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Δ^m-STATISTICAL CONVERGENCE OF ORDER $\tilde{\alpha}$ FOR DOUBLE SEQUENCES OF FUNCTIONS

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Abstract. In this paper, we have introduced and examined the concepts of Δ^m-pointwise and Δ^m-uniform statistical convergence of order $\tilde{\alpha}$ for double sequences of real valued functions. Also, we have given the concept of Δ^m-statistically Cauchy sequence for double sequences of real valued functions and proven that it is equivalent to Δ^m-pointwise statistical convergence of order $\tilde{\alpha}$ for double sequences of real valued functions. Some relations between $S^2_{\tilde{\alpha}}(\Delta^m, f)$-statistical convergence and strong $\left[w^2_p\right]_{\tilde{\alpha}}(\Delta^m, f)-$summability have also been given.
Keywords. Statistical convergence; Cauchy sequence; summability.

1. Introduction

The idea of statistical convergence was given by Zygmund [27]] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [25] and Fast [11] and later reintroduced by Schoenberg [23] independently. Many mathematicians have studied various properties of statistical convergence and applications of this concept in different areas such as Fourier analysis, Ergodic theory, number theory, measure theory, Trigonometric series, Turnpike theory and Banach spaces Cınar *et al.* [1], Colak [2], Colak and Altın [3], Connor [4], Et *et al.* ([7],[8],[9],[10]), Fridy [12], Işık [16], Mohiuddine *et al.* [18], Móricz [19], Mursaleen [21], Et and Şengül [24], Tripathy and Sarma [26] and many authors have examined the relationship between statistical convergence with sequences spaces and summability theory.

Pointwise and uniform statistical convergence of sequences of real valued functions were defined by Gökhan *et al.* ([13], [14], [15]) and independently by Duman and Orhan [5]. The aim of the present paper is to introduce and examine the concepts of Δ^m-pointwise and Δ^m-uniform statistical convergence of order $\tilde{\alpha}$ for

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double sequences of real valued functions. In Section 2 we give a brief overview of statistical convergence of order $\tilde{\alpha}$ and strong $p-$Cesàro summability of double sequences of functions. In Section 3 we give the concepts of Δ^m-pointwise and Δ^m-uniform statistical convergence of order $\tilde{\alpha}$ and the concept of Δ^m-statistically Cauchy sequence for sequences of real valued functions.

2. Definition and Preliminaries

A double sequence $x = (x_{jk})$ is said to be convergent in the Pringsheim [22] sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case, we write $P - \lim x = L$.

A double sequence $x = (x_{jk})_{j,k=0}^{\infty}$ is bounded if there exists a positive real number M such that $|x_{jk}| < M$ for all j and k, that is, $\|x\| = \sup_{j,k\geq 0} |x_{jk}| < \infty$. Although every convergent single sequence is bounded, a convergent double sequence need not be bounded.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K(m,n) = \{(j,k) : j \leq m, k \leq n\}$. The double natural density of K is defined by

$$\delta^2(K) = P - \lim_{m,n} \frac{1}{mn} |K(m,n)|, \quad \text{if the limit exists.}$$

A double sequence $x = (x_{jk})$ is said to be statistically convergent to a number L if for every $\varepsilon > 0$ the set $\{(j,k) : j \leq m, k \leq n : |x_{jk} - L| \geq \varepsilon\}$ has double natural density zero [21].

A convergent double sequence is statistically convergent but the converse is not true in general. Also, a statistically convergent double sequence need not be bounded.

Throughout the paper, we have taken $s, t, u, v \in (0,1]$ and written $\tilde{\alpha}$ instead of (s,t) and $\tilde{\beta}$ instead of (u,v). We have defined

$$\begin{aligned}
\tilde{\alpha} &\preceq \tilde{\beta} \Leftrightarrow s \leq u \text{ and } t \leq v \\
\tilde{\alpha} &\prec \tilde{\beta} \Leftrightarrow s < u \text{ and } t < v \\
\tilde{\alpha} &\cong \tilde{\beta} \Leftrightarrow s = u \text{ and } t = v \\
\tilde{\alpha} &\in (0,1] \Leftrightarrow s, t \in (0,1] \\
\tilde{\beta} &\in (0,1] \Leftrightarrow u, v \in (0,1] \\
\tilde{\alpha} &\cong 1 \text{ in case } s = t = 1 \\
\tilde{\beta} &\cong 1 \text{ in case } u = v = 1 \\
\tilde{\alpha} &\succ 1 \text{ in case } s > 1 \text{ and } t > 1
\end{aligned}$$

