FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 2 (2020), 367–377
https://doi.org/10.22190/FUMI2002367D

NEW FIXED POINT RESULTS FOR T-CONTRACTIVE MAPPING WITH C-DISTANCE IN CONE METRIC SPACES

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Abstract. In this article, we generalize and improve the results of Fadail et al.[Z. M. Fadail and S. M. Abusalim, Int. Jour. of Math. Anal., Vol. 11, No. 8(2017), pp. 397-405.] and Dubey et al.[AnilKumar Dubey and Urmila Mishra, Non. Func. Anal. Appl., Vol. 22, No. 2(2017), pp 275-286.] under the concept of a c-distance in cone metric spaces. We prove the existence and uniqueness of the fixed point for T-contractive type mapping under the concept of c-distance in cone metric spaces.
Keywords: Fixed point; T-contractive mapping; Cone metric space; c-distance.

1. Introduction

In 2007, Huang and Zhang[12] first introduced the concept of cone metric spaces and they established and proved the existence of fixed point theorems which is an extension of the Banach contraction mapping principle in to the cone metric spaces. Recently, Cho et al.[3] introduced the concept of c-distance in a cone metric spaces and proved some fixed point results in ordered cone metric spaces. Afterward, many authors have generalized and studied fixed point theorems under c-distance in cone metric spaces (see [1, 7, 8, 9, 10, 11, 14, 15, 16]). In 2009, Beiranvand et al.[2] introduced new classes of contractive functions and established the Banach principle. Since then, fixed point theorems for T-contraction mapping on cone metric spaces have been appeared, see for instance [4, 5, 6] and [11].

The purpose of this paper is to extend and generalize some results on c-distance in cone metric spaces. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is $intP \neq \phi$. Also, in this paper we assume $\mathbb{R}$ as a set of real numbers and $\mathbb{N}$ as a set of natural numbers.

Received April 08, 2018; accepted May 02, 2020
2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H10

2. Preliminaries

Definition 2.1. ([12]) Let E be a real Banach space and θ denote to the zero element in E. A cone P is a subset of E such that:

(1) P is a non-empty, closed and $P \neq \{\theta\}$;

(2) If a, b are non-negative real numbers and $x, y \in P$ then $ax + by \in P$;

(3) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.

Definition 2.2. ([12]) A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, \theta \preceq x \preceq y$ implies $||x|| \leq K||y||$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose E is a Banach space, P is a cone in E with $intP \neq \phi$ and $\preceq$ is partial ordering with respect to P.

Definition 2.3. ([12]) Let X be a non empty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(i) If $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.1. Let $E = \mathbb{R}^2$, and $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}^2$ and suppose that $d : X \times X \to E$ is defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\})$ where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. It is easy to see that d is a cone metric, and hence (X, d) becomes a cone metric space over (E, P). Also, we have P is a solid and normal cone where the normal constant $K = 1$.

Definition 2.4. ([12]) Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$:

(1) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x.

(2) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) if every Cauchy sequence in X is convergent in X then (X, d) is called a complete cone metric space.

The following Lemma is useful to prove our results.

Lemma 2.1. *([13])*

(1) If E be a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

(2) If $c \in intP$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next, we give the notion of c-distance on a cone metric space (X, d) of Cho et al. in [3].

Definition 2.5. ([3]) Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q_1) $\theta \preceq q(x, y)$ for all $x, y \in X$;

(q_2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q_3) for each $x \in X$ and $n \geq 1$ if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.2. ([3]) Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$, $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ is defined by $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

The following Lemma is very important to prove our results.

Lemma 2.2. ([3]) *Let (X, d) be a cone metric space and q is a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following hold:*

(1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.

(2) If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.

(3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

Remark 2.1. ([3])

(1) $q(x,y) = q(y,x)$ does not necessarily for all $x, y \in X$.

(2) $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Next definition taken from [2]:

Definition 2.6. Let (X,d) be a cone metric space, P a solid cone and $T : X \to X$. Then

(a) T is said to be continuous if $\lim_{n\to\infty} x_n = x^*$ implies that $lim_{n\to\infty} Tx_n = Tx^*$, for all $\{x_n\}$ in X;

(b) T is said to be sequentially convergent if we have, for every sequence $\{x_n\}$, if $\{Tx_n\}$ is convergent, then $\{x_n\}$ is also convergent;

(c) T is said to be subsequentially convergent if we have, for every sequence $\{x_n\}$ that $\{Tx_n\}$ is convergent, implies $\{x_n\}$ has a convergent subsequence.

