$

When $m = 1$, set

$$g_\mu^b(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|}\right)^{n\mu} |F_t^b(f)(x, y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^b(f)(x, y) = \int_{R^n} (b(x) - b(z))\psi_t(y - z)f(z)dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$, we also define that

$$g_\mu(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

which is the Littlewood-Paley function (see [16]).

2. Theorems and Proofs

We shall prove the following theorems.

Theorem 1. Let

$$\mu > 3 + 2/n + 2\varepsilon/n,\ b_i \in BMO,\ 1 \le i \le m,\ \vec{b} = (b_1, \ldots, b_m),\ n/(n+\varepsilon) < p \le 1.$$

Then the multilinear commutator $g_\mu^{\vec{b}}$ is bounded from $H^p_{\vec{b}}(R^n)$ to $L^p(R^n)$.

Proof. It suffices to show that there exists a constant $C > 0$, such that for every $(p, \vec{b})$ atom a,

$$||g_\mu^{\vec{b}}(a)||_{L^p} \le C.$$

Let a be a $(p, \vec{b})$ atom supported on a ball $B = B(x_0, r)$. When $m = 1$ see [8], and now we assume $m > 1$. Write

$$\int_{R^n} |g_\mu^{\vec{b}}(a)(x)|^p dx = \int_{|x-x_0|\le 2r} |g_\mu^{\vec{b}}(a)(x)|^p dx + \int_{|x-x_0|>2r} |g_\mu^{\vec{b}}(a)(x)|^p dx = I + II.$$

For I, taking $q > 1$, by Hölder's inequality and the L^q- boundedness of $g_\psi^{\vec{b}}$, we see that

$$\begin{aligned} I &\le \left(\int_{|x-x_0|\le 2r} |g_\mu^{\vec{b}}(a)(x)|^q dx\right)^{p/q} \cdot |B(x_0, 2r)|^{1-p/q} \\ &\le C||g_\mu^{\vec{b}}(a)(x)||^p_{L^q} \cdot |B(x_0, 2r)|^{1-p/q} \\ &\le C||\vec{b}||^p_{BMO}||a||^p_{L^q}|B|^{1-p/q} \\ &\le C||\vec{b}||^p_{BMO}. \end{aligned}$$

For II, denoting $\lambda = (\lambda_1, \cdots, \lambda_m)$ with $\lambda_i = (b_i)_B,\ 1 \le i \le m$, where

$$(b_i)_B = \frac{1}{|B(x_0, r)|}\int_{B(x_0,r)} b_i(x)dx,$$

by Hölder's inequality and the vanishing moment of a, we get

$$\begin{aligned} II &= \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B} |g_\mu^{\vec{b}}(a)(x)|^p dx \\ &\le C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p}\left(\int_{2^{k+1}B\setminus 2^k B} |g_\mu^{\vec{b}}(a)(x)|dx\right)^p \end{aligned}$$

$$\leq C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p}\left[\int_{2^{k+1}B\backslash 2^kB}\left(\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\right.\right.$$
$$\left.\left.\times\left(\int_B|\psi_t(y-z)-\psi_t(y)|\prod_{j=1}^m|b_j(x)-b_j(z)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}dx\right]^p.$$

noting that $z\in B, y\in 2^{k+1}B\backslash 2^kB$, then

$$\left[\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_B|\psi_t(y-z)-\psi_t(y)|\prod_{j=1}^m|b_j(x)-b_j(z)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$
$$\leq C\left[\iint_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_B t^{-n}|a(z)|\prod_{j=1}^m|b_j(x)-b_j(z)|\frac{(|z|/t)^{\varepsilon}}{(1+|y|/t)^{n+1+\varepsilon}}dy\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$
$$\leq C\left(\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}dydt}{(t+|y|)^{2(n+1+\varepsilon)}}\right)^{1/2}\left(\int_B\prod_{j=1}^m|b_j(x)-b_j(z)||z|^{\varepsilon}|a(z)|dz\right).$$