Let $\tilde{\alpha} \in (0,1]$ be given. The $\tilde{\alpha}-$double density of a subset K of $\mathbb{N} \times \mathbb{N}$ was defined by Çolak and Altın as follows [3]

$$\delta^2_{\tilde{\alpha}}(K) = P - \lim_{n,m} \frac{1}{n^s m^t} |K(n,m)|, \text{ if the limit exists.}$$

$\delta^2(K^c) = 1 - \delta^2(K)$ holds, but $\delta^2_{\tilde{\alpha}}(K^c) = 1 - \delta^2_{\tilde{\alpha}}(K)$ does not hold for $0 < \widetilde{\alpha} < 1$ in general.

A double sequence $x = (x_{jk})$ is said to be statistically convergent order $\widetilde{\alpha}$ to the number L if for each $\varepsilon > 0$, the set

$$\{(j,k) : j \leq n, k \leq m : |x_{jk} - L| \geq \varepsilon\}$$

has double natural density zero, i.e.

$$|x_{jk} - L| < \varepsilon \quad a.a.\,(j,k)\,(\widetilde{\alpha}).$$

A double sequence $x = (x_{jk})$ is said to be strongly Cesàro summable to a number L if

$$P - \lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} |x_{jk} - L| = 0.$$

The idea of difference sequences defined by Kızmaz [17] and the notion was generalized by Et and Çolak [6] such as

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

for $X = \ell_\infty$, c or c_0, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta^m x = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum\limits_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$. Recently difference sequence spaces have been studied in ([7], [8], [20]).

For a double sequence $x = (x_{jk})$ we have generalized difference sequences as follows:

$$\Delta^m x_{jk} = \sum_{v_1=0}^{m} \sum_{v_2=0}^{m} (-1)^{v_1+v_2} \binom{m}{v_1} \binom{m}{v_2} x_{j+v_1 k+v_2}$$

where $\Delta x_{jk} = x_{jk} - x_{jk+1} - x_{j+1k} + x_{j+1k+1}$ for all $j, k \in \mathbb{N}$.

3. Main Result

In this section, we have given the relations between Δ^m-pointwise statistical convergence of order $\tilde{\alpha}$ and Δ^m-pointwise statistical convergence of order $\tilde{\beta}$ and the relations between strong Δ_p^m-pointwise Cesàro summability of order $\tilde{\alpha}$ and strong Δ_p^m-pointwise Cesàro summability of order $\tilde{\beta}$ and the relations between strong Δ_p^m-pointwise Cesàro summability of order $\tilde{\alpha}$ and Δ^m-pointwise statistical convergence of order $\tilde{\beta}$ for double sequences of functions, where $\tilde{\alpha} \preceq \tilde{\beta}$.

Definition 3.1. Let $\tilde{\alpha} \in (0,1]$ be given. A double sequence of functions $\{f_{jk}\}$ is said to be Δ^m-pointwise statistically convergent of order $\tilde{\alpha}$ (or $S^2_{\tilde{\alpha}}(\Delta^m, f)-$summable) to the function f on a set A if for every $\varepsilon > 0$ and for every $x \in A$

$$\lim_{n,m} \frac{1}{n^s m^t} \left| \{(j,k) : j \leq n, k \leq m : |\Delta^m f_{jk}(x) - f(x)| \geq \varepsilon \ \} \right| = 0$$

i.e. for every $x \in A$,

$$|\Delta^m f_{jk}(x) - f(x)| < \varepsilon \qquad a.a. \, (j,k) \;\; (\tilde{\alpha}).$$

In this case, we write $S^2_{\tilde{\alpha}} - \lim \Delta^m f_{jk}(x) = f(x)$ on A. The function f is said to be double Δ^m-statistical limit of order $\tilde{\alpha}$ of the sequence $\{f_{jk}\}$ (or Pringsheim Δ^m-statistical limit of order $\tilde{\alpha}$). The set of all Δ^m-pointwise statistically convergent sequences of functions order $\tilde{\alpha}$ will be denoted by $S^2_{\tilde{\alpha}}(\Delta^m, f)$.