Now, we give our main results in this paper.

3. Main Results

Theorem 3.1. *Let (X,d) be a complete cone metric space, P a solid cone and q be a c-distance on X. Let $T : X \to X$ be an one to one, continuous function and subsequentially convergent and $f : X \to X$ be a mapping. In addition, suppose that there exists mapping $k, l : X \to [0,1)$ such that the following conditions hold:*

(a) $k(fx) \leq k(x), l(fx) \leq l(x)$, *for all* $x \in X$;

(b) $(k+2l)(x) < 1$ *for all* $x \in X$;

(c) $q(Tfx, Tfy) \preceq k(x)q(Tx,Ty) + l(x)[q(Tfx,Ty) + q(Tfy,Tx)]$

for all $x, y \in X$. Then the map f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$, then $q(Tu,Tu) = \theta$.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, ...x_{n+1} = fx_n = f^{n+1}x_0$. Then we have

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &= q(Tfx_{n-1}, Tfx_n) \\ &\preceq k(x_{n-1})q(Tx_{n-1}, Tx_n) + l(x_{n-1})[q(Tfx_{n-1}, Tx_n) \\ &\quad +q(Tfx_n, Tx_{n-1})] \\ &= k(fx_{n-2})q(Tx_{n-1}, Tx_n) + l(fx_{n-2})[q(Tx_n, Tx_n) \\ &\quad +q(Tx_{n+1}, Tx_{n-1})] \\ &\preceq k(x_{n-2})q(Tx_{n-1}, Tx_n) + l(x_{n-2})[q(Tx_{n-1}, Tx_n) \\ &\quad +q(Tx_n, Tx_{n+1})], \end{aligned}$$

continuing in this manner, we can get

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &\preceq k(x_0)q(Tx_{n-1}, Tx_n) + l(x_0)q(Tx_{n-1}, Tx_n) \\ &\quad + l(x_0)q(Tx_n, Tx_{n+1}) \end{aligned}$$

and hence

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &\preceq \frac{k(x_0) + l(x_0)}{1 - l(x_0)} q(Tx_{n-1}, Tx_n) \\ &= hq(Tx_{n-1}, Tx_n) \\ &\preceq h^2 q(Tx_{n-2}, Tx_{n-1}) \\ &\preceq h^n q(Tx_0, Tx_1), \end{aligned}$$

where $h = \frac{k(x_0)+l(x_0)}{1-l(x_0)} < 1$. Note that,

$$q(Tfx_{n-1}, Tfx_n) = q(Tx_n, Tx_{n+1}) \preceq hq(Tx_{n-1}, Tx_n). \tag{3.1}$$

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(Tx_n, Tx_m) &\preceq q(Tx_n, Tx_{n+1}) + q(Tx_{n+1}, Tx_{n+2}) \\ &\quad + ... + q(Tx_{m-1}, Tx_m) \\ &\preceq (h^n + h^{n+1} + ... + h^{m-1}) q(Tx_0, Tx_1) \\ &\preceq \frac{h^n}{1-h} q(Tx_0, Tx_1) \to \theta \quad as \quad n \to \infty. \end{aligned}$$

Thus, Lemma 2.2(3) shows that $\{Tx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $Tx_n \to v$ as $n \to \infty$. Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So, there are $x^* \in X$ and $\{x_{n_i}\}$ such that $x_{n_i} \to x^*$ as $i \to \infty$. Since T is continuous, we obtain $\lim Tx_{n_i} \to Tx^*$. The uniqueness of the limit implies that $Tx^* = v$. Then by (q_3), we have

$$q(Tx_n, Tx^*) \preceq \frac{k^n}{1-k} q(Tx_0, Tx_1). \tag{3.2}$$

Now by using (3.1), we have

$$\begin{aligned} q(Tx_n, Tfx^*) &= q(Tfx_{n-1}, Tfx^*) \\ &\preceq hq(Tx_{n-1}, Tx^*) \\ &\preceq h\frac{k^{n-1}}{1-k} q(Tx_0, Tx_1) \end{aligned}$$