Using the notation $B'=B'(x,t)$, we have

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y|)^{2(n+1+\varepsilon)}}$$
$$\leq t^{-n}\left(\int_{B'}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y|)^{2(n+1+\varepsilon)}}\right.$$
$$\left.+\sum_{k=1}^{\infty}\int_{2^kB'\backslash 2^{k-1}B'}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y|)^{2(n+1+\varepsilon)}}\right)$$
$$\leq Ct^{-n}\left(\int_{B'}\frac{2^{2(n+1+\varepsilon)}dy}{(2t+|y|)^{2(n+1+\varepsilon)}}+\sum_{k=1}^{\infty}\int_{2^kB'}\left(\frac{t}{t+2^{k-1}t}\right)^{n\mu}\frac{2^{2(k+1)(n+1+\varepsilon)}dy}{(2^{k+1}t+|y|)^{2(n+1+\varepsilon)}}\right)$$

$$\leq Ct^{-n}\left(t^n+\sum_{k=1}^{\infty}2^{-kn\mu}2^{2k(n+1+\varepsilon)}(2^kt)^n\right)\frac{1}{(t+|x|)^{2(n+1+\varepsilon)}}$$
$$\leq C\left(1+\sum_{k=1}^{\infty}2^{k(3n+2\varepsilon-n\mu+2)}\right)\frac{1}{(t+|x|)^{2(n+1+\varepsilon)}}$$
$$\leq C\frac{1}{(t+|x|)^{2(n+1+\varepsilon)}},$$

it is easy to calculate that

$$\int_0^{\infty} \frac{tdt}{(t+|x|)^{2n+1+\varepsilon}} = C|x|^{-2(n+\varepsilon)}.$$

So

$$\begin{aligned}
II &\le C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p} \\
&\times\left[\int_{2^{k+1}B\backslash 2^kB}|x|^{(-n+\varepsilon)}\left(\int_B\prod_{j=1}^{m}|b_j(x)-b_j(z)||z^{\varepsilon}||a(z)|dz\right)dx\right]^p \\
&\le C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p} \\
&\times\left[\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}\int_{2^{k+1}B\backslash 2^kB}|x|^{(-n+\varepsilon)}|(\vec{b}(x)-\lambda)_{\sigma}|dx\int_B|(\vec{b}(z)-\lambda)_{\sigma^c}||z^{\varepsilon}||a(z)|dz\right]^p \\
&\le C\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}\left(\int_B|(\vec{b}(y)-\lambda)_{\sigma^c}||z^{\varepsilon}||a(z)|dz\right)^p \\
&\times\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p}\left[\int_{2^{k+1}B\backslash 2^kB}|x|^{(-n+\varepsilon)}|(\vec{b}(x)-\lambda)_{\sigma}|dx\right]^p \\
&\le C\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}||\vec{b}_{\sigma^c}||_{BMO}^p\cdot||\vec{b}_{\sigma}||_{BMO}^p\sum_{k=1}^{\infty}|2^{k+1}B|^{(1-p(n+\varepsilon)/n)}k^p|B|^{(1+\varepsilon/n-1/p)p} \\
&\le C||\vec{b}||_{BMO}^p\sum_{k=1}^{\infty}k^p\cdot 2^{kn(1-p(n+\varepsilon)/n)} \\
&\le C||\vec{b}||_{BMO}.
\end{aligned}$$

This finish the proof of Theorem 1.

Theorem 2. Let

$$\mu > 3+2/n+2\varepsilon/n,\ 0<p<\infty,\ 1<q<\infty,\ n(1-1/q)\le\alpha<n(1-1/q)+\varepsilon$$

and

$$b_i\in BMO(R^n), 1\le i\le m,\ \vec{b}=(b_1,\cdots,b_m).$$

Then $g_{\mu}^{\vec{b}}$ is bounded from $H\dot{K}_{q,\vec{b}}^{\alpha,p}(R^n)$ to $\dot{K}_q^{\alpha,p}(R^n)$.

Proof. Let $f \in H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3, so we write

$$\begin{aligned}
\|g_\mu^{\vec{b}}(f)(x)\|_{\dot{K}_q^{\alpha,p}} &\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|\|g_\mu^{\vec{b}}(a_j)\chi_k\|_{L^q}\Big)^p\right]^{1/p} \\
&\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j|\|g_\mu^{\vec{b}}(a_j)\chi_k\|_{L^q}\Big)^p\right]^{1/p} \\
&\quad + C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-2}^{\infty} |\lambda_j|\|g_\mu^{\vec{b}}(a_j)\chi_k\|_{L^q}\Big)^p\right]^{1/p} \\
&= I + II.
\end{aligned}$$