For $\tilde{\alpha} \in (0,1]$, Δ^m-pointwise statistical convergence of order $\tilde{\alpha}$ is well defined, but is not well defined for $\tilde{\alpha} \succ 1$. For this, a sequence of functions have been defined $\{f_{jk}\}$ by

$$f_{jk}(x) = \begin{cases} 1 & j+k = 2n \\ x^{j+k} & j+k \neq 2n \end{cases} \qquad n = 1,2,3..., x \in \left[0, \tfrac{1}{2}\right].$$

Then we calculate $\Delta f_{jk}(x)$ as follows;

$$\Delta f_{jk}(x) \;\; = \;\; \begin{cases} 2 - 2x^{j+k+1} & j+k = 2n \\ x^{j+k} + x^{j+k+2} - 2 & j+k \neq 2n \end{cases} \qquad n = 1,2,3..., x \in \left[0, \tfrac{1}{2}\right].$$

Then for every $x \in A$, both

$$\lim_{n,m\to\infty} \frac{1}{n^s m^t} \left| \left\{ (j,k) : j \leq n, k \leq m : \left| \Delta f_{jk}(x) - \left(2 - 2x^{j+k+1}\right) \right| \geq \varepsilon \right\} \right|$$

$$\leq \quad \lim_{n,m\to\infty} \frac{\left(\frac{n}{2}+1\right)\left(\frac{m}{2}+1\right)}{n^s m^t} = 0$$

and

$$\lim_{n,m\to\infty} \frac{1}{n^s m^t} \left| \left\{ (j,k) : j \leq n, k \leq m : \left| \Delta f_{jk}(x) - \left(x^{j+k} + x^{j+k+2} - 2\right) \right| \geq \varepsilon \right\} \right|$$

$$\leq \quad \lim_{n,m\to\infty} \frac{\left(\frac{n}{2}+1\right)\left(\frac{m}{2}+1\right)}{n^s m^t} = 0$$

for $\tilde{\alpha} \succ 1$, hence $S^2_{\tilde{\alpha}} - \lim \Delta f_{jk}(x) = 2$ and $S^2_{\tilde{\alpha}} - \lim \Delta f_{jk}(x) = -2$ which is impossible.

Theorem 3.1. *Let* $\tilde{\alpha} \in (0,1]$, $\{f_{jk}\}$ *and* $\{g_{jk}\}$ *be two double sequences of real valued functions defined on a set* A.

(*i*) *If* $S^2_{\tilde{\alpha}} - \lim \Delta^m f_{jk}(x) = f(x)$ *and* $c \in \mathbb{R}$, *then* $S^2_{\tilde{\alpha}} - \lim c\Delta^m f_{jk}(x) = cf(x)$,

(*ii*) *If* $S^2_{\tilde{\alpha}} - \lim \Delta^m f_{jk}(x) = f(x)$ *and* $S^2_{\tilde{\alpha}} - \lim \Delta^m g_{jk}(x) = g(x)$, *then* $S^2_{\tilde{\alpha}} - \lim(\Delta^m f_{jk}(x) + \Delta^m g_{jk}(x)) = f(x) + g(x)$.

Proof. Omitted. □

It is easy to see that every Δ^m-pointwise convergent sequences of function is Δ^m-pointwise statistically convergent of order $\tilde{\alpha}$, but the converse does not hold. To see this, a sequence $\{f_{jk}\}$ is defined by

$$f_{jk}(x) = \begin{cases} 1 & j,k = n^2 \\ \frac{jkx}{2+j^2k^2x^2} & j,k \neq n^2 \end{cases}.$$

Then we calculate $\Delta f_{jk}(x)$ as follows:

$$\Delta f_{jk}(x) = \begin{cases} 1 - \frac{j(k+1)x}{2+j^2(k+1)^2x^2} - \frac{(j+1)kx}{2+(j+1)^2k^2x^2} + \frac{(j+1)(k+1)x}{2+(j+1)^2(k+1)^2x^2} & j,k = n^2 \\ \frac{jkx}{2+j^2k^2x^2} - \frac{j(k+1)x}{2+j^2(k+1)^2x^2} - \frac{(j+1)kx}{2+(j+1)^2k^2x^2} - 1 & j,k = n^2 - 1 \\ \frac{jkx}{2+j^2k^2x^2} - \frac{j(k+1)x}{2+j^2(k+1)^2x^2} - \frac{(j+1)kx}{2+(j+1)^2k^2x^2} + \frac{(j+1)(k+1)x}{2+(j+1)^2(k+1)^2x^2} & j,k \neq n^2 \end{cases}$$

The sequence $\{f_{jk}\}$ is $\Delta-$pointwise statistically convergent of order $\tilde{\alpha}$ with $S^2_{\tilde{\alpha}} - \lim \Delta f_{jk}(x) = 0$ for $\tilde{\alpha} \succ \frac{1}{2}$, but it is not $\Delta-$pointwise convergent.