$$= \frac{h^n}{1-h} q(Tx_0, Tx_1). \tag{3.3}$$

By Lemma 2.2(1), (3.2) and (3.3), we have $Tx^* = Tfx^*$. Since T is one to one, then $x^* = fx^*$. Thus, x^* is fixed point of f. Suppose that $u = fu$, then we have

$$\begin{aligned} q(Tu, Tu) &= q(Tfu, Tfu) \\ &\preceq k(u)q(Tu, Tu) + l(u)[q(Tfu, Tu) + q(Tfu, Tu)] \\ &= k(u)q(Tu, Tu) + l(u)[q(Tu, Tu) + q(Tu, Tu)] \\ &\preceq (k + 2l)(x_0)q(Tu, Tu). \end{aligned}$$

Since $(k+2l)(x_0) < 1$, Lemma 2.1(1) shows that $q(Tu, Tu) = \theta$. Finally, suppose there is another fixed point y^* of f, then we have

$$\begin{aligned} q(Tx^*, Ty^*) &= q(Tfx^*, Tfy^*) \\ &\preceq k(x^*)q(Tx^*, Ty^*) + l(x^*)[q(Tfx^*, Ty^*) + q(Tfy^*, Tx^*)] \\ &= k(x^*)q(Tx^*, Ty^*) + l(x^*)[q(Tx^*, Ty^*) + q(Ty^*, Tx^*)] \\ &= (k + 2l)(x^*)q(Tx^*, Ty^*). \end{aligned}$$

Since $(k+2l)(x^*) < 1$, Lemma 2.1(1) shows that $q(Tx^*, Ty^*) = \theta$. Also we have $q(Tx^*, Tx^*) = \theta$. Thus Lemma 2.2(1), $Tx^* = Ty^*$. Since T is one to one, then $x^* = fx^*$. Therefore, the fixed point is unique □

Corollary 3.1. *Let (X, d) be a complete cone metric space, P a solid cone and q be a c-distance on X. Let $T : X \to X$ be one to one, continuous function and subsequentially convergent and $f : X \to X$ be a mapping. In addition, suppose that there exists mapping $k, l : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \le k(x), l(fx) \le l(x)$, *for all* $x \in X$;

(b) $(k + 2l)(x) < 1$ *for all* $x \in X$;

(c) $q(Tfx, Tfy) \preceq k(x)q(Tx, Ty) + l(x)[q(Tfx, Tx) + q(Tfy, Ty)]$

for all $x, y \in X$. Then the map f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$, then $q(Tu, Tu) = \theta$.

Theorem 3.2. *Let (X, d) be a complete cone metric space, P a solid cone and q be a c-distance on X. Let $T : X \to X$ be an one to one, continuous function and subsequentially convergent and $f : X \to X$ be a mapping. In addition suppose that there exists mapping $k, l, r : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \le k(x), l(fx) \le l(x), r(fx) \le r(x)$ *for all* $x \in X$;

(b) $(k+2l+2r)(x) < 1$ *for all* $x \in X$;

(c) $q(Tfx, Tfy) \preceq k(x)q(Tx, Ty) + l(x)[q(Tfy, Tx) + q(Tfx, Ty)]$
$+ r(x)[q(Tfx, Tx) + q(Tfy, Ty)]$

for all $x, y \in X$. *Then the map* f *has a unique fixed point* $x^* \in X$ *and for any* $x \in X$, *iterative sequence* $\{fx_n\}$ *converges to the fixed point. If* $u = fu$, *then* $q(Tu, Tu) = \theta$.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, ...x_{n+1} = fx_n = f^{n+1}x_0$. Then we have

$$\begin{aligned}
q(Tx_n, Tx_{n+1}) &= q(Tfx_{n-1}, Tfx_n) \\
&\preceq k(x_{n-1})q(Tx_{n-1}, Tx_n) + l(x_{n-1})[q(Tfx_n, Tx_{n-1}) \\
&\quad +q(Tfx_{n-1}, Tx_n)] + r(x_{n-1})[q(Tfx_{n-1}, Tx_{n-1}) \\
&\quad +q(Tfx_n, Tx_n)] \\
&= k(fx_{n-2})q(Tx_{n-1}, Tx_n) + l(fx_{n-2})[q(Tx_{n+1}, Tx_{n-1}) \\
&\quad +q(Tx_n, Tx_n)] + r(fx_{n-2})[q(Tx_n, Tx_{n-1}) + q(Tx_{n+1}, Tx_n)] \\
&\preceq k(x_{n-2})q(Tx_{n-1}, Tx_n) + l(x_{n-2})[q(Tx_{n-1}, Tx_n) \\
&\quad +q(Tx_n, Tx_{n+1})] + r(x_{n-2})[q(Tx_{n-1}, Tx_n) + q(Tx_n, Tx_{n+1})],
\end{aligned}$$