For II, by the boundedness of $g_\mu^{\vec{b}}$ on L^q and the Hölder's inequality, we have

$$\begin{aligned}
II &\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|\|g_\mu^{\vec{b}}(a_j)\chi_j\|_{L^q}\right)^p\right]^{1/p} \\
&\le C\left[\sum_{-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|\|a_j\|_{L^q}\right)^p\right]^{1/p} \\
&\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|\cdot 2^{-j\alpha}\right)^p\right]^{1/p} \\
&\le C\begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right]^{1/p}, & 0 < p \le 1 \\ \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)p/2}\right]^{1/p}, & 1 < p < \infty \end{cases} \\
&\le C\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \\
&\le C\|f\|_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}.
\end{aligned}$$

For I, when $m = 1$, let

$$\begin{aligned}
&C_k = B_k \backslash B_{k-1}, \\
&\chi_k = \chi_{A_k},\ b_j^i = |B_j|^{-1}\int_{B_j} b_i(x)dx,\ 1 \le i \le m,\ \vec{b'} = (b_j^1, \cdots, b_j^m).
\end{aligned}$$

Similar to the proof of II in Theorem 1, we have

$$\begin{aligned}
g_\mu^{b_1}(a_j)(x) &= \left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left|\int_{B_j}(b_1(x)-b_1(z))\psi_t(y-z)a_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\le C\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_{B_j}|\psi_t(y-z)-\psi_t(y)||b_1(x)-b_1(z)||a_j(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\le C\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_{B_j}t^{-n}|a_j(z)||b_1(x)-b_1(z)|\frac{(|z|/t)^\varepsilon}{(1+|y|/t)^{n+1+\varepsilon}}dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\le C\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}dydt}{(t+|y|)^{2(n+1+\varepsilon)}}\right)^{1/2}\left(\int_{B_j}|b_1(x)-b_1(z)||z|^\varepsilon|a_j(z)|dz\right)\\
&\le C\left(\int_0^\infty\frac{tdt}{(t+|x|)^{2n+1+\varepsilon}}\right)^{1/2}\left(\int_{B_j}|b_1(x)-b_1(z)||z|^\varepsilon|a(z)|dz\right)\\
&\le C|x|^{-n+\varepsilon}\int_{B_j}|z|^\varepsilon|a_j(z)||b_1(x)-b_1(z)|dz\\
&\le C|x|^{-n+\varepsilon}\int_{B_j}|z|^\varepsilon|a_j(z)||b_1(x)-b_j^1|dz+C|x|^{-(n+\varepsilon)}\int_{B_j}|z|^\varepsilon|a_j(z)||b_1(z)-b_j^1|dz\\
&\le C|x|^{-(n+\varepsilon)}(|b_1(x)-b_j^1|2^{j(\varepsilon+n(1-1/q)-\alpha)}+2^{j(\varepsilon+n(1-1/q)-\alpha)}\|b_1\|_{BMO}).
\end{aligned}$$

So

$$\begin{aligned}
&\|g_\mu^{b_1}(a_j)\chi_k\|_{L_q}\\
&\le C2^{j(\varepsilon+n(1-1/q)-\alpha)}\left[\left(\int_{B_k}|x|^{-(n+\varepsilon)q}|b_1(x)-b_j^1|^q dx\right)^{1/q}+\left(\int_{B_k}|x|^{-(n+\varepsilon)q}dx\right)^{1/q}\|b_1\|_{BMO}\right]\\
&\le C2^{j(\varepsilon+n(1-1/q)-\alpha)}\left[2^{-k(n+\varepsilon)}\cdot|B_k|^{1/q}\|b_1\|_{BMO}+2^{-k(n+\varepsilon)}\cdot|B_k|^{1/q}\|b_1\|_{BMO}\right]\\
&\le C\|b_1\|_{BMO}2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}.
\end{aligned}$$