Theorem 3.2. *Let* $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ *be given such that* $\tilde{\alpha} \preceq \tilde{\beta}$, *then* $S^2_{\tilde{\alpha}}(\Delta^m, f) \subseteq S^2_{\tilde{\beta}}(\Delta^m, f)$ *and the inclusion is strict.*

Proof. The inclusion part of the proof is easy. To show that the inclusion is strict, a double sequence $\{f_{jk}\}$ is defined by

$$f_{jk}(x) = \begin{cases} 1 & j,k = n^2 \\ \frac{j^2k^2x}{1+j^3k^3x^2} & j,k \neq n^2 \end{cases}.$$

So we have

$$\Delta f_{jk}(x) = \begin{cases} 1 - \frac{j^2(k+1)^2x}{1+j^3(k+1)^3x^2} - \frac{(j+1)^2k^2x}{1+(j+1)^3k^3x^2} + \frac{(j+1)^2(k+1)^2x}{1+(j+1)^3(k+1)^3x^2} & j,k = n^2 \\ \frac{j^2k^2x}{1+j^3k^3x^2} - \frac{j^2(k+1)^2x}{1+j^3(k+1)^3x^2} - \frac{(j+1)^2k^2x}{1+(j+1)^3k^3x^2} - 1 & j,k = n^2 - 1 \\ \frac{j^2k^2x}{1+j^3k^3x^2} - \frac{j^2(k+1)^2x}{1+j^3(k+1)^3x^2} - \frac{(j+1)^2k^2x}{1+(j+1)^3k^3x^2} + \frac{(j+1)^2(k+1)^2x}{1+(j+1)^3(k+1)^3x^2} & j,k \neq n^2 \end{cases}.$$

Then $S^2_{\tilde{\beta}} - \lim \Delta f_{jk}(x) = 0$ for $\tilde{\beta} \in \left(\frac{1}{2}, 1\right]$, but $f \notin x \in S^2_{\tilde{\alpha}}(\Delta, f)$ for $\tilde{\alpha} \in \left(0, \frac{1}{2}\right]$. □

Corollary 3.1. *If a double sequence of functions $\{f_{jk}\}$ is Δ^m-pointwise statistically convergent of order $\tilde{\alpha}$ to the function f, then it is Δ^m-pointwise statistically convergent to the function f.*

Definition 3.2. Let $\tilde{\alpha} \in (0,1]$. The sequence $\{f_{jk}\}$ is a Δ^m-pointwise statistically Cauchy sequence of order $\tilde{\alpha}$, provided that for every $\varepsilon > 0$ there are two numbers $N\,(= N(\varepsilon))\,, M\,(= M(\varepsilon))$ such that

$$|\Delta^m f_{jk}(x) - \Delta^m f_{N,M}(x)| < \varepsilon \quad a.a.\,(j,k) \;\; (\tilde{\alpha}) \text{ and for each } x \in A$$

i.e.

$$\lim_{n,m\to\infty} \frac{1}{n^s m^t} \left|\{(j,k) : j \le n, k \le m : |\Delta^m f_{jk}(x) - \Delta^m f_{N,M}(x)| \ge \varepsilon\}\right| = 0$$

for each $x \in A$.

Using the same technique in proof of [1][Thorem3.4], we obtain the proof of the following theorem.

Theorem 3.3. *Let $\{f_{jk}\}$ be a double sequence of functions defined on a set A. The following statements are equivalent:*

(i) $\{f_{jk}\}$ is Δ^m-pointwise statistically convergent of order $\tilde{\alpha}$ to $f(x)$ on A;

(ii) $\{f_{jk}\}$ is Δ^m-pointwise statistically Cauchy sequence of order $\tilde{\alpha}$ on A;

(iii) $\{f_{jk}\}$ is a double sequence of functions for which there is a Δ^m-pointwise convergent sequence of functions $\{g_{jk}\}$ such that $\Delta^m f_{jk}(x) = \Delta^m g_{jk}(x)$ a.a. (j,k) $(\tilde{\alpha})$ for every $x \in A$.