continuing in this manner, we can get

$$\begin{aligned}
q(Tx_n, Tx_{n+1}) &\preceq (k(x_0) + l(x_0) + r(x_0))q(Tx_{n-1}, Tx_n) + (l(x_0) \\
&\quad +r(x_0))q(Tx_n, Tx_{n+1})
\end{aligned}$$

and hence

$$\begin{aligned}
q(Tx_n, Tx_{n+1}) &\preceq \frac{k(x_0) + l(x_0) + r(x_0)}{1 - l(x_0) - r(x_0)} q(Tx_{n-1}, Tx_n) \\
&= hq(Tx_{n-1}, Tx_n) \\
&\preceq h^2 q(Tx_{n-2}, Tx_{n-1}) \\
&\preceq h^n q(Tx_0, Tx_1),
\end{aligned}$$

where $h = \frac{k(x_0)+l(x_0)+r(x_0)}{1-l(x_0)-r(x_0)} < 1$. Note that,

$$q(Tfx_{n-1}, Tfx_n) = q(Tx_n, Tx_{n+1}) \preceq hq(Tx_{n-1}, Tx_n). \tag{3.4}$$

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned}
q(Tx_n, Tx_m) &\preceq q(Tx_n, Tx_{n+1}) + q(Tx_{n+1}, Tx_{n+2}).... + q(Tx_{m-1}, Tx_m) \\
&\preceq (h^n + h^{n+1} + ... + h^{m-1})q(Tx_0, Tx_1) \\
&\preceq \frac{h^n}{1-h} q(Tx_0, Tx_1) \to \theta \quad as \quad n \to \infty.
\end{aligned}$$

Thus, Lemma 2.2(3) shows that $\{Tx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $Tx_n \to v$ as $n \to \infty$. Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So there are $x^* \in X$ and $\{x_{n_i}\}$ such that $x_{n_i} \to x^*$ as $i \to \infty$. Since T is continuous, we obtain $\lim Tx_{n_i} = Tx^*$. The uniqueness of the limit implies that $Tx^* = v$. Then by (q_3), we have

$$q(Tx_n, Tx^*) \preceq \frac{k^n}{1-k} q(Tx_0, Tx_1). \tag{3.5}$$

Now by using (3.4), we have

$$\begin{aligned} q(Tx_n, Tfx^*) &= q(Tfx_{n-1}, Tfx^*) \\ &\preceq hq(Tfx_{n-1}, Tx^*) \\ &\preceq h\frac{k^{n-1}}{1-k} q(Tx_0, Tx_1) \\ &= \frac{h^n}{1-h} q(Tx_0, Tx_1). \end{aligned} \tag{3.6}$$

By Lemma 2.2(1), (3.5) and (3.6), we have $Tx^* = Tfx^*$. Since T is one to one, then $x^* = fx^*$. Thus, x^* is a fixed point of f. Suppose that $u = fu$, then we have

$$\begin{aligned} q(Tu, Tu) &= q(Tfu, Tfu) \\ &\preceq k(u)q(Tu, Tu) + l(u)[q(Tfu, Tu) + q(Tfu, Tu)] \\ &\quad + r(u)[q(Tfu, Tu) + q(Tfu, Tu)] \\ &= k(u)q(Tu, Tu) + l(u)[q(Tu, Tu) + q(Tu, Tu)] \\ &\quad + r(u)[q(Tu, Tu) + q(Tu, Tu)] \\ &\preceq (k + 2l + 2r)(x_0)q(Tu, Tu). \end{aligned}$$