Thus

$$
\begin{aligned}
I &= C\left[\sum_{j=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3} |\lambda_j| \| g_\mu^{b_1}(a_j)\chi_k\|_{L^q}\right)^p\right]^{1/p} \\
&\le C\|b_1\|_{BMO}\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}\right)^p\right]^{1/p} \\
&\le C\|b_1\|_{BMO}\begin{cases}\left[\sum\limits_{k=-\infty}^{\infty} 2^{k\alpha p}\sum\limits_{j=-\infty}^{k-3} |\lambda_j|^p 2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]p}\right]^{1/p}, & 0<p\le 1 \\ \left[\sum\limits_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum\limits_{j=-\infty}^{k-3} |\lambda_j|^p 2^{p/2[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}\right)\right. \\ \qquad \left.\times\left(\sum\limits_{j=-\infty}^{k-3} 2^{p'/2[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases} \\
&\le C\|b_1\|_{BMO}\begin{cases}\left[\sum\limits_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p}\right]^{1/p}, & 0<p\le 1 \\ \left[\sum\limits_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p/2}\right]^{1/p}, & 1<p<\infty\end{cases} \\
&\le C\|b_1\|_{BMO}\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \\
&\le C\|f\|_{H\dot{K}_{q,\vec{b}}^{\alpha,p}}.
\end{aligned}
$$

When $m > 1$, similarly to the proof of $g_\mu^b(a_j)(x)$, we have

$$
\begin{aligned}
& g_\mu^{\vec{b}}(a_j)(x) \\
&= \left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu} \left|\int_{B_j}\prod_{i=1}^{m}(b_i(x)-b_i(z))\psi_t(y-z)a_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&\le C|x|^{-(n+\varepsilon)}\int_{B_j}|z|^{\varepsilon}|a_j(z)|\prod_{i=1}^{m}|b_i(x)-b_i(z)|dz \\
&\le C|x|^{-(n+\varepsilon)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma|\int_{B_j}|z|^{\varepsilon}|a_j(z)||(\vec{b}(x)-\vec{b})_{\sigma^c}|dz
\end{aligned}
$$

$$\le C|x|^{-(n+\varepsilon)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma|2^{j\varepsilon}\cdot 2^{-j\alpha}\cdot 2^{jn(1-1/q)}\|\vec{b}_{\sigma^c}\|_{BMO}$$

$$\le C|x|^{-(n+\varepsilon)}\cdot 2^{j(\varepsilon+n(1-1/q)-\alpha)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma|\|\vec{b}_{\sigma^c}\|_{BMO}.$$

So

$$\begin{aligned}
&\|g_\mu^{\vec{b}}(a_j)\chi_k\|_{L^q}\\
&\le C2^{j(\varepsilon+n(1-1/q)-\alpha)}\|\vec{b}_{\sigma^c}\|_{BMO}\left[\int_{B_k}\left(|x|^{-(n+\varepsilon)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma|\right)\right]^{1/q}\\
&\le C\|\vec{b}_{\sigma^c}\|_{BMO}2^{j(\varepsilon+n(1-1/q)-\alpha)}\cdot 2^{-k(n+\varepsilon)+kn/q}\\
&\le C\|\vec{b}\|_{BMO},
\end{aligned}$$

then

$$\begin{aligned}
I&=C\left[\sum_{j=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|\|g_\mu^{\vec{b}}(a_j)\chi_k\|_{L^q}\right)^p\right]^{1/p}\\
&\le C\|\vec{b}\|_{BMO}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}\right)^p\right]^{1/p}\\
&\le C\|\vec{b}\|_{BMO}\begin{cases}\left[\sum\limits_{k=-\infty}^{\infty}2^{k\alpha p}\sum\limits_{j=-\infty}^{k-3}|\lambda_j|^p2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]p}\right]^{1/p}, & 0<p\le 1\\
\left[\sum\limits_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum\limits_{j=-\infty}^{k-3}|\lambda_j|^p2^{p[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]/2}\right)\right.\\
\quad\left.\times\left(\sum\limits_{j=-\infty}^{k-3}2^{p'[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}\\
&\le C\|\vec{b}\|_{BMO}\begin{cases}\left[\sum\limits_{j=-\infty}^{\infty}|\lambda_j|^p\sum\limits_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p}\right]^{1/p}, & 0<p\le 1\\
\left[\sum\limits_{j=-\infty}^{\infty}|\lambda_j|^p\sum\limits_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p/2}\right]^{1/p}, & 1<p<\infty\end{cases}\\
&\le C\|\vec{b}\|_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\le C\|f\|_{H\dot{K}_{q,\vec{b}}^{\alpha,p}}.
\end{aligned}$$

Remark. Theorem 2 also holds for nonhomogeneous Herz-type spaces, so we omitted the details.

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