Definition 3.3. Let $\tilde{\alpha} \in (0,1]$ and p be a positive real number. A double sequence of functions $\{f_{jk}\}$ is said to be strongly Δ^m_p-pointwise Cesàro summable of order $\tilde{\alpha}$ (or $\left[w^2_p\right]_{\tilde{\alpha}}(\Delta^m, f)-$summable) if there is a function f such that

$$\lim_{n,m\to\infty} \frac{1}{n^s m^t} \sum_{j=1}^{n} \sum_{k=1}^{m} |\Delta^m f_{jk}(x) - f(x)|^p = 0.$$

In this case, we write $\left[w^2_p\right]_{\tilde{\alpha}} - \lim \Delta^m f_{jk}(x) = f(x)$ on A. The set of all strongly Δ^m_p-Cesàro summable double sequences of functions of order $\tilde{\alpha}$ will be denoted by $\left[w^2_p\right]_{\tilde{\alpha}}(\Delta^m, f)$.

Theorem 3.4. *Let p be a positive real number and $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$. Then $\left[w^2_p\right]_{\tilde{\alpha}}(\Delta^m, f) \subseteq \left[w^2_p\right]_{\tilde{\beta}}(\Delta^m, f)$ and the inclusion is strict for some $\tilde{\alpha} = (s,t)$ and $\tilde{\beta} = (u,v)$ such that $\tilde{\alpha} \prec \tilde{\beta}$.*

Proof. The inclusion part of the proof is easy. To show that the inclusion is strict define a double sequence $\{f_{jk}\}$ by

$$f_{jk}(x) = \begin{cases} \frac{jkx}{1+jkx} & j,k = n^2 \\ 0 & j,k \neq n^2 \end{cases} \quad x \in [1,2].$$

Then we calculate $\Delta f_{jk}(x)$ as follows

$$\Delta f_{jk}(x) = \begin{cases} \frac{jkx}{1+jkx} & j,k = n^2 \\ \frac{-(j+1)(k+1)x}{1+(j+1)(k+1)x} & j,k = n^2 - 1 \\ 0 & j,k \neq n^2 \end{cases}.$$

Therefore we get

$$\frac{1}{n^s m^t} \sum_{j=1}^{n} \sum_{k=1}^{m} |\Delta f_{jk}(x) - f(x)|^p \leq \frac{2\sqrt{n}\sqrt{m}}{n^s m^t} = \frac{1}{n^{s-\frac{1}{2}} m^{t-\frac{1}{2}}} \to 0 \; as \; n, m \to \infty$$

and so the sequence $\{f_{jk}\}$ is strongly Δ_p-pointwise Cesàro summable of order $\tilde{\alpha}$, for $\tilde{\alpha}, \tilde{\beta} \in \left(\frac{1}{2}, 1\right]$, but since

$$\frac{1}{n^s m^t} \sum_{j=1}^{n} \sum_{k=1}^{m} |\Delta f_{jk}(x) - f(x)|^p \geq \frac{2\sqrt{n}\sqrt{m}}{2n^s m^t} \to \infty \quad as \; n, m \to \infty.$$

the sequence $\{f_{jk}\}$ is not strongly Δ_p-pointwise Cesàro summable of order $\tilde{\alpha}$, for $\tilde{\alpha}, \tilde{\beta} \in \left(0, \frac{1}{2}\right]$. □

Corollary 3.2. *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ and p be a positive real number. Then*

(i) if $\tilde{\alpha} \cong \tilde{\beta}$, then $\left[w_p^2\right]_{\tilde{\alpha}}(\Delta^m, f) = \left[w_p^2\right]_{\tilde{\beta}}(\Delta^m, f)$,

(ii) $\left[w_p^2\right]_{\tilde{\alpha}}(\Delta^m, f) \subseteq \left[w_p^2\right](\Delta^m, f)$ for each $\tilde{\alpha} \in (0,1]$ and $0 < p < \infty$.