Since $(k + 2l + 2r)(x_0) < 1$, Lemma 2.1(1) shows that $q(Tu, Tu) = \theta$. Finally, suppose there is another fixed point y^* of f, then we have

$$\begin{aligned} q(Tx^*, Ty^*) &= q(Tfx^*, Tfy^*) \\ &\preceq k(x^*)q(Tx^*, Tx^*) + l(x^*)[q(Tfy^*, Tx^*) + q(Tfx^*, Ty^*)] \\ &\quad + r(x^*)[q(Tfx^*, Tx^*) + q(Tfy^*, Ty^*)] \\ &= k(x^*)q(Tx^*, Ty^*) + l(x^*)[q(Ty^*, Tx^*) + q(Tx^*, Ty^*)] \\ &\quad + r(x^*)[q(Tx^*, Tx^*) + q(Ty^*, Ty^*)] \\ &= (k + 2l)(x^*)q(Tx^*, Ty^*) \\ &\preceq (k + 2l + 2r)(x^*)q(Tx^*, Ty^*). \end{aligned}$$

Since $(k + 2l + 2r)(x^*) < 1$, Lemma 2.1(1) shows that $q(Tx^*, Ty^*) = \theta$. Also we have, $q(Tx^*, Tx^*) = \theta$. Thus, by Lemma 2.2(1), $Tx^* = Ty^*$. Since T is one to one, then $x^* = fx^*$. Therefore, the fixed point is unique. $\square$

Theorem 3.3. *Let (X, d) be a complete cone metric space, P a solid cone and q be a c-distance on X. Let $T : X \to X$ be an one to one, continuous function and subsequentially convergent and $f : X \to X$ be a mapping. In addition suppose that there exists mapping $k, r, l, t : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \leq k(x), r(fx) \leq r(x), l(fx) \leq l(x), t(fx) \leq t(x)$ *for all* $x \in X$;

(b) $(k + r + l + 2t)(x) < 1$ *for all* $x \in X$;

(c) $q(Tfx, Tfy) \preceq k(x)q(Tx, Ty) + r(x)q(Tfx, Tx) + l(x)q(Tfy, Ty)$
$+ t(x)[q(Tfx, Ty) + q(Tfy, Tx)]$

for all $x, y \in X$. Then map f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$, then $q(Tu, Tu) = \theta$.

Proof. The proof of this theorem is same as Theorem 3.1. □

Now we give an example which illustrates our Theorems 3.1.

Example 3.1. Let $E = \mathbb{R}$ and $P = \{x \in E, x \geq 0\}$, let $X = [0, 1]$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|e^t$ where $e^t \in E$. Then (X, d) is complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = ye^t$ for all $x, y \in X$. Then q is a c-distance on X. Define the mapping $T, f : X \to X$ by $fx = \frac{x^2}{4}$ and $T(x) = x^4$ for all $x \in X$. Take mapping $k, l : X \to [0, 1)$ by $k(x) = \frac{x+1}{4}$ and $l(x) = \frac{x}{8}$, for all $x \in X$. Observe that

(i) $k(fx) = k\left(\frac{x^2}{4}\right) = \left(\frac{\frac{x^2}{4}+1}{4}\right) = \frac{1}{4}\left(\frac{x^2}{4} + 1\right) \leq (\frac{1}{4})(x + 1) = k(x)$ for all $x \in X$.

(ii) $l(fx) = l(\frac{x^2}{4}) = \left(\frac{\frac{x^2}{4}}{8}\right) = \frac{1}{8}\left(\frac{x^2}{4}\right) \leq \frac{1}{8}(x) = l(x)$, for all $x \in X$.

(iii) $(k + 2l)(x) = \frac{x+1}{4} + \frac{x}{4} = \frac{1}{4}(2x + 1) < 1$, for all $x \in X$.

Now, we have

$$\begin{aligned} q(Tfx, Tfy) &= Tfye^t \\ &= \frac{y^8}{256}e^t \\ &\preceq \left(\frac{y+1}{4}\right)y^4e^t \\ &= k(x)q(Tx, Ty) \\ &\preceq k(x)q(Tx, Ty) + l(x)[q(Tfx, Ty) + q(Tfy, Tx)]. \end{aligned}$$

Therefore, all conditions of Theorem 3.1 are satisfied. Hence f has a unique fixed point $x = 0$ with $q(0, 0) = \theta$.

Competing Interests

The authors declare that they have no competing interests.

Authors' contribution
All authors contributed equally and significantly in writing this article. All authors read and approved final manuscript.

Acknowledgments
The authors are thankful to the learned referee for his/her deep observations and their suggestions, which greatly helped us to improve the paper significantly.

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