Definition 3.4. A double sequence of functions $\{f_{jk}\}$ is said to be $\Delta^m_{(C,1,1)}-$pointwise statistically summable of order $\tilde{\alpha}$ (or $(C,1,1)_{S^2_{\tilde{\alpha}}}$ $-$summable) to the function f if for every $\varepsilon > 0$ and $x \in A$, the set $K_\epsilon(\sigma_{mn})$ has double natural density zero. In this case we write $(C,1,1)_{S^2_{\tilde{\alpha}}} - \lim \Delta^m f_{jk} = f$, where

$$K_\epsilon(\sigma_{mn}) = \{(j,k) : j \leq n, k \leq m : |\sigma_{mn}(\Delta^m, f) - f(x)| \geq \varepsilon\}$$

and

$$\sigma_{mn}(\Delta^m, f) = \frac{1}{(n+1)(m+1)} \sum_{j=0}^{n} \sum_{k=0}^{m} \Delta^m f_{jk}(x).$$

Theorem 3.5. *If a double sequence of functions $\{f_{jk}\}$ is bounded and $S_{\tilde{\alpha}}^2(\Delta^m, f)-$summable to f then it is statistically $(C,1,1)_{S_{\tilde{\alpha}}^2}-$summable to f, but the converse does not hold.*

Proof. Let $\{f_{jk}\}$ be bounded and $S_{\tilde{\alpha}}^2(\Delta^m, f)-$summable to f, we can write $\sup_{j,k}|\Delta^m f_{jk} - f| = M$ and K_ϵ has double natural density zero, where

$$K_\epsilon = \{(j,k) : j \le n, k \le m : |\Delta^m f_{jk}(x) - f(x)| \ge \varepsilon\}.$$

Then

$$\left|\frac{1}{(n+1)(m+1)}\sum_{j=0}^{n}\sum_{k=0}^{m}\Delta^m f_{jk}(x) - f(x)\right| \le \frac{1}{(n+1)(m+1)}\sum_{j=0}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)|$$

$$= \frac{1}{(n+1)(m+1)}\sum_{\substack{j=0 \\ (j,k)\in K(\varepsilon)}}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)| + \frac{1}{(n+1)(m+1)}\sum_{\substack{j=0 \\ (j,k)\notin K(\varepsilon)}}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)|$$

$$\le \frac{1}{(n+1)(m+1)}\sum_{\substack{j=0 \\ (j,k)\in K(\varepsilon)}}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)| + \frac{1}{(n+1)(m+1)}\sum_{\substack{j=0 \\ (j,k)\in K(\varepsilon)}}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)|$$

$$= \frac{1}{(n+1)(m+1)}M|K(\varepsilon)| + \varepsilon \to 0 \ as \ n,m \to \infty$$

which implies that $P-\lim \sigma_{mn}(\Delta^m, f) = f$.

For the converse if we define $f_{jk}(x) = (-1)^{j+k}x$, $x \in (0,1)$ then we get $\Delta f_{jk}(x) = 4(-1)^{j+k}x$. The sequence of functions $\{f_{jk}\}$ is statistically $(C,1,1)_{S_{\tilde{\alpha}}^2}-$summable of order $\tilde{\alpha}$ to 0 but neither bounded nor statistically convergent. □

Definition 3.5. Let $\tilde{\alpha}$ be any real number such that $\tilde{\alpha} \in (0,1]$. A double sequence of functions $\{f_{jk}\}$ is said to be Δ^m-uniformly statistically convergent of order $\tilde{\alpha}$ to the function f on a set A if, for every $\varepsilon > 0$

$$P-\lim_{n,m\to\infty}\frac{1}{n^s m^t}|\{(j,k) : j \le n, k \le m : |\Delta^m f_{jk}(x) - f(x)| \ge \varepsilon \text{ for all } x \in A\}| = 0.$$

i.e., for all $x \in A$,

$$|\Delta^m f_{jk}(x) - f(x)| < \varepsilon \ a.a.(j,k)(\tilde{\alpha})$$

In this case we write

$$S_{\tilde{\alpha}}^2 - \lim \Delta^m f_{jk}(x) = f(x) \text{ uniformly on } A \text{ or } S_{\tilde{\alpha},u}^2 - \lim \Delta^m f_{jk}(x) = f(x) \text{ on } A.$$

The set of all Δ^m-uniformly statistically convergent sequences of order $\tilde{\alpha}$ will be denoted by $S_{\tilde{\alpha},u}^2(\Delta^m, f)$.

We can give this definition as follows:

f_{jk}, Δ^m-uniformly statistically of order $\tilde{\alpha}$ converges to $f \iff$ for all $\varepsilon > 0$, $\exists$K$\subset \mathbb{N}\times\mathbb{N}$, $\delta_{\tilde{\alpha}}^2(K) = 1$ and exists $(n_0, m_0) \in K$, $n_0 = n_0(\varepsilon)$,$m_0 = m_0(\varepsilon)\ni$,$\forall j > n_0$, $k > m_0$ and$(j,k)\in$ K and $\forall \in A$, $|\Delta^m f_{jk}(x) - f(x)| < \varepsilon$.

Theorem 3.6. *Let f and f_{jk} (for all $j,k\in\mathbb{N}$) be continuous functions on $A=[a,b]\subset\mathbb{R}$ and $\tilde{\alpha}\in(0,1]$. Then $S_{\tilde{\alpha}}^2-\lim\Delta^m f_{jk}(x)=f(x)$ uniformly on A if and only if $S_{\tilde{\alpha}}^2-\lim c_{j,k}=0$, where $c_{j,k}=\max_{x\in A}|\Delta^m f_{jk}(x)-f(x)|$.*

Proof. Omitted. □

It follows from (3.2) that, if $\lim\Delta^m f_{jk}(x)=f(x)$ uniformly on A, then $S_{\tilde{\alpha}}^2-\lim\Delta^m f_{jk}(x)=f(x)$ uniformly on A. But the converse is not true, for this consider a sequence defined by

$$f_{jk}(x)=\begin{cases}3 & j=m^2,k=n^2\\ \frac{j}{1+j^2x^2} & otherwise\end{cases}\quad j,k=1,2,3...,x\in[0,1].$$

So we have

$$\Delta f_{j,k}(x)=\begin{cases}3-\frac{j}{1+j^2x^2} & j=m^2,k=n^2\\ -3+\frac{(j+1)}{1+(j+1)^2x^2} & j=m^2-1,k=n^2\\ -\frac{(j+1)}{1+(j+1)^2x^2}+3 & j=m^2-1,k=n^2-1\\ \frac{j}{1+j^2x^2}-3 & j=m^2,k=n^2-1\\ 0 & otherwise\end{cases}.$$

Then $\{f_{jk}\}$ is $\Delta-$uniformly statistically convergent sequences of order $\tilde{\alpha}$ to $f(x)=0$ on $[0,1]$ for $\tilde{\alpha}\in\left[\frac{1}{2},1\right]$ since $S_{\tilde{\alpha}}^2-\lim c_{j,k}=0$, where

$$c_{jk}=\max_{x\in[0,1]}|\Delta f_{jk}(x)-0|=\begin{cases}3-\frac{1}{2\sqrt{j}} & j=m^2,k=n^2\\ 0 & otherwise\end{cases},$$

but $\{f_{jk}\}$ is not $\Delta-$uniformly convergent on $[0,1]$ since $\lim_{k\to\infty}c_{j,k}$ does not exist.

It can be shown that if a sequence $\{f_{jk}\}$ is Δ^m-uniformly statistically convergent of order $\tilde{\alpha}$, then it is Δ^m-pointwise statistically convergent of order $\tilde{\alpha}$, but the converse does not hold. For this consider a sequence defined by

$$f_{jk}(x)=\begin{cases}1 & j=m^2,k=n^2\\ \frac{j^2x}{1+j^3x^2} & otherwise\end{cases}\quad j,k=1,2,3...,\ k\in\mathbb{N},x\in[0,1].$$

then we have

$$\Delta f_{j,k}(x) = \begin{cases} 1-\frac{j^2x}{1+j^3x^2} & j=m^2, k=n^2 \\ -1+\frac{(j+1)^2x}{1+(j+1)^3x^2} & j=m^2-1, k=n^2 \\ -\frac{j^2x}{1+j^3x^2}+1 & j=m^2-1, k=n^2-1 \\ \frac{(j+1)^2x}{1+(j+1)^3x^2}-1 & j=m^2, k=n^2-1 \\ 0 & otherwise \end{cases}$$

The sequence $\{f_{jk}\}$ is $\Delta-$pointwise statistically convergent of order $\tilde{\alpha}$ to $f(x) = 0$ on $[0,1]$ but $\{f_{jk}\}$ is not $\Delta-$uniformly statistically convergent of order $\tilde{\alpha}$ to $f(x) = 0$ on $[0,1]$ by Theorem 3.13, because

$$c_{jk} = \max x \in [0,1] \left|\Delta f_{jk}(x) - 0\right| = \begin{cases} 1-\frac{\sqrt{j}}{2} & j=m^2, k=n^2 \\ -1+\frac{\sqrt{j+1}}{2} & j=m^2-1, k=n^2 \\ -\frac{\sqrt{j}}{2}+1 & j=m^2-1, k=n^2-1 \\ \frac{\sqrt{j+1}}{2}-1 & j=m^2, k=n^2-1 \\ 0 & otherwise \end{cases}$$

and $S^2_{\tilde{\alpha}} - \lim c_{j,k}$ does not exist.

REFERENCES

1. M.Çinar, M. Karakaş and M. Et: *On pointwise and uniform statistical convergence of order α for sequences of functions*. Fixed Point Theory Appl. **2013:33** (2013), 11 pp.
2. R. Çolak: *Statistical convergence of order α,*. Modern Methods in Analysis and Its Applications, New Delhi, India, Anamaya Pub. (2010), 121–129.
3. R. Çolak and Y. Altin: *Statistical convergence of double sequences of order $\tilde{\alpha}$*. J. Funct. Spaces Appl. (2013), Art. ID 682823, 5 pp.
4. J. S. Connor: *The statistical and strong p-Cesaro convergence of sequences*. Analysis **8** (1988), 47-63.
5. O. Duman and C. Orhan: *μ−statistically convergent function sequences*. Czechoslovak Math. J. **54(2)** (2003), 413-422.
6. M. Et and R. Çolak : *On generalized difference sequence spaces*. Soochow J. Math. **21(4)** (1995), 377-386.
7. M. Et: *Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences.*Appl. Math. Comput. **219(17)** (2013), 9372-9376.
8. M. Et , H. Altinok and Y. Altin: *On some generalized sequence spaces.*Appl. Math. Comput. **154(1)** (2004), 167-173.

9. M. Et , M. Çinar and M. Karakaş: *On $\lambda-$statistical convergence of order α of sequences of function*. J. Inequal. Appl. **2013:204** (2013), 8 pp.

10. M. Et and H. Şengül : *Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α*. Filomat **28:8** (2014), 1593-1602.

11. H. Fast M. Et: *Sur la convergence statistique*. Colloq. Math. **2** (1951), 241-244.

12. J. Fridy: *On statistical convergence*. Analysis **5** (1985), 301-313.

13. A. Gökhan and M. Güngör : *On pointwise statistical convergence*. Indian J. Pure Appl. Math. **233** (2002), 1379-1384.

14. A. Gökhan , M. Güngör and M. Et: *Statistical convergence of double sequences of real-valued functions*. Int. Math. Forum **2** (2007), 365-374.

15. M. Güngör and A. Gökhan : *On uniform statistical convergence*. Int. J. Pure Appl. Math. **19(1)** (2005), 17-24.

16. M. Işik: *Strongly almost $(w, \lambda, q)-$summable sequences*. Math. Slovaca **61(5)** (2011), 779–788.

17. H. Kizmaz: *On certain sequence spaces*. Canad. Math.Bull. **24(2)** (1981), 169-176.

18. S. A. Mohiuddine ; A. Alotaibi and M. Mursaleen: *Statistical convergence of double sequences inlocally solid Riesz spaces.*Abstract and Applied Analysis (2012), Article ID 719729, 9 pages.

19. F. Móricz: *Tauberian conditions, under which statistical convergence follows from statistical summability* $(C, 1)$. J. Math. Anal. Appl. **275(1)** (2002), 277-287.

20. M. Mursaleen , R. Çolak and M. Et: *Some geometric inequalities in a new Banach sequence space*. J. Inequal. Appl. (2007), Art. ID 86757, 6 pp.

21. M. Mursaleen and Osama H. H. Edely : *Statistical convergence of double sequences*. J. Math. Anal. Appl. **288(1)** (2003), 223-231.

22. A. Pringsheim: *Zur Theorie der zweifach unendlichen Zahlenfolgen, (German)*. Math. Ann. **53(3)** (1900), 289-.321

23. I. J. Schoenberg: *The integrability of certain functions and related summability methods*. Amer. Math. Monthly **66** (1959), 361-375.

24. H. Şengül and M. Et : *On lacunary statistical convergence of order α*. Acta Math. Sci. Ser. B Engl. Ed. **34(2)** (2014), 473-482.

25. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*. Colloq. Math. **2** (1951), 73-74.

26. B. C. Tripathy and B. Sarma : *Statistically convergent difference double sequence spaces*. Acta Math. Sin. (Engl. Ser.) **24(5)** (2008), 737-742.

27. A. Zygmund: *Trigonometric Series*. Cambridge University Press, Cambridge, UK, (1979).

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